A new six-component super soliton hierarchy and its self-consistent sources and conservation laws*

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A new six-component super soliton hierarchy is obtained based on matrix Lie super algebras. Super trace identity is used to furnish the super Hamiltonian structures for the resulting nonlinear super integrable hierarchy. After that, the self-consistent sources of the new six-component super soliton hierarchy are presented. Furthermore, we establish the infinitely many conservation laws for the integrable super soliton hierarchy.

Keywords: super Hamiltonian structures, self-consistent sources, conservation laws, Fermi variable

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1. Introduction

In recent years, with the development of integrable systems, super integrable systems have attracted a great deal of attention. Since 1986, Chowdhury and Roy have already considered the super trace identity.[1] Hu presented the super trace identity in 1997, but he did not give its rigorous proof.[2] Later, Ma gave a systematic proof of the super trace identity and the expression of its constant γ.[3] For application, the super Hamiltonian structures of the super AKNS hierarchy and the super Dirac hierarchy have been obtained. Afterwards, super C-KdV hierarchy[4] and super Boussinesq hierarchy[5] as well as their super Hamiltonian structures are presented. The Bargmann symmetry constraint and binary nonlinearization of the super Dirac systems were given in Ref. [6].

Soliton equations with self-consistent sources have been paid considerable attention in recent years. The study on this topic has important physical applications. For example, the nonlinear Schrödinger equation with self-consistent sources is relevant to some problems of plasma physics and solid state physics. There are many methods to obtain exact solutions of soliton equations with self-consistent sources, such as the Darboux transformation method[7] and the inverse scattering transform method.[8] There are also some results to consider hierarchies with self-consistent sources.[9,10] Very recently, self-consistent sources for the super Broer–Kaup–Kupershmidt equation hierarchy was presented in Ref. [11].

The conservation laws play an important role in discussing the integrability for soliton hierarchy. An infinite number of conservation laws for the KdV equation was first discovered by Miura, Gardner, and Kruscal in 1968,[12] then lots of methods have been developed to find them. This may be mainly due to the contributions of Wadati *et al.*[13] Many papers dealing with symmetries and conservation laws were presented. The direct construction method of multipliers for the conservation laws was presented in Ref. [14].

In this paper, starting from a Lie super algebra, a 3×3 matrix super spectral problem is designed. With the help of variational identity, we get a new six-component super soliton hierarchy and its super Hamiltonian structures. In Section 4, the self-consistent sources of the new six-component super soliton hierarchy are obtained. In Section 5, we present the conservation laws for the new super soliton hierarchy.

2. A new six-component super soliton hierarchy

A soliton hierarchy is based on the spectral problem[15]

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_x=\frac{1}{2}\begin{pmatrix}\lambda^{-1} & u_1+u_2\\ u_1-u_2 & -\lambda^{-1}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{1}$$

and a super extension of the soliton hierarchy can be constructed by the matrix super spectral[16]

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}_x=\frac{1}{2}\begin{pmatrix}\lambda^{-1} & u_1+u_2 & u_3\\ u_1-u_2 & -\lambda^{-1} & u_4\\ u_4 & -u_3 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}, \tag{2}$$

where u_3, u_4, and φ_3 are fermion fields. It reduces to the system (1) as $u_3=u_4=0$.

Based on the Lie super algebra G,

$$e_1=\frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\quad e_2=\frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

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$$e_3=\frac{1}{2}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad e_4=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},$$
$$e_5=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix},\tag{3}$$

where e_1, e_2, e_3 are even and e_4, e_5 are odd, that is along with the communicative operation

$$\begin{aligned}&[e_1,e_2]=2e_2,\quad [e_1,e_3]=-2e_3,\\&[e_2,e_3]=e_1,\quad [e_1,e_4]=[e_2,e_5]=e_4,\\&[e_1,e_5]=[e_4,e_3]=-e_5,\quad [e_4,e_5]_+=e_1,\\&[e_4,e_4]_+=-2e_2,\quad [e_5,e_5]_+=2e_3.\end{aligned}$$

