# Rigidity based formation tracking for multi-agent networks*

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This paper considers the formation tracking problem under a rigidity framework, where the target formation is specified as a minimally and infinitesimally rigid formation and the desired velocity of the group is available to only a subset of the agents. The following two cases are considered: the desired velocity is constant, and the desired velocity is time-varying. In the first case, a distributed linear estimator is constructed for each agent to estimate the desired velocity. The velocity estimation and a formation acquisition term are employed to design the control inputs for the agents, where the rigidity matrix plays a central role. In the second case, a distributed non-smooth estimator is constructed to estimate the time-varying velocity, which is shown to converge in a finite time. Theoretical analysis shows that the formation tracking problem can be solved under the proposed control algorithms and estimators. Simulation results are also provided to show the validity of the derived results.

**Keywords:** multi-agent system, formation control, graph rigidity, distributed estimator

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## 1. Introduction

Multi-agent systems have attracted much attention in recent years. They offer many advantages over single agent systems, such as efficiency, robustness, scalability, versatility, adaptability, and lower cost.[1] A considerable amount of research effort has been devoted to the distributed control of multi-agent systems. Examples of interesting research directions include consensus,[2–4] formation control,[5–7] coordinated tracking,[8,9] flocking,[10,11] and so on.

In the formation control problem, the agents are required to form a pre-defined formation shape, and then maintain the shape during translational or rotation movements. Rigid formation can keep the distance between any agent pair (the shape of the whole group) unchanged, which is not guaranteed by an arbitrary formation. Another feature of rigid formation is that it can be used to stabilize the distances between all of the agent pairs while employing the information of only a few agent pairs. This feature is of great significance in a large group of agents. Rigid formation control has found applications in various fields, including a wireless sensor network, surveillance, industrial processes, and so on.[1,12,13] The objective of rigid formation tracking is to stabilize the relative positions of the agents according to a target rigid formation, while the agents can at the same time track a desired velocity. The incorporation of the desired velocity in formation control gives the system an extra ability to suit a complicated environment. However, this also complicates the design and analysis of the control algorithm. This paper is interested in the formation tracking problem for multi-agent systems via a rigidity framework.

The are many studies in the literature of formation control. In Ref. [14], the global asymptotic stability of triangular formations is investigated. Local asymptotic stability of minimally infinitesimally rigid formations is analyzed by means of Lyapunov-based stability analysis.[15] In Ref. [16], a distributed coordinated tracking problem for multiple networked Euler–Lagrange systems is solved under the constraint that the leader is a neighbor of only a subset of the followers. A distance-based formation maintenance controller is designed for cycle-free persistent formations under the condition that the velocity of the trajectory is sufficiently low.[17] For multi-agent formation maintenance and target tracking, a control law is developed in Ref. [7], where the desired translational velocity and the leader's relative position and velocity are required by all of the followers. Using the relative positions of all of the agents, distance-based formation maintenance and target tracking schemes are solved by a modified gradient-of-potential-function law.[18] Reference [19] presents the design of control protocols and distributed observers for the leader-following formation control, which also deals with the formation as well as the tracking problem. In Ref. [20], the consensus problem with a periodic intermittent communication and fixed directed topology is investigated, and then the consensus algorithm is extended to solve the formation control problem as well as the consensus tracking problem with switching directed topologies. Graph theory is a natural tool to describe multi-agent formation, a detailed discussion on the rigid graph theory and its applications is given in Ref. [1].

The contributions of this paper are stated as follows. First, a desired group velocity is incorporated into the rigid formation control problem. When the desired velocity is time-

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varying, a distributed nonsmooth estimator is constructed to estimate the velocity and a Lyapunov function is provided to show the finite-time convergence of the estimator. Second, a formation acquisition term is derived from the rigidity matrix, which is used to solve the rigid formation control problem with the aid of the velocity estimator. Theoretical proofs are provided to show the convergence of the proposed formation control algorithms.

The rest of this paper is organized as follows. In Section 2, notation and mathematical preliminaries are presented. The formation control problem is defined in Section 3. Control algorithm design and stability analysis are presented in Sections 4 and 5, respectively, for the case of constant desired velocity and the case of time-varying desired velocity. Simulation results are given in Section 6. Finally, Section 7 summarizes the investigation.

## 2. Preliminaries

Some basic knowledge on graph theory will be given in this section. An undirected graph is defined as $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} \triangleq \{1,2,...,n\}$ is a set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of undirected edges. The order and size of $\mathcal{G}$ are denoted, respectively, by $n \triangleq |\mathcal{V}|$ and $m \triangleq |\mathcal{E}|$. Denote the set of neighbors of vertex $i$ by

$$\mathcal{N}_i(\mathcal{E}) \triangleq \{j \in \mathcal{V} | (i,j) \in \mathcal{E}\}.$$

Let $p_i \in \mathbb{R}^2$ be the position of vertex $i$. A formation can be denoted by a pair $(\mathcal{G}, p)$ where $p \triangleq (p_1^{\mathrm{T}}, ..., p_n^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2n}$. Based on an arbitrary ordering of the edges in $\mathcal{E}$, an edge function $\Phi_{\mathcal{G}}: \mathbb{R}^{2n} \to \mathbb{R}^m$ associated with $(\mathcal{G}, p)$ is defined as

$$\Phi_{\mathcal{G}}(p) \triangleq (..., \|p_i - p_j\|^2, ...)^{\mathrm{T}}, \quad (i,j) \in \mathcal{E},$$

where $\|\cdot\|$ denotes the Euclidean norm. The rigidity of a formation is defined as follows.

