Quantum walks with coins undergoing different quantum noisy channels*

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Quantum walks have significantly different properties compared to classical random walks, which have potential applications in quantum computation and quantum simulation. We study Hadamard quantum walks with coins undergoing different quantum noisy channels and deduce the analytical expressions of the first two moments of position in the long-time limit. Numerical simulations have been done, the results are compared with the analytical results, and they match extremely well. We show that the variance of the position distributions of the walks grows linearly with time when enough steps are taken and the linear coefficient is affected by the strength of the quantum noisy channels.

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1. Introduction

The classical random walk (CRW) is well known due to its diverse applications in many fields, including designing algorithms in mathematics, describing diffusion processes in physics and biology, and stimulating the price fluctuation of stock market in finance.[1–3] Emerging as the quantum analogue of the CRW, the quantum walk (QW) has drawn intense investigation interest in recent years and exhibits striking nonclassical properties, among which the most important one is the ballistic spreading. Hereto, studies have shown that the QW has implications in various fields, such as quantum algorithm engineering,[4–8] quantum simulation,[9–17] universal quantum computing,[18,19] and energy transport in biology.[20] Experimentally, several different physical systems have been used to implement QWs, for instance, systems of trapped atoms or ions,[21–29] nuclear magnetic resonance,[30,31] and linear optics.[32–43] As real physical systems inevitably suffer from unwanted interactions with the environment, which show up as noise, research taking noise into account on QWs[44,45] is of practical value. QWs with noise have been extensively studied recently, mainly including the noise existing in the graphs on which the walker walks,[46–49] the noise caused by the interaction between the walker or the coin and the environment,[50–54] and the noise imposed on different coins used in the QWs.[55–58]

In this paper, we study discrete-time Hadamard QWs with coins undergoing different quantum noisy channels, including dephasing channel, bit flip channel, and depolarizing channel, which are of significance in quantum systems. By deducing the analytical expressions of the first two moments of position in the long-time limit and direct numerical simulations, we find that the variance of the position distribution of the walks grows linearly with time in the long-time limit, which means the QW is transmitted into the CRW. It indicates that noise which leads to decoherence plays a fundamental role in the transition from the quantum to the classical regime. Besides, the influence of the strength of quantum noise is found to be responsible for the linear coefficient.

This paper is organized as follows. In Section 2, we briefly review the perfect Hadamard QW. In Section 3, we present discrete-time QWs with coins undergoing three different types of quantum noisy channels (dephasing channel, bit flip channel, and depolarizing channel) respectively, and give the analytical expressions of the first two moments of position in the long-time limit. In Section 4, we analyze the results of numerical simulations compared with the analytical results. Finally, in Section 5, we summarize and give a conclusion.

2. Perfect Hadamard quantum walk

The perfect Hadamard QW is usually considered to be the simplest version of QW, which includes only one walker walking on a line and a two-dimensional quantum coin with no decoherence. Before each step is taken, the Hadamard operator $\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is used for coin flipping. The walker takes a step left or right according to the coin state after the flipping, head or tail. The state of the walker is $|x_t\rangle$, where x_t is the coordinate of the walker's position on the line after t steps. Starting from the origin of the line gives the initial state of the walker $|x_0\rangle = |0\rangle$. The initial coin state is usually chosen to be $|\phi_0\rangle = (|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$. The initial state of the system

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becomes $|\Phi_0\rangle = |x_0\rangle \otimes |\phi_0\rangle$. The step operator applied to the position state of the walker takes the form

$$\hat{F} = \hat{S} \otimes \hat{P}_0 + \hat{S}^\dagger \otimes \hat{P}_1, \tag{1}$$

where $\hat{S}, \hat{S}^\dagger$ are the conditional position shift operators on the walk's position state

$$\hat{S}\,|x\rangle = |x+1\rangle\,, \quad \hat{S}^\dagger\,|x\rangle = |x-1\rangle\,, \tag{2}$$

and $\hat{P}_0, \hat{P}_1$ are projectors applied to the coin

$$\hat{P}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \hat{P}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