We will derive a generalized super soliton hierarchy. To this end, we take a 3×3 matrix super spectral problem

$$\begin{cases}\varphi_x=U\varphi,\\ \varphi_t=V\varphi,\end{cases}\tag{4}$$

where

$$U=\frac{1}{2}\begin{pmatrix}\lambda^{-1}&u_1+u_2&u_3\\u_1-u_2&-\lambda^{-1}&u_4\\u_6&u_5&0\end{pmatrix},$$
$$V=\frac{1}{2}\begin{pmatrix}A+G&B+C&\delta\\B-C&-A+G&\rho\\\varepsilon&\tau&G\end{pmatrix},$$

and $A=\sum\limits_{m\geq 0}a_m\lambda^m$, $B=\sum\limits_{m\geq 0}b_m\lambda^m$, $C=\sum\limits_{m\geq 0}c_m\lambda^m$, $G=\sum\limits_{m\geq 0}g_m\lambda^m$, $\delta=\sum\limits_{m\geq 0}\delta_m\lambda^m$, $\rho=\sum\limits_{m\geq 0}\rho_m\lambda^m$, $\varepsilon=\sum\limits_{m\geq 0}\varepsilon_m\lambda^m$, and $\tau=\sum\limits_{m\geq 0}\tau_m\lambda^m$. As u_3, u_4, u_5, and u_6 are Fermi variables, they constitute Grassmann algebra.

Starting from the stationary zero curvature equation

$$V_x=[U,V],$$

we have

$$\begin{aligned}a_{mx}&=u_2b_m-u_1c_m+\frac{1}{4}u_3\varepsilon_m-\frac{1}{4}u_4\tau_m-\frac{1}{4}u_5\rho_m+\frac{1}{4}u_6\delta_m,\\b_{mx}&=c_{m+1}-u_2a_m+\frac{1}{4}u_3\tau_m+\frac{1}{4}u_4\varepsilon_m+\frac{1}{4}u_5\delta_m+\frac{1}{4}u_6\rho_m,\\c_{mx}&=b_{m+1}-u_1a_m+\frac{1}{4}u_3\tau_m-\frac{1}{4}u_4\varepsilon_m+\frac{1}{4}u_5\delta_m-\frac{1}{4}u_6\rho_m,\\g_{mx}&=\frac{1}{2}u_3\varepsilon_m+\frac{1}{2}u_4\tau_m+\frac{1}{2}u_5\rho_m+\frac{1}{2}u_6\delta_m,\\\delta_{mx}&=\frac{1}{2}\delta_{m+1}+\frac{1}{2}(u_1+u_2)\rho_m-\frac{1}{2}u_3a_m-\frac{1}{2}u_4(b_m+c_m),\\\rho_{mx}&=-\frac{1}{2}\rho_{m+1}+\frac{1}{2}(u_1-u_2)\delta_m-\frac{1}{2}u_3(b_m-c_m)+\frac{1}{2}u_4a_m,\\\tau_{mx}&=\frac{1}{2}\tau_{m+1}-\frac{1}{2}(u_1+u_2)\varepsilon_m-\frac{1}{2}u_5a_m+\frac{1}{2}u_6(b_m+c_m),\\\varepsilon_{mx}&=-\frac{1}{2}\varepsilon_{m+1}-\frac{1}{2}(u_1-u_2)\tau_m+\frac{1}{2}u_5(b_m-c_m)+\frac{1}{2}u_6a_m.\end{aligned}\tag{5}$$

Choose the initial data

$$\begin{aligned}&a_0=1,\quad b_0=c_0=\delta_0=\rho_0=\tau_0=\varepsilon_0,\\&g_0=-\alpha=\text{constant}.\end{aligned}\tag{6}$$

From the recursion relations in Eqs. (5), the first few results can be obtained

$$\begin{aligned}&a_1=0,\quad b_1=u_1,\quad c_1=u_2,\quad \delta_1=u_3,\\&\rho_1=u_4,\quad \tau_1=u_5,\quad \varepsilon_1=u_6,\quad g_1=0,\\&a_2=u_2^2-u_1^2-u_3u_6-u_4u_5,\quad b_2=u_{2x},\\&c_2=u_{1x},\quad \delta_2=2u_{3x},\quad \rho_2=-2u_{4x},\\&\tau_2=2u_{5x},\quad \varepsilon_2=-2u_{6x},\quad g_2=u_4u_5-u_3u_6,\ldots\end{aligned}$$

Next, we consider the auxiliary spectral problem

$$\varphi_{t_n}=V^{(n)}\varphi=(\lambda^{-n}V)_-\varphi,\tag{7}$$

where

$$V^{(n)}=\sum_{m=0}^{n}\frac{1}{2}\begin{pmatrix}a_m+g_m&b_m+c_m&\delta_m\\b_m-c_m&-a_m+g_m&\rho_m\\\varepsilon_m&\tau_m&g_m\end{pmatrix}\lambda^{-n+m}.$$