**Definition 1**[6] A formation $(\mathcal{G}, p)$ is rigid if there exists a neighborhood $\mathcal{U}_p$ of $p \in \mathbb{R}^{2n}$ such that $\Phi_{\mathcal{G}}^{-1}[\Phi_{\mathcal{G}}(p)] \cap \mathcal{U}_p = \Phi_{\mathcal{K}}^{-1}[\Phi_{\mathcal{K}}(p)] \cap \mathcal{U}_p$, where $\Phi^{-1}$ is the set of the positions $q \in \mathbb{R}^{2n}$ satisfying $\Phi_{\mathcal{G}}(q) = \Phi_{\mathcal{G}}(p)$ and $\mathcal{K}$ is a complete graph of $n$ vertices.

For a rigid formation, the distance between any two vertices will not change if the distances of the vertices corresponding to the edges which belong to the rigid graph remain unchanged during translation and rotation movements. The rigidity matrix $\boldsymbol{R}$ of a formation $F \triangleq (\mathcal{G}, p)$ is given by

$$R(p) \triangleq \frac{1}{2} \frac{\partial \Phi_{\mathcal{G}}(p)}{\partial p} \in \mathbb{R}^{m \times 2n}.$$

Obviously, the rigidity matrix only depends on the relative positions of the agents. Thus, we write the rigidity matrix as $R(\tilde{p})$ in the following, where $\tilde{p}_{ij} = p_i - p_j, (i,j) \in \mathcal{E}$. Based on the rank property of $R(\tilde{p})$, the infinitesimal rigidity of $F$ is defined as follows.

**Definition 2**[6] A formation $F = (\mathcal{G}, p)$, $p \in \mathbb{R}^{2n}$ and $n \geq 3$, is infinitesimally rigid in a two-dimensional space if $\mathrm{rank}[R(\tilde{p})] = 2n - 3$.

Typically, formations that are rigid but fail to be infinitesimally rigid have collinear or parallel edges. In the following, the minimal rigidity of a formation is introduced.

**Definition 3**[1] A formation is minimally rigid if it is rigid and no single interagent distance constraint can be removed without causing the formation to lose rigidity.

It can be verified that the graph in Fig. 1(a) is rigid. However, if any edge is removed from the graph, the graph is no longer rigid (see Fig. 1(b)). Thus, the graph in Fig. 1(a) satisfies Definition 3, and is said to be minimally rigid. In a two-dimensional space, a graph is minimally rigid if and only if $m = 2n - 3$.[1]

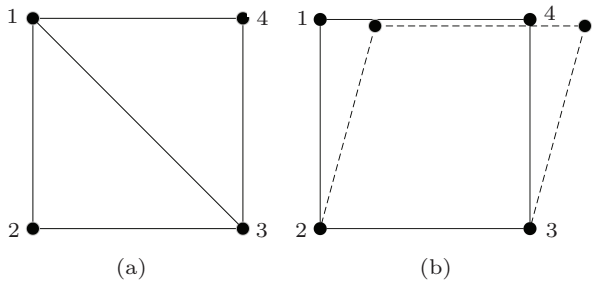



**Fig. 1.** Illustration of minimal rigidity.

**Lemma 1** If a real matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has full row rank, then $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is positive definite.

**Proof** For any $\boldsymbol{x} \in \mathbb{R}^m$, it follows that $\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})\boldsymbol{x} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x}) \geq 0$. In particular, if $\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = 0$. Because $\boldsymbol{A}$ has a full-row rank, the equation $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0}$ has one and only one solution $\boldsymbol{x} = \boldsymbol{0}$. That is, $\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})\boldsymbol{x} = 0$ has one and only has one solution $\boldsymbol{x} = \boldsymbol{0}$, which suggests that $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ is positive definite.

## 3. Problem description

Consider a system of $n$ agents governed by the single integrator dynamics

$$\dot{q}_i = u_i, \tag{1}$$

where $\boldsymbol{q}_i \triangleq (x_i, y_i)^{\mathrm{T}} \in \mathbb{R}^2$ is the position and $u_i \in \mathbb{R}^2$ is the control input to be designed. Consider an infinitesimally and minimally rigid formation in $\mathbb{R}^2$ described by $F^* \triangleq (\mathcal{G}, q^*)$, where $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$, $|\mathcal{E}| = m$, $\boldsymbol{d} \triangleq (..., d_{ij}, ...)^{\mathrm{T}}$ and $d_{ij} \triangleq \|\boldsymbol{q}_i^* - \boldsymbol{q}_j^*\|, (i,j) \in \mathcal{E}$. Let $F^*$ be the desired formation, and $F(t) \triangleq [\mathcal{G}, q(t)]$ denote the actual formation of the agents, where $\boldsymbol{q}(t) \triangleq [\boldsymbol{q}_1(t)^{\mathrm{T}}, ..., \boldsymbol{q}_n(t)^{\mathrm{T}}]^{\mathrm{T}}$.