The evolution operator of the system for one step of the walk then becomes

$$\hat{E} = \hat{F}\left(\hat{I}_{\rm w} \otimes \hat{H}\right), \tag{4}$$

where $\hat{I}_{\rm w}$ is the identity operator applied to the walker. Therefore, the final state of the system after t steps is

$$|\Phi_t\rangle = \left(\hat{E}\right)^t |\Phi_0\rangle\,. \tag{5}$$

In order to obtain an analytical result, we perform a Fourier transformation from the position space to the momentum space

$$|k\rangle = \sum_x \mathrm{e}^{\mathrm{i}kx}\,|x\rangle. \tag{6}$$

The state $|k\rangle$ is the eigenstate of $\hat{S}, \hat{S}^\dagger$ and satisfies the relations

$$\hat{S}\,|k\rangle = \mathrm{e}^{-\mathrm{i}k}\,|k\rangle\,, \quad \hat{S}^\dagger\,|k\rangle = \mathrm{e}^{\mathrm{i}k}\,|k\rangle\,. \tag{7}$$

The inverse Fourier transformation is given by

$$|x\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}kx}\,|k\rangle. \tag{8}$$

Thus, the initial state of the walker becomes

$$|0\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\,|k\rangle. \tag{9}$$

The evolution operator in the momentum space becomes

$$\hat{E}\left(|k\rangle \otimes |\phi_0\rangle\right) = |k\rangle \otimes \hat{U}_k\,|\phi_0\rangle\,, \tag{10}$$

where $\hat{U}_k$ is a unitary operator applied to the coin state and takes the form

$$\hat{U}_k = \left(\mathrm{e}^{-\mathrm{i}k}\hat{P}_0 + \mathrm{e}^{\mathrm{i}k}\hat{P}_1\right)\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}k} & \mathrm{e}^{-\mathrm{i}k} \\ \mathrm{e}^{\mathrm{i}k} & -\mathrm{e}^{\mathrm{i}k} \end{pmatrix}. \tag{11}$$

Meanwhile, the final state of the system in the momentum space can be expressed as

$$\rho(t) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,|k\rangle\langle k'| \otimes \left(\hat{U}_k\right)^t |\phi_0\rangle\langle\phi_0| \left(\hat{U}_{k'}^\dagger\right)^t. \tag{12}$$

The solution to the QW of this case has been shown in Ref. [59] and the variance of the position distribution of the walk grows quadratically with time in the long-time limit, which is a general symbol for QWs.

3. Quantum walks with coins undergoing different noisy channels

Now we discuss the Hadamard QW with decoherence introduced via the coin undergoing quantum noisy channels on the basis of the reviewed content in the above section. The general form of quantum noise applied to the coin state is

$$\rho_{\rm c}' = \sum_n \hat{A}_n \rho_{\rm c} \hat{A}_n^\dagger, \tag{13}$$

where $\rho_{\rm c} = |\phi\rangle\langle\phi|$, which is the density matrix of the initial coin state, $\rho_{\rm c}'$ is the density matrix of the coin state with noise involved, and the ensemble of $\hat{A}_n$ represents the operation of noise, which satisfies

$$\sum_n \hat{A}_n^\dagger \hat{A}_n = \sum_n \hat{A}_n \hat{A}_n^\dagger = \hat{I}_{\rm c}. \tag{14}$$

For the initial state of the system,

$$\rho(0) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,|k\rangle\langle k'| \otimes |\phi_0\rangle\langle\phi_0|\,, \tag{15}$$

after one step, the state becomes

$$\begin{aligned}\rho(1) \ = \ & \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,|k\rangle\langle k'| \\ & \otimes \sum_n \hat{U}_k \hat{A}_n\,|\phi_0\rangle\langle\phi_0|\hat{A}_n^\dagger \hat{U}_{k'}^\dagger.\end{aligned} \tag{16}$$

Similarly, after t steps, the state evolves to

$$\begin{aligned}\rho(t) \ = \ & \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,|k\rangle\langle k'| \otimes \sum_{n_1,n_2,\cdots,n_t} \Big(\hat{U}_k \hat{A}_{n_t} \cdots \hat{U}_k \hat{A}_{n_2} \\ & \times \hat{U}_k \hat{A}_{n_1}\,|\phi_0\rangle\langle\phi_0|\hat{A}_{n_1}^\dagger \hat{U}_{k'}^\dagger \hat{A}_{n_2}^\dagger \hat{U}_{k'}^\dagger \cdots \hat{A}_{n_t}^\dagger \hat{U}_{k'}^\dagger\Big).\end{aligned} \tag{17}$$