Substituting Eq. (7) into the zero curvature equation

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0,\tag{8}$$

we obtain a new six-component super soliton hierarchy

$$\begin{aligned}u_{t_n}=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_{t_n}&=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&&0&1\\0&0&0&0&-1&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\end{pmatrix}\\&\times\begin{pmatrix}b_{n+1}\\-c_{n+1}\\-\frac{1}{2}\varepsilon_{n+1}\\-\frac{1}{2}\tau_{n+1}\\\frac{1}{2}\rho_{n+1}\\\frac{1}{2}\delta_{n+1}\end{pmatrix}=J\begin{pmatrix}b_{n+1}\\-c_{n+1}\\-\frac{1}{2}\varepsilon_{n+1}\\-\frac{1}{2}\tau_{n+1}\\\frac{1}{2}\rho_{n+1}\\\frac{1}{2}\delta_{n+1}\end{pmatrix}.\end{aligned}\tag{9}$$

When $n=2$, the hierarchy (9) can be reduced to the second-

order super nonlinear integrable couplings equations

$$\begin{cases} u_{1t_2} = \frac{1}{2}u_{2xx} + u_2^3 - u_2u_1^2 - u_2u_3u_6 - u_2u_4u_5 \\ \quad - \frac{1}{2}u_3u_{5x} + \frac{1}{2}u_4u_{6x} - \frac{1}{2}u_5u_{3x} + \frac{1}{2}u_6u_{4x}, \\ u_{2t_2} = \frac{1}{2}u_{1xx} - u_1^3 + u_1u_2^2 - u_1u_3u_6 - u_1u_4u_5 \\ \quad - \frac{1}{2}u_3u_{5x} - \frac{1}{2}u_4u_{6x} - \frac{1}{2}u_5u_{3x} - \frac{1}{2}u_6u_{4x}, \\ u_{3t_2} = 2u_{3xx} + (u_1+u_2)u_{4x} + \frac{1}{2}u_3u_2^2 - \frac{1}{2}u_3u_1^2 \\ \quad - \frac{1}{2}u_3u_4u_5 + \frac{1}{2}u_4u_{1x} + \frac{1}{2}u_4u_{2x}, \\ u_{4t_2} = -2u_{4xx} - (u_1-u_2)u_{3x} + \frac{1}{2}u_4u_3u_6 \\ \quad + \frac{1}{2}u_3(u_1-u_2)_x - \frac{1}{2}u_4u_2^2 + \frac{1}{2}u_4u_1^2, \\ u_{5t_2} = 2u_{5xx} - (u_1+u_2)u_{6x} + \frac{1}{2}u_5u_2^2 - \frac{1}{2}u_5u_1^2 \\ \quad - \frac{1}{2}u_5u_3u_6 - \frac{1}{2}u_6(u_1+u_2)_x, \\ u_{6t_2} = -2u_{6xx} + (u_1-u_2)u_{5x} - \frac{1}{2}u_6u_2^2 + \frac{1}{2}u_6u_1^2 \\ \quad + \frac{1}{2}u_6u_4u_5 - \frac{1}{2}u_5(u_1-u_2)_x. \end{cases} \tag{10}$$

By choosing $u_3 = -u_5$ and $u_4 = u_6$, equations (10) can be reduced to the second-order super soliton equations[16]

$$\begin{cases} u_{1t_2} = \frac{1}{2}u_{2xx} + u_2^3 - u_2u_1^2 + u_3u_{3x} + u_4u_{4x} - 2u_2u_3u_4, \\ u_{2t_2} = \frac{1}{2}u_{1xx} - u_1^3 + u_1u_2^2 + u_3u_{3x} - u_4u_{4x} - 2u_1u_3u_4, \\ u_{3t_2} = 2u_{3xx} + \frac{1}{2}u_3(u_2^2 - u_1^2) \\ \quad + \frac{1}{2}u_4(u_1+u_2)_x + (u_1+u_2)u_{4x}, \\ u_{4t_2} = -2u_{4xx} + \frac{1}{2}u_4(u_1^2 - u_2^2) \\ \quad - \frac{1}{2}u_3(u_1-u_2)_x - (u_1-u_2)u_{3x}. \end{cases} \tag{11}$$