The objective of this paper is to design distributed controls $u_i$ based on local information such that,

$$\|\boldsymbol{q}_i(t)-\boldsymbol{q}_j(t)\|\to d_{ij},\ \text{as}\ t\to\infty,\quad \forall(i,j)\in\mathscr{E}, \tag{2}$$

$$\dot{\boldsymbol{q}}_i(t)-\boldsymbol{v}_{\rm d}(t)\to 0,\ \text{as}\ t\to\infty,\quad i=1,...,n, \tag{3}$$

where $\boldsymbol{v}_{\rm d}(t)\in\mathbb{R}^2$ is any bounded, continuous function representing the desired velocity of the multi-agent system, and is available to only a subset of the agents.

## 4. Formation maintenance with constant translational velocity

In this section, we consider a case where the desired velocity $v_{\rm d}$ is constant. Define the relative position of two agents as

$$\tilde{\boldsymbol{q}}_{ij}\triangleq\boldsymbol{q}_i-\boldsymbol{q}_j,\quad (i,j)\in\mathscr{E},$$

and let $\tilde{\boldsymbol{q}}\triangleq(...,\tilde{\boldsymbol{q}}_{ij}^{\rm T},...)^{\rm T}\in\mathbb{R}^{2m}$. The distance error is given by

$$e_{ij}\triangleq\|\tilde{\boldsymbol{q}}_{ij}\|-d_{ij},\quad (i,j)\in\mathscr{E}.$$

The dynamics of the distance error are

$$\begin{aligned}\dot{e}_{ij}&=\frac{\rm d}{{\rm d}t}(\sqrt{\tilde{\boldsymbol{q}}_{ij}^{\rm T}\tilde{\boldsymbol{q}}_{ij}})\\&=(\tilde{\boldsymbol{q}}_{ij}^{\rm T}\tilde{\boldsymbol{q}}_{ij})^{1/2}\tilde{\boldsymbol{q}}_{ij}^{\rm T}(u_i-u_j)\\&=\frac{\tilde{\boldsymbol{q}}_{ij}^{\rm T}(u_i-u_j)}{e_{ij}+d_{ij}}.\end{aligned}\tag{4}$$

Define the following positive definite, radially unbounded function:

$$\begin{aligned}W&\triangleq\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}(\|\tilde{\boldsymbol{q}}_{ij}\|^2-d_{ij}^2)^2\\&=\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}[(\|\tilde{\boldsymbol{q}}_{ij}\|-d_{ij})(\|\tilde{\boldsymbol{q}}_{ij}\|+d_{ij})]^2\\&=\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}[e_{ij}(e_{ij}+2d_{ij})]^2.\end{aligned}\tag{5}$$

The derivative of Eq. (5) along Eq. (4) is given by

$$\begin{aligned}\dot{W}&=\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}\frac{d[e_{ij}(e_{ij}+2d_{ij})]^2}{dt}\\&=\frac{1}{2}\sum_{(i,j)\in\mathscr{E}}e_{ij}(e_{ij}+2d_{ij})[\dot{e}_{ij}(e_{ij}+2d_{ij})+e_{ij}\dot{e}_{ij}]\\&=\frac{1}{2}\sum_{(i,j)\in\mathscr{E}}e_{ij}(e_{ij}+2d_{ij})\\&\quad\times\left[\frac{\tilde{\boldsymbol{q}}_{ij}^{\rm T}(u_i-u_j)}{e_{ij}+d_{ij}}(e_{ij}+2d_{ij})+e_{ij}\frac{\tilde{\boldsymbol{q}}_{ij}^{\rm T}(u_i-u_j)}{e_{ij}+d_{ij}}\right]\\&=\sum_{(i,j)\in\mathscr{E}}e_{ij}(e_{ij}+2d_{ij})\tilde{\boldsymbol{q}}_{ij}^{\rm T}(u_i-u_j).\end{aligned}$$

From the definition of $\boldsymbol{R}(\tilde{\boldsymbol{q}})$, we know that each row of $\boldsymbol{R}(\tilde{\boldsymbol{q}})$ consists of $\tilde{\boldsymbol{q}}_{ij}$ and $\tilde{\boldsymbol{q}}_{ji}$. Thus, $\dot{\boldsymbol{W}}$ can be written in a vector form as

$$\dot{\boldsymbol{W}}=\boldsymbol{\sigma}^{\rm T}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{u},\tag{6}$$

where $\boldsymbol{\sigma}(e)\triangleq[...,e_{ij}(e_{ij}+2d_{ij}),...]^{\rm T}\in\mathbb{R}^m$ and $\boldsymbol{u}\triangleq(\boldsymbol{u}_1^{\rm T},...,\boldsymbol{u}_n^{\rm T})^{\rm T}\in\mathbb{R}^{2n}$.

For each agent $i$, an estimator is designed to track the desired velocity $v_{\rm d}$

$$\begin{aligned}\dot{\hat{\boldsymbol{v}}}_i(t)&=-\sum_{j\in\mathscr{N}_i(\mathscr{E})}(\hat{\boldsymbol{v}}_i(t)-\hat{\boldsymbol{v}}_j(t))-a_{i0}(\hat{\boldsymbol{v}}_i(t)-v_{\rm d}),\\&\quad i=1,...,n,\end{aligned}\tag{7}$$

where $\hat{v}_i(t)$ is the $i$-th agent's estimate of $v_{\rm d}$, $a_{i0}>0$ if agent $i$ has access to $v_{\rm d}$, and $a_{i0}=0$ otherwise. We assume that there is at least one agent which has access to $v_{\rm d}$.