Making use of the method in Ref. [58], we define a superoperator $\mathscr{L}_{kk'}$ for the evolution of the QW system with noise, and $\mathscr{L}_{kk'}$ satisfies

$$\mathscr{L}_{kk'}\rho_{\rm c} = \sum_n \hat{U}_k \hat{A}_n\,|\phi_0\rangle\langle\phi_0|\hat{A}_n^\dagger \hat{U}_{k'}^\dagger. \tag{18}$$

Moreover, $\mathscr{L}_{kk'}$ possesses the property of trace preserving,

$$k = k', \quad \mathrm{Tr}\{\mathscr{L}_{kk}\rho\} = \mathrm{Tr}\{\rho\}. \tag{19}$$

Then we rewrite Eq. (17) with $\mathscr{L}_{kk'}$

$$\rho(t) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,|k\rangle\langle k'| \otimes \mathscr{L}_{kk'}^t\,|\phi_0\rangle\langle\phi_0|\,. \tag{20}$$

The probability of the walker reaching position x after t steps is

$$\begin{aligned} p(x,t) \ &= \ \mathrm{Tr}\{\left(|x\rangle\langle x| \otimes \hat{I}_{\rm c}\right)\rho(t)\} \\ &= \ \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}k'}{2\pi}\,\mathrm{e}^{-\mathrm{i}x(k-k')} \end{aligned}$$

$$\times \mathrm{Tr}\{\mathscr{L}_{kk'}^{t}|\phi_0\rangle\langle\phi_0|\}. \quad (21)$$

The moment of the position distribution for the QW is

$$\begin{aligned}\langle\hat{x}^m\rangle_t &= \sum_x x^m p(x,t)\\ &= \frac{(-\mathrm{i})^m}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\int_{-\pi}^{\pi}\mathrm{d}k'\delta^{(m)}(k-k')\\ &\quad\times\mathrm{Tr}\{\mathscr{L}_{kk'}^{t}|\phi_0\rangle\langle\phi_0|\}. \quad (22)\end{aligned}$$

We use the relations below for further evaluating:

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{Tr}\{\mathscr{L}_{kk'}\rho\} = \sum_n \mathrm{Tr}\Big\{\Big(\frac{\mathrm{d}\hat{U}_k}{\mathrm{d}k}\Big)\hat{A}_n\rho\hat{A}_n^{\dagger}\hat{U}_{k'}^{\dagger}\Big\}, \quad (23)$$

$$\frac{\mathrm{d}\hat{U}_k}{\mathrm{d}k} = -\mathrm{i}(\hat{P}_0-\hat{P}_1)\hat{U}_k = -\mathrm{i}\hat{Z}\hat{U}_k, \quad (24)$$

$$\frac{\mathrm{d}\hat{U}_k^{\dagger}}{\mathrm{d}k} = \mathrm{i}\hat{U}_k^{\dagger}(\hat{P}_0-\hat{P}_1) = \mathrm{i}\hat{U}_k^{\dagger}\hat{Z}, \quad (25)$$

$$\hat{Z} = \hat{P}_0-\hat{P}_1, \quad (26)$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{Tr}\{\mathscr{L}_{kk'}\rho\} &= -\mathrm{i}\mathrm{Tr}\{\hat{Z}\mathscr{L}_{kk'}\rho\}\\ &= -\mathrm{i}\mathrm{Tr}\{(\mathscr{L}_{kk'}\rho)\hat{Z}\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}k'}\mathrm{Tr}\{\mathscr{L}_{kk'}\rho\}. \quad (27)\end{aligned}$$

Besides, we have to use the formula

$$\frac{1}{2\pi}\sum_x\big(x^m\mathrm{e}^{-\mathrm{i}x(k-k')}\big) = (-\mathrm{i})^m\delta^{(m)}(k-k'). \quad (28)$$