3. Super Hamiltonian structures of the new six-component super soliton hierarchy

In this section, we will establish super Hamiltonian structures of the new six-component super soliton hierarchy by super trace identity[2,3]

$$\frac{\delta}{\delta u}\int \mathrm{Str}\left(V\frac{\partial U}{\partial \lambda}\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{Str}\left(V\frac{\partial U}{\partial u}\right), \tag{12}$$

where the constant γ is determined by

$$\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\mathrm{Str}(VV)|. \tag{13}$$

According to super trace identity on Lie super algebras, a direct calculation reads

$$\begin{aligned} &\mathrm{Str}\left(V\frac{\partial U}{\partial\lambda}\right) = -\frac{1}{2}\lambda^{-2}A, \quad \mathrm{Str}\left(V\frac{\partial U}{\partial u_1}\right) = \frac{1}{2}B, \\ &\mathrm{Str}\left(V\frac{\partial U}{\partial u_2}\right) = -\frac{1}{2}C, \quad \mathrm{Str}\left(V\frac{\partial U}{\partial u_3}\right) = -\frac{1}{4}\varepsilon, \\ &\mathrm{Str}\left(V\frac{\partial U}{\partial u_4}\right) = -\frac{1}{4}\tau, \quad \mathrm{Str}\left(V\frac{\partial U}{\partial u_5}\right) = \frac{1}{4}\rho, \\ &\mathrm{Str}\left(V\frac{\partial U}{\partial u_6}\right) = \frac{1}{4}\delta. \end{aligned} \tag{14}$$

Substituting the above formula into the super trace identity (12) yields

$$\begin{aligned} &\frac{\delta}{\delta u}\int\left(-\frac{A}{2\lambda^2}\right)\mathrm{d}x \\ &= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(\frac{1}{2}B, -\frac{1}{2}C, -\frac{1}{4}\varepsilon, -\frac{1}{4}\tau, \frac{1}{4}\rho, \frac{1}{4}\delta\right)^{\mathrm{T}}. \end{aligned} \tag{15}$$

Comparing the coefficients of λ^{n+2} on both sides of Eq. (15) gives rise to

$$\begin{aligned} &\frac{\delta}{\delta u}\int\left(-\frac{a_{n+2}}{2}\right)\mathrm{d}x \\ &= (\gamma+n+1)\left(\frac{1}{2}b_{n+1}, -\frac{1}{2}c_{n+1}, -\frac{1}{4}\varepsilon_{n+1}, \right. \\ &\quad \left. -\frac{1}{4}\tau_{n+1}, \frac{1}{4}\rho_{n+1}, \frac{1}{4}\delta_{n+1}\right)^{\mathrm{T}}. \end{aligned} \tag{16}$$

By employing the computing formula (13) on the constant γ, we obtain $\gamma = 0$. Therefore, we conclude that

$$\begin{aligned} &\frac{\delta H_n}{\delta u} = \left(b_{n+1}, -c_{n+1}, -\frac{1}{2}\varepsilon_{n+1}, -\frac{1}{2}\tau_{n+1}, \frac{1}{2}\rho_{n+1}, \frac{1}{2}\delta_{n+1}\right)^{\mathrm{T}}, \\ &H_n = \int -\frac{a_{n+2}}{n+1}\mathrm{d}x, \quad n \geqslant 0. \end{aligned} \tag{17}$$

From the recursion relations in Eq. (5), we can obtain the hereditary recursion operator L which satisfies that

$$\frac{\delta H_n}{\delta u} = L\frac{\delta H_{n-1}}{\delta u}, \tag{18}$$

where

$$L = \begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} u_1\partial^{-1}u_2 & -\partial + u_1\partial^{-1}u_1 \\ -\partial - u_2\partial^{-1}u_2 & -u_2\partial^{-1}u_1 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} -\frac{1}{2}u_4 - \frac{1}{2}u_1\partial^{-1}u_3 & \frac{1}{2}u_3 + \frac{1}{2}u_1\partial^{-1}u_4 \\ -\frac{1}{2}u_4 + \frac{1}{2}u_2\partial^{-1}u_3 & -\frac{1}{2}u_3 - \frac{1}{2}u_2\partial^{-1}u_4 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} \frac{1}{2}u_6 - \frac{1}{2}u_1\partial^{-1}u_5 & -\frac{1}{2}u_5 + \frac{1}{2}u_1\partial^{-1}u_6 \\ \frac{1}{2}u_6 + \frac{1}{2}u_2\partial^{-1}u_5 & \frac{1}{2}u_5 - \frac{1}{2}u_2\partial^{-1}u_6 \end{pmatrix},$$