The control input $\boldsymbol{u}$ is then designed as

$$\boldsymbol{u}=-k\boldsymbol{R}^{\rm T}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e)+\hat{\boldsymbol{v}}(t),\tag{8}$$

where $\hat{\boldsymbol{v}}(t)=(\hat{\boldsymbol{v}}_1^{\rm T}(t),...,\hat{\boldsymbol{v}}_n^{\rm T}(t))^{\rm T}$ and $k>0$. Let $\tilde{\boldsymbol{v}}_i(t)\triangleq\hat{\boldsymbol{v}}_i(t)-v_{\rm d}$ and $\tilde{\boldsymbol{v}}(t)\triangleq[\tilde{\boldsymbol{v}}_1^{\rm T}(t),...,\tilde{\boldsymbol{v}}_n^{\rm T}(t)]^{\rm T}$. Note that $v_{\rm d}$ is constant. Equation (7) can be written in a vector form as

$$\begin{aligned}\dot{\tilde{\boldsymbol{v}}}(t)&=-(\mathscr{L}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}(t)-(\boldsymbol{A}_0\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}(t)\\&=-(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}(t),\end{aligned}\tag{9}$$

where $\mathscr{L}$ is the Laplacian matrix associated with $\mathscr{G}$, $\boldsymbol{A}_0=\text{diag}(a_{10},...,a_{n0})$, $a_{i0}\geq 0$ and there is at least one nonzero $a_{i0}$, and $\boldsymbol{H}\triangleq\mathscr{L}+\boldsymbol{A}_0$.

For a rigid graph, it is at least connected because otherwise if one or more nodes are not connected with other nodes, then this part of nodes can move freely without keeping distance to other nodes, which is in conflict with the nature of rigidity. Thus, $\mathscr{G}$ is connected. It follows that $\boldsymbol{H}$ is positive definite.[16] A Lyapunov function candidate $V$ can be defined as follows:

$$\begin{aligned}V&=\boldsymbol{W}+\frac{1}{2k}\tilde{\boldsymbol{v}}^{\rm T}(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{-1}\tilde{\boldsymbol{v}}\\&=\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}(\|\tilde{\boldsymbol{q}}_{ij}\|^2-d_{ij}^2)^2+\frac{1}{2k}\tilde{\boldsymbol{v}}^{\rm T}(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{-1}\tilde{\boldsymbol{v}},\end{aligned}$$

where $k$ is defined in Eq. (8). Let $\boldsymbol{w}=(\tilde{\boldsymbol{q}}^{\rm T}(t),\tilde{\boldsymbol{v}}^{\rm T}(t))^{\rm T}$. Define a level set $\Omega_{\rm c}$ as

$$\Omega_{\rm c}=\{\boldsymbol{w}|V\leq c\},$$

where $c$ is a sufficiently small positive number.

The main result of this section is stated in the following theorem.

**Theorem 1** For system (1) under the control law (8) and the estimator (7), if $F^*$ is the desired formation and $w(0)\in\Omega(c)$, then the control objective (2) and (3) can be achieved, and the formation error and the estimation error converge to zero exponentially.

**Proof** The derivative of $V$ along (6) and (9) is given by

$$\dot{V} = \boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{u} - \frac{1}{k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}}. \tag{10}$$

Substituting Eq. (8) into Eq. (10) yields

$$\dot{V} = -k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) + \boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\hat{\boldsymbol{v}} - \frac{1}{k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}}.$$

Because $\hat{\boldsymbol{v}} = \tilde{\boldsymbol{v}} + \mathbf{1}_n \otimes v_{\mathrm{d}}$, and it is straightforward to verify that $\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})(\mathbf{1}_n \otimes v_{\mathrm{d}}) = 0$. It follows that

$$\begin{aligned}\dot{\boldsymbol{V}} &= -k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) + \boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{v}} - \frac{1}{k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}} \\ &\le -k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) - \frac{1}{k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}} \\ &\quad + \frac{1}{2}(k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) + \frac{1}{k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}}) \\ &\le -\frac{1}{2}k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) - \frac{1}{2k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}} \\ &\le 0.\end{aligned}$$

Because $\dot{\boldsymbol{V}} \le 0$, for $w(0) \in \Omega_{\mathrm{c}}$, we can conclude that $w(t) \in \Omega_{\mathrm{c}}$ for all $t \ge 0$. Because the target formation $\boldsymbol{F}^*$ is infinitesimally and minimally rigid, $\boldsymbol{R}(\tilde{\boldsymbol{q}}^*)$ has a full row rank. From Lemma 1, we know that the matrix $\boldsymbol{R}(\tilde{\boldsymbol{q}}^*)\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}}^*)$ is positive definite. The eigenvalues of $\boldsymbol{R}(\tilde{\boldsymbol{q}}^*)\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}}^*)$ are continuous functions of the elements of the matrix. This is because the coefficients of the characteristic polynomial of a matrix is a continuous function of the elements of the matrix, and the eigenvalues of the matrix are continuous functions of the coefficients. Thus, for a sufficiently small positive number $c$, $w \in \Omega(c)$ can ensure that $\tilde{\boldsymbol{q}}$ belongs to the open neighborhood where $\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})$ has a positive minimal eigenvalue. Define the set $\boldsymbol{S} = \{\tilde{\boldsymbol{q}} | \lambda_{\min}[\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})] > 0\}$, where $\lambda_{\min}$ denotes the minimum eigenvalue. It follows that