Due to the trace preserving property of $\mathscr{L}_{kk'}$, we redefine $\mathscr{L}_{kk} = \mathscr{L}_k$ for simplicity. Thus, we have the following expressions for the first two moments of position:

$$\begin{aligned}\langle\hat{x}\rangle_t &= \frac{(-\mathrm{i})}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{Tr}\{\mathscr{L}_k^t|\phi_0\rangle\langle\phi_0|\}\\ &= \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\sum_{j=1}^{t}\mathrm{Tr}\{\hat{Z}\mathscr{L}_k^j|\phi_0\rangle\langle\phi_0|\}, \quad (29)\end{aligned}$$

$$\begin{aligned}\langle\hat{x}^2\rangle_t &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\Bigg[\sum_{j=1}^{t}\sum_{j'=1}^{j}\mathrm{Tr}\{\hat{Z}\mathscr{L}_k^{j-j'}(\hat{Z}\mathscr{L}_k^{j'}|\phi_0\rangle\langle\phi_0|)\}\\ &\quad+\sum_{j=1}^{t}\sum_{j'=1}^{j-1}\mathrm{Tr}\{\hat{Z}\mathscr{L}_k^{j-j'}[(\mathscr{L}_k^{j'}|\phi_0\rangle\langle\phi_0|)\hat{Z}]\}\Bigg]. \quad (30)\end{aligned}$$

In this paper, we consider three different quantum noisy channels. The general form of noise mapping on the coin state is

$$\varepsilon\rho_{\mathrm{c}} = \sum_{i=0}^{3}p_i(t)\hat{\sigma}_i\rho_{\mathrm{c}}\hat{\sigma}_i, \quad (31)$$

where $\hat{\sigma}_i$ is the Pauli matrix, $\hat{\sigma}_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\hat{\sigma}_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Corresponding to $\sum_n\hat{A}_n\hat{A}_n^{\dagger} = \hat{I}_{\mathrm{c}}$, we have $\hat{A}_i = \sqrt{p_i(t)}\hat{\sigma}_i$. The parameters for the dephasing channel take the form

$$\begin{aligned}p_0^{\mathrm{I}}(t) &= \frac{1+\mathrm{e}^{-\gamma t}}{2},\\ p_1^{\mathrm{I}}(t) &= p_2^{\mathrm{I}}(t) = 0,\\ p_3^{\mathrm{I}}(t) &= \frac{1-\mathrm{e}^{-\gamma t}}{2}.\end{aligned}$$

The parameters for the bit flip channel take the form

$$\begin{aligned}p_0^{\mathrm{II}}(t) &= \frac{1+\mathrm{e}^{-\gamma t}}{2},\\ p_1^{\mathrm{II}}(t) &= \frac{1-\mathrm{e}^{-\gamma t}}{2},\\ p_2^{\mathrm{II}}(t) &= p_3^{\mathrm{II}}(t) = 0.\end{aligned}$$

The parameters for the depolarizing channel take the form

$$\begin{aligned}p_0^{\mathrm{III}}(t) &= \frac{1+3\mathrm{e}^{-\gamma t}}{4},\\ p_1^{\mathrm{III}}(t) = p_2^{\mathrm{III}}(t) &= p_3^{\mathrm{III}}(t) = \frac{1-\mathrm{e}^{-\gamma t}}{4}.\end{aligned}$$

Here γ is the parameter related to the strength of the noisy channel and t is the time of the coin in the noisy channel. We choose γ and t both to be constant and define $p = \mathrm{e}^{-\gamma t}$, which means that the coin will be in the identical strength noisy channel for the same duration of time when each flipping is made. The new parameter p ($0 \le p \le 1$) also indicates the strength of the noisy channel involved. The larger the p, the smaller the strength of the noisy channel. From the definitions, it is obvious that if $p = 1$, there is no noise at all. An arbitrary two-by-two matrix can be written in the representation of the usual Pauli matrixes

$$\rho = r_0\hat{\sigma}_0 + r_1\hat{\sigma}_1 + r_2\hat{\sigma}_2 + r_3\hat{\sigma}_3. \quad (32)$$

For simplicity, the matrix can be represented as a column vector with $\hat{\sigma}_i$ ($i = 0,1,2,3$) being the base

$$\rho = \begin{pmatrix}r_0\\r_1\\r_2\\r_3\end{pmatrix}. \quad (33)$$

Then $\mathscr{L}_k$ turns into

$$\mathscr{L}_{k1} = \begin{pmatrix}1&0&0&0\\0&0&p\sin 2k&\cos 2k\\0&0&-p\cos 2k&\sin 2k\\0&p&0&0\end{pmatrix}, \quad (34)$$

$$\mathscr{L}_{k2} = \begin{pmatrix}1&0&0&0\\0&0&p\sin 2k&p\cos 2k\\0&0&-p\cos 2k&p\sin 2k\\0&1&0&0\end{pmatrix}, \quad (35)$$