$$L_4=\begin{pmatrix} -\frac{1}{2}u_5-\frac{1}{2}u_6\partial^{-1}u_2 & -\frac{1}{2}u_5-\frac{1}{2}u_6\partial^{-1}u_1 \\ \frac{1}{2}u_6-\frac{1}{2}u_5\partial^{-1}u_2 & -\frac{1}{2}u_6-\frac{1}{2}u_5\partial^{-1}u_1 \end{pmatrix},$$

$$L_5=\begin{pmatrix} -2\partial+\frac{1}{4}u_6\partial^{-1}u_3 & u_2-u_1-\frac{1}{4}u_6\partial^{-1}u_4 \\ u_1+u_2+\frac{1}{4}u_5\partial^{-1}u_3 & 2\partial-\frac{1}{4}u_5\partial^{-1}u_4 \end{pmatrix},$$

$$L_6=\begin{pmatrix} \frac{1}{4}u_6\partial^{-1}u_5 & -\frac{1}{4}u_6\partial^{-1}u_6 \\ \frac{1}{4}u_5\partial^{-1}u_5 & -\frac{1}{4}u_5\partial^{-1}u_6 \end{pmatrix},$$

$$L_7=\begin{pmatrix} -\frac{1}{2}u_3+\frac{1}{2}u_4\partial^{-1}u_2 & -\frac{1}{2}u_3+\frac{1}{2}u_4\partial^{-1}u_1 \\ \frac{1}{2}u_4+\frac{1}{2}u_3\partial^{-1}u_2 & -\frac{1}{2}u_4+\frac{1}{2}u_3\partial^{-1}u_1 \end{pmatrix},$$

$$L_8=\begin{pmatrix} -\frac{1}{4}u_4\partial^{-1}u_3 & \frac{1}{4}u_4\partial^{-1}u_4 \\ -\frac{1}{4}u_3\partial^{-1}u_3 & \frac{1}{4}u_3\partial^{-1}u_4 \end{pmatrix},$$

$$L_9=\begin{pmatrix} -2\partial-\frac{1}{4}u_4\partial^{-1}u_5 & u_1-u_2+\frac{1}{4}u_4\partial^{-1}u_6 \\ -u_1-u_2-\frac{1}{4}u_3\partial^{-1}u_5 & 2\partial+\frac{1}{4}u_3\partial^{-1}u_6 \end{pmatrix}.$$

Therefore, the new super soliton hierarchy (9) possesses the following super Hamiltonian structures

$$u_{t_n}=K_n(u)=J\frac{\delta H_n}{\delta u},\quad n\geq 0. \tag{19}$$

4. Self-consistent sources of the new six-component super soliton hierarchy

Consider the linear system

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}_x=U\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix},\quad \begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}_t=V\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}. \tag{20}$$

Based on the result in Ref. [17], we can show that the following equation:

$$\frac{\delta H_k}{\delta u}+\sum_{j=1}^{N}\alpha_j\frac{\delta\lambda_j}{\delta u}=0, \tag{21}$$

holds true, where α_j are constants, determines a finite-dimensional invariant set for the flows

$$\frac{\delta H_n}{\delta u}=L\frac{\delta H_{n-1}}{\delta u}=\cdots=L^n\frac{\delta H_0}{\delta u},\quad n=1,2,\ldots,$$
$$\frac{\delta}{\delta u}=\left(\frac{\delta}{\delta u_1},\cdots,\frac{\delta}{\delta u_6}\right)^{\mathrm{T}}. \tag{22}$$

From Eq. (21), we may know that

$$\frac{\delta\lambda_j}{\delta u_i}=\frac{1}{3}\mathrm{Str}\left(\Psi_j\frac{\partial U(u,\lambda_j)}{\delta u_i}\right),\quad i=1,2,\ldots,6, \tag{23}$$

where Str denotes the trace of a matrix and

$$\Psi_j=\begin{pmatrix}\varphi_{1j}\varphi_{2j} & -\varphi_{1j}^2 & \varphi_{1j}\varphi_{3j}\\ \varphi_{2j}^2 & -\varphi_{1j}\varphi_{2j} & \varphi_{2j}\varphi_{3j}\\ \varphi_{2j}\varphi_{3j} & -\varphi_{1j}\varphi_{3j} & 0\end{pmatrix},\quad j=1,2,\ldots,N. \tag{24}$$