$$\begin{aligned}\dot{\boldsymbol{V}} &\le -\frac{1}{2}k\inf(\boldsymbol{S})\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{\sigma}(e) - \frac{1}{2k}\tilde{\boldsymbol{v}}^{\mathrm{T}}\tilde{\boldsymbol{v}} \\ &\le -\lambda \boldsymbol{V}, \end{aligned} \tag{11}$$

where $\inf(\boldsymbol{S})$ denotes the infimum of set $\boldsymbol{S}$, and $\lambda = \min\{2k\inf(\boldsymbol{S}), 1/\lambda_{\max}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{-1}]\}$. $\lambda_{\max}$ denotes the maximum eigenvalue. Thus, from Eq. (11), it follows that $\boldsymbol{\sigma}(e) = 0, \tilde{\boldsymbol{v}} = 0$ is exponentially stable.[21]

The exponential stability of $\boldsymbol{\sigma}(e) = 0, \tilde{\boldsymbol{v}} = 0$ implies that $\|\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t)\| \to d_{ij}$ as $t \to \infty$ for all $(i,j) \in \mathscr{E}$ and $\hat{\boldsymbol{v}}_i \to v_{\mathrm{d}}$ as $t \to \infty$ for all $i \in \mathscr{V}$. Finally, since $\boldsymbol{\sigma}(e) \to 0$ as $t \to \infty$ and $\boldsymbol{R}(\tilde{\boldsymbol{q}})$ is bounded, we have that $\boldsymbol{u} \to \hat{\boldsymbol{v}}$ as $t \to \infty$. Because we have proven that $\hat{\boldsymbol{v}}_i \to v_{\mathrm{d}}$ exponentially as $t \to \infty$, it follows from Eq. (1) that $\dot{\boldsymbol{q}}_i(t) - v_{\mathrm{d}}(t) \to 0$ as $t \to \infty, i = 1, \ldots, n$. The proof is thus completed.

## 5. Formation maintenance with time-varying translational velocity

In this section, we consider a case where $v_{\mathrm{d}}(t)$ is time-varying. It is assumed that the derivative of the desired velocity $v_{\mathrm{d}}(t)$ is bounded, and this bound is denoted by $\gamma$.

Inspired by Ref. [8], the following non-smooth algorithm is proposed to estimate the time-varying desired translational velocity $v_{\mathrm{d}}(t)$:

$$\begin{aligned}\dot{\hat{\boldsymbol{v}}}_i(t) &= -\beta\,\mathrm{sgn}\Big\{\sum_{j\in\mathscr{N}_i}[\hat{\boldsymbol{v}}_i(t) - \hat{\boldsymbol{v}}_j(t)] + a_{i0}[\hat{\boldsymbol{v}}_i(t) - v_{\mathrm{d}}(t)]\Big\}, \\ &\quad i = 1, \ldots, n, \end{aligned} \tag{12}$$

where $\hat{\boldsymbol{v}}_i(t)$ is the $i$-th agent's estimate of $v_{\mathrm{d}}(t)$, $\beta$ is a positive constant, and $\mathrm{sgn}(\cdot)$ is the signum function. Define $\hat{\boldsymbol{v}}(t) = [\hat{\boldsymbol{v}}_1(t)^{\mathrm{T}}, \ldots, \hat{\boldsymbol{v}}_n(t)^{\mathrm{T}}]^{\mathrm{T}}$. The objective of (12) is to guarantee that $\hat{\boldsymbol{v}}_i(t) \to v_{\mathrm{d}}(t)$ in finite time. The control law can be designed as

$$\boldsymbol{u} = -k\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e) + \hat{\boldsymbol{v}}(t). \tag{13}$$

**Theorem 2** For system (1) under the control law (13) and the estimator (12), if $\boldsymbol{F}^*$ is the desired formation, $\boldsymbol{w}(0) \in \boldsymbol{\Omega}(c)$, and $\beta > \gamma$, then the control objective (2) and (3) can be achieved, and the formation error converges to zero asymptotically.

**Proof** In the following, we first prove that $\hat{\boldsymbol{v}}(t) \to 1_n \otimes v_{\mathrm{d}}(t)$ in finite time. From Eq. (12), it follows that

$$\begin{aligned}\dot{\tilde{\boldsymbol{v}}}_i(t) &= -\beta\,\mathrm{sgn}\Big\{\sum_{j\in\mathscr{N}_i}[\tilde{\boldsymbol{v}}_i(t) - \tilde{\boldsymbol{v}}_j(t) + a_{i0}\tilde{\boldsymbol{v}}_i(t)]\Big\} - \dot{v}_{\mathrm{d}}(t), \\ &\quad i = 1, \ldots, n. \end{aligned} \tag{14}$$

Rewrite Eq. (14) in a vector form as

$$\dot{\tilde{\boldsymbol{v}}}(t) = -\beta\,\mathrm{sgn}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}] - \mathbf{1}_n \otimes \dot{v}_{\mathrm{d}}(t),$$

where $\mathrm{sgn}(\boldsymbol{x}) = [\mathrm{sgn}(x_1), \ldots, \mathrm{sgn}(x_n)]^{\mathrm{T}}$. Design the following Lyapunov function as:

$$\Psi(\tilde{\boldsymbol{v}}) = \frac{1}{2}\tilde{\boldsymbol{v}}^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{v}.$$