$$\mathscr{L}_{k3} = \begin{pmatrix}1&0&0&0\\0&0&p\sin 2k&p\cos 2k\\0&0&-p\cos 2k&p\sin 2k\\0&p&0&0\end{pmatrix}, \quad (36)$$

for the dephasing channel, bit flip channel, and depolarizing channel, respectively. With the representation of the Pauli matrixes, the processes of left multiplying and right multiplying

by $\hat{Z}$ are given by matrixes $Z_{\rm L}$ and $Z_{\rm R}$,

$$Z_{\rm L}=\begin{pmatrix}0&0&0&1\\0&0&-{\rm i}&0\\0&{\rm i}&0&0\\1&0&0&0\end{pmatrix},\quad Z_{\rm R}=\begin{pmatrix}0&0&0&1\\0&0&{\rm i}&0\\0&-{\rm i}&0&0\\1&0&0&0\end{pmatrix},\tag{37}$$

$$Z_{\rm L}+Z_{\rm R}=\begin{pmatrix}0&0&0&2\\0&0&0&0\\0&0&0&0\\2&0&0&0\end{pmatrix}.\tag{38}$$

Then we go back to Eq. (29) for the first moment of position

$$\begin{aligned}&\langle\hat{x}\rangle_t\\&=\frac{(-1)}{2\pi}\int_{-\pi}^{\pi}{\rm d}k\begin{pmatrix}0&0&0&1\end{pmatrix}\left[(1-\mathscr{L}_k)^{-1}(\mathscr{L}_k-\mathscr{L}_k^{t+1})\right]\begin{pmatrix}r_0\\r_1\\r_2\\r_3\end{pmatrix}.\end{aligned}\tag{39}$$

We use the form of the column vector from Eq. (33) to represent the initial coin state and the values for r_0,r_1,r_2,r_3 are decided by the initial coin state chosen. In the long-time limit, $\mathscr{L}_k^t\to\infty$ because the eigenvalues of $\mathscr{L}_k$ satisfy $0<|\lambda|<1$. Thus, the first moment approximately turns to

$$\begin{aligned}&\langle\hat{x}\rangle_t\\&\approx\frac{(-1)}{2\pi}\int_{-\pi}^{\pi}{\rm d}k\begin{pmatrix}0&0&0&1\end{pmatrix}\left[(1-\mathscr{L}_k)^{-1}\mathscr{L}_k\right]\begin{pmatrix}r_0\\r_1\\r_2\\r_3\end{pmatrix}.\end{aligned}\tag{40}$$

Similarly, we apply Eqs. (37) and (38) to Eq. (30) for the second moment

$$\begin{aligned}\langle\hat{x}^2\rangle_t&=t+\frac{1}{2\pi}\int_{-\pi}^{\pi}{\rm d}k\begin{pmatrix}0&0&0&1\end{pmatrix}\Big[t-(1-\mathscr{L}_k)^{-1}\mathscr{L}_k\\&\quad+(1-\mathscr{L}_k)^{-1}\mathscr{L}_k^t\Big](1-\mathscr{L}_k)^{-1}\mathscr{L}_k\begin{pmatrix}0\\0\\0\\2\end{pmatrix},\end{aligned}\tag{41}$$

where $\mathscr{L}_k^t$ approaches zero in the long-time limit. For different quantum noisy channels, we obtain different forms of $\mathscr{L}_k$ and respectively substitute $\mathscr{L}_{k1}$, $\mathscr{L}_{k2}$, $\mathscr{L}_{k3}$ for $\mathscr{L}_k$ in Eqs. (40) and (41). For the dephasing channel, we obtain

$$\langle\hat{x}\rangle_t=\frac{p(r_1+pr_3)}{1-p^2},\tag{42}$$

$$\langle\hat{x}^2\rangle_t=t-\frac{p^2[5-2t+2p^2(1+t)]}{(1-p^2)^2}.\tag{43}$$

Furthermore, the variance of the position distribution of the QW is given by

$$\begin{aligned}\sigma_{\rm var}&=\langle\hat{x}^2\rangle_t-\langle\hat{x}\rangle_t^2\\&=t-\frac{p^2[5-2t+2p^2(1+t)-(r_1+pr_3)]}{(1-p^2)^2}.\end{aligned}\tag{44}$$