Based on Eq. (21), for system (9), we set

$$\frac{\delta H_n}{\delta u}=\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}, \tag{25}$$

and obtain the following $\delta\lambda_j/\delta u$

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u_i}=\sum_{j=1}^{N}\begin{pmatrix}\mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_1}\right)\\ \mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_2}\right)\\ \mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_3}\right)\\ \mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_4}\right)\\ \mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_5}\right)\\ \mathrm{Str}\left(\Psi_j\frac{\delta U}{\delta u_6}\right)\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\langle\Phi_1,\Phi_1\rangle+\frac{1}{2}\langle\Phi_2,\Phi_2\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_1\rangle+\frac{1}{2}\langle\Phi_2,\Phi_2\rangle\\ -\frac{1}{2}\langle\Phi_2,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_2,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_3\rangle\end{pmatrix},$$

where $\Phi_i=(\varphi_{i1},\ldots,\varphi_{iN})^{\mathrm{T}}$, $i=1,2,3$.

Therefore, the new six-component integrable super soliton hierarchy with self-consistent sources is proposed

$$u_{t_n}=\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix}_{t_n}=J\frac{\delta H_n}{\delta u_i}+J\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u_i}=J\begin{pmatrix}b_{n+1}\\ -c_{n+1}\\ -\frac{1}{2}\varepsilon_{n+1}\\ -\frac{1}{2}\tau_{n+1}\\ \frac{1}{2}\rho_{n+1}\\ \frac{1}{2}\delta_{n+1}\end{pmatrix}+J\begin{pmatrix}-\frac{1}{2}\langle\Phi_1,\Phi_1\rangle+\frac{1}{2}\langle\Phi_2,\Phi_2\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_1\rangle+\frac{1}{2}\langle\Phi_2,\Phi_2\rangle\\ -\frac{1}{2}\langle\Phi_2,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_2,\Phi_3\rangle\\ \frac{1}{2}\langle\Phi_1,\Phi_3\rangle\end{pmatrix}. \tag{26}$$

For $n=2$, we obtain the super soliton equation with self-consistent sources

$$\begin{cases}
u_{1t_2} = \frac{1}{2}u_{2xx}+u_2^3-u_2u_1^2-u_2u_3u_6-u_2u_4u_5 \\
\quad -\frac{1}{2}u_3u_{5x}+\frac{1}{2}u_4u_{6x}-\frac{1}{2}u_5u_{3x} \\
\quad +\frac{1}{2}u_6u_{4x}-\frac{1}{2}\sum_{j=1}^{N}(\varphi_{1j}^2+\varphi_{2j}^2), \\
u_{2t_2} = \frac{1}{2}u_{1xx}-u_1^3+u_1u_2^2-u_1u_3u_6-u_1u_4u_5-\frac{1}{2}u_3u_{5x} \\
\quad -\frac{1}{2}u_4u_{6x}-\frac{1}{2}u_5u_{3x}-\frac{1}{2}u_6u_{4x}-\frac{1}{2}\sum_{j=1}^{N}(\varphi_{1j}^2-\varphi_{2j}^2), \\
u_{3t_2} = 2u_{3xx}+(u_1+u_2)u_{4x}+\frac{1}{2}u_3u_2^2-\frac{1}{2}u_3u_1^2 \\
\quad -\frac{1}{2}u_3u_4u_5+\frac{1}{2}u_4u_{1x}+\frac{1}{2}u_4u_{2x}+\frac{1}{2}\sum_{j=1}^{N}\varphi_{1j}\varphi_{3j}, \\
u_{4t_2} = -2u_{4xx}-(u_1-u_2)u_{3x}+\frac{1}{2}u_4u_3u_6 \\
\quad +\frac{1}{2}u_3(u_1-u_2)_x-\frac{1}{2}u_4u_2^2+\frac{1}{2}u_4u_1^2-\frac{1}{2}\sum_{j=1}^{N}\varphi_{2j}\varphi_{3j}, \\
u_{5t_2} = 2u_{5xx}-(u_1+u_2)u_{6x}+\frac{1}{2}u_5u_2^2-\frac{1}{2}u_5u_1^2 \\
\quad -\frac{1}{2}u_5u_3u_6-\frac{1}{2}u_6(u_1+u_2)_x-\frac{1}{2}\sum_{j=1}^{N}\varphi_{1j}\varphi_{3j}, \\
u_{6t_2} = -2u_{6xx}+(u_1-u_2)u_{5x}-\frac{1}{2}u_6u_2^2+\frac{1}{2}u_6u_1^2 \\
\quad +\frac{1}{2}u_6u_4u_5-\frac{1}{2}u_5(u_1-u_2)_x-\frac{1}{2}\sum_{j=1}^{N}\varphi_{2j}\varphi_{3j},
\end{cases} \tag{27}$$