The derivative of $\Psi(\tilde{\boldsymbol{v}})$ is given by

$$\begin{aligned}\dot{\Psi}(\tilde{\boldsymbol{v}}) &= \tilde{\boldsymbol{v}}^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)\dot{\tilde{\boldsymbol{v}}} \\ &= -\beta\tilde{\boldsymbol{v}}^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)\mathrm{sgn}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}] \\ &\quad - \tilde{\boldsymbol{v}}^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)[\mathbf{1}_n \otimes \dot{v}_{\mathrm{d}}(t)] \\ &= -\beta\sum_{i=1}^{2n}\Big|[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i\Big| - \dot{v}_{\mathrm{d}}(t)\sum_{i=1}^{2n}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i,\end{aligned}$$

where $[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i$ represents the $i$-th element of $(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}$. Recall that $\dot{v}_{\mathrm{d}}(t)$ is bounded, and this bound is $\gamma$. It thus follows that

$$\begin{aligned}\dot{\Psi}(\tilde{\boldsymbol{v}}) &= -\beta\sum_{i=1}^{2n}\Big|[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i\Big| - \dot{v}_{\mathrm{d}}(t)\sum_{i=1}^{2n}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i \\ &\le -\beta\sum_{i=1}^{2n}\Big|[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i\Big| + \gamma\sum_{i=1}^{2n}\Big|[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i\Big|\end{aligned}$$

$$
\begin{aligned}
&= -(\beta-\gamma)\sum_{i=1}^{2n}\left|[(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}]_i\right| \\
&= -(\beta-\gamma)\|(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}\|_1 \\
&\leq -(\beta-\gamma)\|(\boldsymbol{H}\otimes\boldsymbol{I}_2)\tilde{\boldsymbol{v}}\|_2 \\
&\leq -(\beta-\gamma)\sqrt{\lambda_{\min}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)]}\|\tilde{\boldsymbol{v}}\|_2 \\
&\leq -(\beta-\gamma)\sqrt{\frac{2\lambda_{\min}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)]}{\lambda_{\max}(\boldsymbol{H}\otimes\boldsymbol{I}_2)}}\Psi^{\frac{1}{2}} \\
&= -\alpha\Psi^{1/2},
\end{aligned}
$$

where $\alpha=(\beta-\gamma)\sqrt{\frac{2\lambda_{\min}[(\boldsymbol{H}\otimes\boldsymbol{I}_2)^{\mathrm{T}}(\boldsymbol{H}\otimes\boldsymbol{I}_2)]}{\lambda_{\max}(\boldsymbol{H}\otimes\boldsymbol{I}_2)}}>0$. It can be verified that there exists $\varepsilon$, such that

$$\int_0^{\varepsilon}\frac{\mathrm{d}\Psi}{\alpha\Psi^{\frac{1}{2}}}=\frac{2}{\alpha}\varepsilon^{1/2}<\infty.$$

According to Theorem 3 in Ref. [22], we can conclude that $\tilde{\boldsymbol{v}}\to 0$ in finite time $T$, that is $\hat{v}_i(t)=v_{\mathrm{d}}(t)$ for $i=1,\ldots,n$ after time $T$.

After time $T$, the control law of (13) becomes

$$\boldsymbol{u}=-k\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e)+\mathbf{1}_n\otimes v_{\mathrm{d}}(t). \tag{15}$$

Substituting Eq. (15) into Eq. (6) yields

$$
\begin{aligned}
\dot{\boldsymbol{W}} &= -k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e)+\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})(\mathbf{1}_n\otimes v_{\mathrm{d}}(t)) \\
&= -k\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})\boldsymbol{\sigma}(e).
\end{aligned}
$$

Following a similar proof to that of Theorem 1, it can be shown that $\boldsymbol{R}(\tilde{\boldsymbol{q}})\boldsymbol{R}^{\mathrm{T}}(\tilde{\boldsymbol{q}})$ is positive definite for $w(0)\in\Omega_{\mathrm{c}}$. It thus follows that

$$
\begin{aligned}
\dot{\boldsymbol{W}} &\leq -k\inf(\boldsymbol{S})\boldsymbol{\sigma}^{\mathrm{T}}(e)\boldsymbol{\sigma}(e) \\
&= -4k\inf(\boldsymbol{S})\boldsymbol{W}.
\end{aligned}
$$

The rest of the proof is similar to that of Theorem 1, and is hence omitted.

## 6. Simulation results

In this section, we present two simulation examples for the formation problem with constant and time-varying desired velocity, respectively. In both examples, we consider a group of four agents. The desired formation is a square shown in Fig. 2 where the agents are located at the four vertices. There are five edges in the graph. It can be verified that the formation given in Fig. 2 is minimally rigid. Let $a_{10}=1$, $a_{20}=a_{30}=a_{40}=0$. The desired distances between the agents are given by $d_{12}=d_{23}=d_{34}=d_{41}=10$, $d_{13}=10\sqrt{2}$. The initial conditions of the agents are selected as $\boldsymbol{q}(0)=(\boldsymbol{q}_1^{\mathrm{T}}(0)\ \boldsymbol{q}_2^{\mathrm{T}}(0)\ \boldsymbol{q}_3^{\mathrm{T}}(0)\ \boldsymbol{q}_4^{\mathrm{T}}(0))^{\mathrm{T}}=(0.3\ 1.2\ 0.1\ 0.1\ 0.9\ -0.1\ 1.2\ 1.2)^{\mathrm{T}}$ and $\hat{\boldsymbol{v}}(0)=2\times\mathrm{rand}(1,8)-1$. The control gain $k$ in Eqs. (8) and (13) is set to 0.5, and $\beta$ in Eq. (12) is set to 1.2.