For the bit flip channel, we obtain

$$\langle\hat{x}\rangle_t=\frac{r_1+p^2r_3}{1-p^2},\tag{45}$$

$$\langle\hat{x}^2\rangle_t=t-\frac{p^2[5-2t+2p^2(1+t)]}{(1-p^2)^2}.\tag{46}$$

The variance for the bit flip channel turns out to be

$$\sigma_{\rm var}=t-\frac{p^2[5-2t+2p^2(1+t)]-(r_1+p^2r_3)^2}{(1-p^2)^2}.\tag{47}$$

For the depolarizing channel, we obtain

$$\begin{aligned}\langle\hat{x}\rangle_t&=\frac{p}{1-p}\left[\left(1-\frac{p}{\sqrt{1+p^2}}\right)r_1\right.\\&\quad\left.+\left(1-\frac{1}{\sqrt{1+p^2}}\right)r_3\right],\end{aligned}\tag{48}$$

$$\begin{aligned}\langle\hat{x}^2\rangle_t&=t-\frac{2p}{(1-p)^2\sqrt{(1+p^2)^3}}\Big[t-\sqrt{(1+p^2)}t\\&\quad+p(\sqrt{(1+p^2)}-1)(1+t)\\&\quad+p^2(1+t-\sqrt{(1+p^2)}t)\\&\quad+p^2(\sqrt{(1+p^2)}-2+(\sqrt{(1+p^2)}-1)t)\Big].\end{aligned}\tag{49}$$

Then the variance for the depolarizing channel is

$$\begin{aligned}&\sigma_{\rm var}\\&=t-\frac{2p}{(1-p)^2\sqrt{(1+p^2)^3}}\Big[t-\sqrt{(1+p^2)}t\\&\quad+p(\sqrt{(1+p^2)}-1)(1+t)+p^2(1+t-\sqrt{(1+p^2)}t)\\&\quad+p^2(\sqrt{(1+p^2)}-2+(\sqrt{(1+p^2)}-1)t)\Big]\\&\quad-\frac{p^2}{(1-p)^2}\left[\left(1-\frac{p}{\sqrt{1+p^2}}\right)r_1\right.\\&\quad\left.+\left(1-\frac{1}{\sqrt{1+p^2}}\right)r_3\right]^2.\end{aligned}\tag{50}$$

Although the first two moments of position have different forms for different quantum noisy channels, we find some properties in common from the results above. The first moment is dependent on the initial state while the second moment is not. The variance grows linearly with time in the long-time limit for all three quantum noisy channels, which indicates that the QW has transmitted to the CRW. However, the rate of growth is different, which also depends on the strength of the noisy channel and will be discussed in the next section.

4. Numerical simulations and analysis

We simulate the QWs according to the expression

$$\begin{aligned}\rho(t)&=\sum_{n_1,n_2,\ldots,n_t}\Big(\hat{E}\hat{B}_{n_t}\cdots\hat{E}\hat{B}_{n_2}\hat{E}\hat{B}_{n_1}|\Phi_0\rangle\langle\Phi_0|\\&\quad\otimes\hat{B}_{n_1}^{\dagger}\hat{E}^{\dagger}\hat{B}_{n_2}^{\dagger}\hat{E}^{\dagger}\cdots\hat{B}_{n_t}^{\dagger}\hat{E}^{\dagger}\Big),\end{aligned}\tag{51}$$

where $\hat{B}_n = \hat{I}_w \otimes \hat{A}_n$ denotes the operator of noise in the system. Further, we calculate the variance of the position distribution of the QW with coins undergoing different noisy channels. We also choose four different noisy strengths, $p^2 = 0.2, 0.4, 0.6, 0.8$, for each noisy channel. For simplicity and fair comparison among different noisy channels, we choose the initial state of the coin to be $|\phi_0\rangle = (|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$ for all cases. The results of numerical simulations are shown in Figs. 1–3 for the dephasing channel, bit-flip channel, and depolarizing channel, respectively.

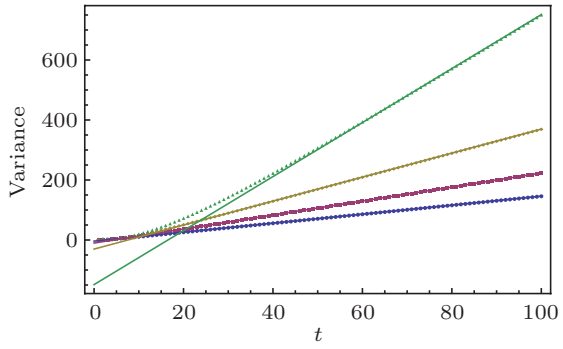


Fig. 1. (color online) Variance versus t for the Hadamard QWs with coins undergoing dephasing channel. The strengths of the noisy channel are $p^2 = 0.2$ (blue), 0.4 (purple), 0.6 (yellow), and 0.8 (green). Solid lines are results from the analytical solutions while the points are from numerical simulations. For all cases, the initial state of the coin is $|\phi_0\rangle = (|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$.