where $\Phi_i=(\varphi_{i1},\ldots,\varphi_{iN})^{\mathrm{T}}$, $i=1,2,3$ satisfy

$$\begin{cases}
\varphi_{1jx} = \frac{1}{2}\lambda^{-1}\varphi_{1j}+\frac{1}{2}(u_1+u_2)\varphi_{2j}+\frac{1}{2}u_3\varphi_{3j}, \\
\varphi_{2jx} = \frac{1}{2}(u_1-u_2)\varphi_{1j}-\frac{1}{2}\lambda^{-1}\varphi_{2j}+\frac{1}{2}u_4\varphi_{3j}, \\
\varphi_{3jx} = \frac{1}{2}u_6\varphi_{1j}+\frac{1}{2}u_5\varphi_{2j}, \quad j=1,\cdots,N.
\end{cases} \tag{28}$$

5. Conservation laws of the new six-component super soliton hierarchy

In the following, we will construct conservation laws of the new six-component super soliton hierarchy. We introduce the variables

$$E=\frac{\varphi_2}{\varphi_1}, \quad K=\frac{\varphi_3}{\varphi_1}. \tag{29}$$

From Eq. (4), we have

$$\begin{aligned}
E_x &= \frac{1}{2}(u_1-u_2)-\lambda^{-1}E+\frac{1}{2}u_4K \\
&\quad -\frac{1}{2}(u_1+u_2)E^2-\frac{1}{2}u_3EK, \\
K_x &= \frac{1}{2}u_6+\frac{1}{2}u_5E-\frac{1}{2}\lambda^{-1}K \\
&\quad -\frac{1}{2}(u_1+u_2)KE+\frac{1}{2}u_3K^2.
\end{aligned} \tag{30}$$

Expand E, K in the power of λ

$$E=\sum_{j=1}^{\infty}e_j\lambda^j, \quad K=\sum_{j=1}^{\infty}k_j\lambda^j. \tag{31}$$

Substituting Eqs. (31) into Eqs. (30) and comparing the coefficients of the same power of λ, we obtain

$$\begin{aligned}
e_1 &= \frac{1}{2}(u_1-u_2), \quad k_1=u_6, \quad e_2=\frac{1}{2}(u_2-u_1)x+\frac{1}{2}u_4u_6, \\
k_2 &= -2u_{6x}+\frac{1}{2}u_5(u_1-u_2), \\
e_3 &= -\frac{1}{2}(u_2-u_1)_{xx}-\frac{1}{2}u_{4x}u_6-\frac{3}{2}u_4u_{6x} \\
&\quad +\frac{1}{4}(u_1-u_2)u_4u_5, \\
k_3 &= 4u_{6xx}+\frac{3}{2}(u_2-u_1)_xu_5 \\
&\quad -(u_1-u_2)u_{5x}+\frac{1}{2}u_5u_4u_6, \\
&\cdots,
\end{aligned} \tag{32}$$

and a recursion formula for e_j and k_j

$$\begin{aligned}
e_{j+1} &= -e_{jx}+\frac{1}{2}u_4k_j-\frac{1}{2}(u_1+u_2)\sum_{l=1}^{j-1}e_le_{j-l} \\
&\quad -\frac{1}{2}u_3\sum_{l=1}^{j-1}e_lk_{j-l}, \\
k_{j+1} &= -2k_{jx}+u_5e_j-(u_1+u_2)\sum_{l=1}^{j-1}k_le_{j-l} \\
&\quad +u_3\sum_{l=1}^{j-1}k_lk_{j-l}, \quad j\geq 2.
\end{aligned} \tag{33}$$

Because of

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}\lambda^{-1}+\frac{1}{2}(u_1+u_2)E+\frac{1}{2}u_3K\right] \\
&= \frac{\partial}{\partial x}\left[\frac{1}{2}(A+G)+\frac{1}{2}(B+C)E+\frac{1}{2}\delta K\right],
\end{aligned} \tag{34}$$

we assume that

$$\begin{aligned}
\theta &= \frac{1}{2}\lambda^{-1}+\frac{1}{2}(u_1+u_2)E+\frac{1}{2}u_3K, \\
\nu &= \frac{1}{2}(A+G)+\frac{1}{2}(B+C)E+\frac{1}{2}\delta K.
\end{aligned}$$