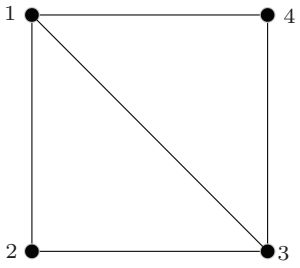



**Fig. 2.** Desired formation.

For the constant desired velocity case, let $v_{\mathrm{d}}=1$. Figure 3 shows the agents' trajectories, which shows that the agents can achieve the formation asymptotically. Figure 4 shows that the inter-agent distance errors converge to zero. Figure 5 describes the velocities of all of the agents which converge to the desired velocity $v_{\mathrm{d}}=1$.

For the time-varying desired velocity case, let $v_{\mathrm{d}}(t)=\cos(t)$. Figure 6 shows that the agents can achieve the formation asymptotically. Figure 7 describes the inter-agent distance errors which converge to zero. Figure 8 shows the velocities of all the agents which converge to the given time-varying velocity $v_{\mathrm{d}}(t)=\cos(t)$.

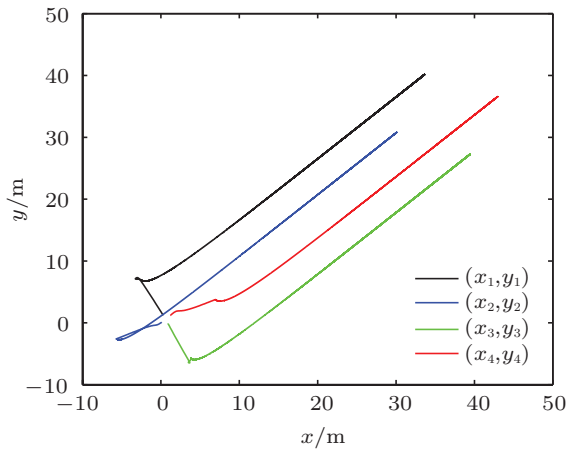



**Fig. 3.** (color online) Agents' trajectories in the constant velocity case.

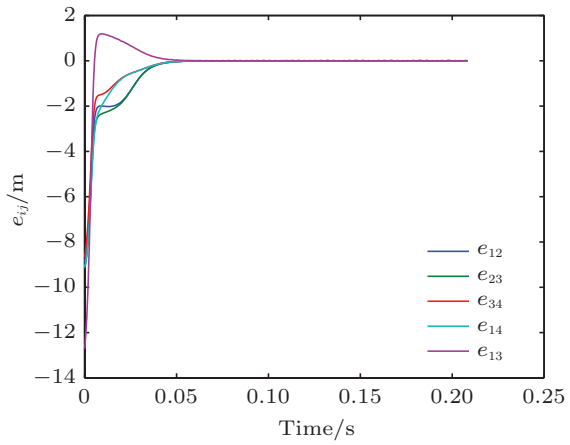



**Fig. 4.** (color online) Inter-agent distance errors for $(i,j)\in\mathscr{E}$ in the constant velocity case.

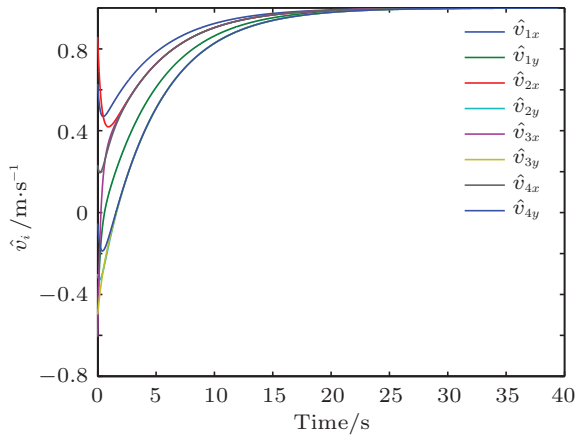



**Fig. 5.** (color online) Agents' velocities in the constant velocity case.

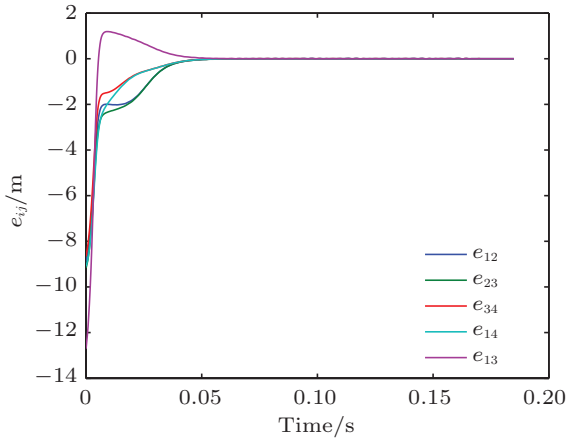



**Fig. 7.** (color online) Inter-agent distance errors for $(i, j) \in \mathcal{E}$ in the time-varying velocity case.

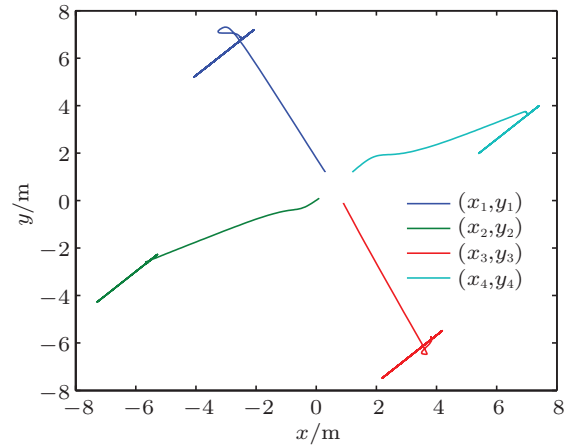



**Fig. 6.** (color online) Agents' trajectories in the time-varying velocity case.

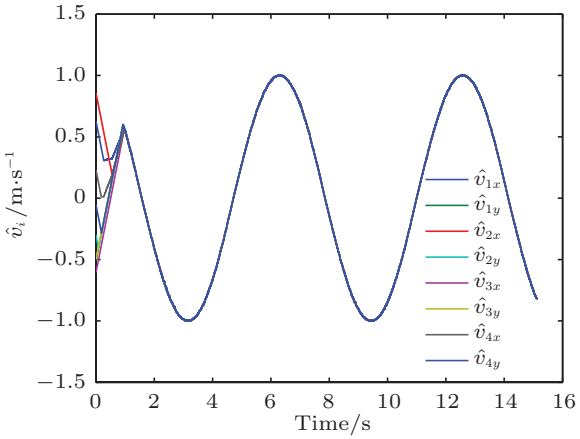



**Fig. 8.** (color online) Agents' velocities in the time-varying velocity case.

## 7. Conclusion

In this paper, the distributed tracking problem of a minimally and infinitesimally rigid formation has been investigated. Two control algorithms have been designed to stabilize the inter-agent distances while allowing the formation to follow a constant or a time-varying velocity. In both cases, the desired trajectory velocity is assumed to be available to a subset of the agents. Distributed estimators have been constructed for each agent to estimate the desired velocity. The estimates are further employed to construct the control inputs of the agents where the rigidity matrix plays a central role. It has been proven that both the formation error and the estimator error can converge to zero. Finally, numerical examples have been proposed to verify the derived results.