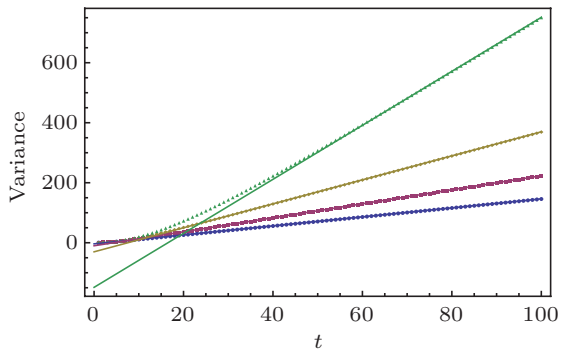


Fig. 2. (color online) Variance versus t for the Hadamard QWs with coins undergoing the bit-flip channel. The strengths of the noisy channel are $p^2 = 0.2$ (blue), 0.4 (purple), 0.6 (yellow), and 0.8 (green). Solid lines are results from the analytical solutions while the points are from numerical simulations. For all cases, the initial state of the coin is $|\phi_0\rangle = (|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$.

We can clearly see that the variance of the position distribution of the QW grows linearly with time when enough steps are taken. However, we can still see a quadratic growth of the variance in the first several steps, which is significantly different from that in the CRW and is usually considered as a qualitative mark for the quantumness of the QW system. On one hand, the quantumness of the walks stays for longer time with larger p^2, which means the weakening of the strength of different noisy channels. On the other hand, for the same p^2, the walk undergoing depolarizing channel takes less steps in the transition from the quantum to the classical regime than that undergoing dephasing channel and bit-flip channel, the results of which are identical for the initial state we choose. Although the steps for the quantum-to-classical transition of the walks are dependent on both the value of p^2 and the type of noisy channel, the number of steps is 100 in our simulations and is obviously big enough for all cases we choose to see the linear growth of variance with time of the walks, which is also confirmed by the fact that the analytical expressions we find in the long time limit match extremely well with the results of numerical simulations. Besides, the linear growth coefficient of variance versus t increases gradually with p^2, and the linear growth coefficient of variance for the dephasing channel and bit-flip channel is larger than that for the depolarizing channel with the same noisy strength.

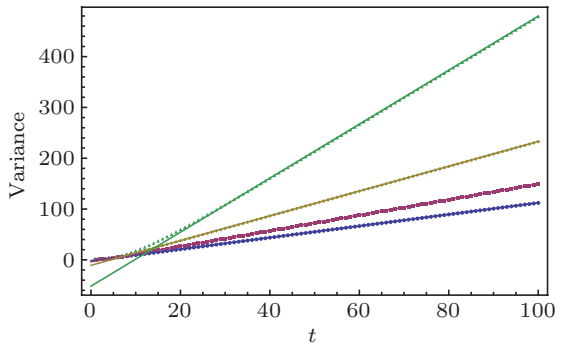


Fig. 3. (color online) Variance versus t for the Hadamard QWs with coins undergoing depolarizing channel. The strengths of the noisy channel are $p^2 = 0.2$ (blue), 0.4 (purple), 0.6 (yellow), and 0.8(green). Solid lines are results from the analytical solutions while the points are from numerical simulations. For all cases, the initial state of the coin is $|\phi_0\rangle = (|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$.

5. Conclusion

We have studied the Hadamard QWs with coins undergoing different quantum noisy channels, including dephasing channel, bit-flip channel, and depolarizing channel. We find the analytical expressions of the first two moments of position in the long-time limit, which match extremely well with the results of the numerical simulations. We show that the variance of the position distribution of the QW grows linearly with time when enough steps are taken, and still exhibits a quadratic growth in the first several steps, which is an important sign of QWs. The linear growth coefficient of variance versus the steps of the walks increases gradually with the weakening of the noise in different noisy channels. It is decoherence transmitting the QW to the classical.

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