Then equation (34) can be written as $\theta_t=\nu_x$, which is the right form of conservation laws. With regard to Eq. (10), we have

$$\begin{aligned}
A &= \lambda^{-2}+u_2^2-u_1^2-u_3u_6-u_4u_5, \\
B &= u_1\lambda^{-1}+u_{2x}, \\
C &= u_2\lambda^{-1}+u_{1x}, \\
G &= -\alpha\lambda^{-2}+u_4u_5-u_3u_6, \\
\delta &= u_3\lambda^{-1}+2u_{3x}.
\end{aligned} \tag{35}$$

We expand θ and ν as a series in powers of λ with the coefficients

$$\theta = \frac{1}{2}\lambda^{-1} + \sum_{j=1}^{\infty} \theta_j \lambda^j,$$
$$\nu = \frac{1}{2}(1-\alpha)\lambda^{-2} + \frac{1}{2}(u_2^2 - u_1^2 - 2u_3u_6) + \sum_{j=1}^{\infty} \nu_j \lambda^j, \quad (36)$$

the first two conserved densities and currents of Eqs. (10) read

$$\begin{aligned}
\theta_1 &= \frac{1}{4}(u_1+u_2)^2 + \frac{1}{2}u_3u_6,\\
\nu_1 &= \frac{1}{4}(u_1+u_2)(u_2-u_1)_x + \frac{1}{4}(u_1+u_2)u_4u_6\\
&\quad + \frac{1}{4}(u_1+u_2)_x(u_1-u_2) + \frac{1}{4}(u_1-u_2)u_3u_5\\
&\quad + u_{3x}u_6 - u_3u_{6x},\\
\theta_2 &= \frac{1}{4}(u_1+u_2)(u_2-u_1)_x + \frac{1}{4}(u_1+u_2)u_4u_6\\
&\quad + \frac{1}{4}(u_1-u_2)u_3u_5 - u_3u_{6x},\\
\nu_2 &= \frac{1}{4}(u_1+u_2)_x(u_{2x}-u_{1x}+u_4u_6)\\
&\quad + u_{3x}\left(\frac{1}{2}u_1u_5 - \frac{1}{2}u_2u_5 - 2u_{6x}\right)\\
&\quad + \frac{1}{4}(u_1+u_2)[(u_1-u_2)_{xx}\\
&\quad - u_{4x}u_6 - 3u_4u_{6x} + \frac{1}{2}(u_1-u_2)u_4u_5]\\
&\quad + \frac{1}{2}u_3\left[4u_{6xx} + \frac{1}{2}u_5u_4u_6\right.\\
&\quad \left. + \frac{3}{2}u_5(u_2-u_1)_x - (u_1-u_2)u_{5x}\right]. \qquad (37)
\end{aligned}$$

The recursion relations for θ_j and ν_j are

$$\begin{aligned}
\theta_j &= \frac{1}{2}(u_1+u_2)e_j + \frac{1}{2}u_3k_j,\\
\nu_j &= \frac{1}{2}(u_1+u_2)e_{j+1} + \frac{1}{2}(u_1+u_2)_xe_j\\
&\quad + \frac{1}{2}u_3k_{j+1} + u_{3x}k_j, \qquad (38)
\end{aligned}$$

where e_j and k_j can be calculated from Eq. (33). We can display the first two conservation laws of Eq. (10) as

$$\theta_{1t} = \nu_{1x}, \quad \theta_{2t} = \nu_{2x}, \qquad (39)$$

where θ_1, ν_1, θ_2, and ν_2 are defined in Eq. (37). Then, the infinitely many conservation laws of Eq. (9) can be easily obtained from Eqs. (30)–(39), respectively.

6. Conclusions

Starting from Lie super algebras, we obtain a new six-component super soliton hierarchy. With the help of variational identity, the Hamiltonian structures of the six-component super soliton hierarchy can be presented. Meanwhile, the super soliton hierarchy with self-consistent sources is established. Finally, we also derive the conservation laws of the six-component super soliton hierarchy. It is worth noting that the coupling terms of super integrable hierarchies involve Fermi variables, they satisfy the Grassmann algebra which is different from the ordinary one. Sometime we call super integrable system Eq. (9) as super Fermi extension. In fact, in Refs. [18]–[20], the authors gave many Lie algebras which have more complex mathematical structures, which deserves more meaningful work in the near future.

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