## References


[1] Anderson B D O, Yu C, Fidan B and Hendrickx J 2008 *IEEE Contr. Syst. Mag.* **28** 48

[2] Chen F, Chen Z, Xiang L Y, Liu Z and Yuan Z 2009 *Automatica* **45** 1215

[3] Su H, Chen M Z Q, Wang X and Lam J 2014 *IEEE T. Ind. Electron.* **61** 2842

[4] Ji L and Liao X 2013 *Chin. Phys. B* **22** 040203

[5] Bai L, Chen F and Lan W 2014 *Proceedings of* 2014 *Chinese Control Conference*, July 28–30, 2014, Nanjing, China, p. 1288

[6] Oh K K and Ahn H S 2011 *Automatica* **47** 2306

[7] Cai X and de Queiroz M 2013 *Proceedings of the IEEE American Control Conference*, June 17–19, 2013, Washington, DC, USA, p. 2521

[8] Cao Y, Ren W and Meng Z 2010 *Syst. Control Lett.* **59** 522

[9] Chen F, Cao Y and Ren W 2012 *IEEE T. Automat. Contr.* **57** 3169

[10] Reza O S 2006 *IEEE T. Automat. Contr.* **51** 401

[11] Su H, Wang X and Lin Z 2009 *IEEE T. Automat. Contr.* **54** 293

[12] Wang X, Li X and Zhang Z 2013 *Control and Decision* **11** 001

[13] Wen L, Zailani S and Fernando Y 2009 *J. Technol. Manage. Innovation* 4 1

[14] Cao M, Morse A, Yu C, Anderson B and Dasgupta S 2011 *Communications in Information and Systems* 11 1

[15] Dörfler F and Francis B 2009 *Proceedings of the* 2009 *European Control Conference*, 2009, Budapest, Hungary, p. 2432

[16] Mei J, Ren W and Ma G 2011 *IEEE T. Automat. Contr.* **56** 1415

[17] Oh K K and Ahn H S 2011 *Proceedings of IEEE International Symposium on Intelligent Control*, September 28–30, 2011, Denver, USA, p. 816

[18] Gazi V and Passino K M 2011 *Swarm Stability and Optimization* (Berlin Heidelberg: Springer)

[19] Luo X, Han N and Guan X 2010 *Chin. Phys. B* **19** 100202

[20] Wen G, Duan Z, Ren W and Chen G 2014 *Int. J. Robust. Nonlinear Control* **24** 2438

[21] Khalil H K 2002 *Nonlinear Systems*, 3rd edn. (New Jersey: Prentice Hall)

[22] Bernuau E, Polyakov A, Efimov D and Perruquetti W 2013 *Joint SSSC, TDS, FDA*, Feberuary 2013, Grenoble, France, hal-00745673