A new model for algebraic Rossby solitary waves in rotation fluid and its solution*

Chen Yao-Deng(陈耀登)[a)b)], Yang Hong-Wei(杨红卫)[c)†], Gao Yu-Fang(高玉芳)[a)b)], Yin Bao-Shu(尹宝树)[d)e)], and Feng Xing-Ru(冯兴如)[d)e)]

[a)] *Key Laboratory of Meteorological Disaster of Ministry of Education, Nanjing University of Information Science & Technology, Nanjing 210044, China*
[b)] *Collaborative Innovation Center on Forecast and Evaluation of Meteorological Disasters, Nanjing University of Information Science & Technology, Nanjing 210044, China*
[c)] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*
[d)] *Institute of Oceanology, Chinese Academy of Sciences, Qingdao 266071, China*
[e)] *Key Laboratory of Ocean Circulation and Wave, Chinese Academy of Sciences, Qingdao 266071, China*

(Received 4 February 2015; revised manuscript received 26 March 2015; published online 20 July 2015)

A generalized Boussinesq equation that includes the dissipation effect is derived to describe a kind of algebraic Rossby solitary waves in a rotating fluid by employing perturbation expansions and stretching transformations of time and space. Using this equation, the conservation laws of algebraic Rossby solitary waves are discussed. It is found that the mass, the momentum, the energy, and the velocity of center of gravity of the algebraic solitary waves are conserved in the propagation process. Finally, the analytical solution of the equation is generated. Based on the analytical solution, the properties of the algebraic solitary waves and the dissipation effect are discussed. The results point out that, similar to classic solitary waves, the dissipation can cause the amplitude and the speed of solitary waves to decrease; however, unlike classic solitary waves, the algebraic solitary waves can split during propagation and the decrease of the detuning parameter can accelerate the occurrence of the solitary waves fission phenomenon.

Keywords: generalized Boussinesq equation, algebraic Rossby solitary waves, dissipation effect, solitary waves fission

PACS: 02.30.Jr, 47.35.Fg, 92.10.Hm **DOI:** 10.1088/1674-1056/24/9/090205

1. Introduction

Recently, Rossby solitary waves which determine the ocean's response to atmosphere and climate changes to a large extent have attracted increasing attention from meteorologists and oceanologists. As Russell observed, solitary waves are steady nonlinear waves.[1] Two major theories to describe solitary waves have been generally formed, which are: classical solitary waves theory[2–6] and algebraic solitary waves theory.[7–11] The former was established by a KdV (Korteweg-de Vries) type model to explain the evolution of classical solitary waves. The outstanding character of this type of solitary wave is that the waveform and the speed remain invariant during propagation and interaction. Meanwhile, algebraic solitary waves indicate the behaviors of solitary waves governed by the BO (Benjamin–Ono) model. In spite of its invariant of total mass, momentum and energy, its waveform and speed may vary because fission phenomenon may take place. These two kinds of models have allowed the remarkable development of solitary waves in a wide variety of physical contexts. However, most of the previous research about Rossby solitary waves was established in the zonal area, and we cannot apply them directly to the spherical earth; that is to say, solitary waves in a rotating fluid might be the next direction for our studies.

The main purpose of this paper is to construct a new model to describe algebraic Rossby solitary waves in a rotating fluid. Starting from rotational potential vorticity conserved equation with dissipation effect, by employing perturbation method and time-space stretching transformations, the model is established and is greatly different from the previous algebraic Rossby solitary wave models. Because of its similarity to the Boussinesq equation, we call it the generalized Boussinesq equation. As we are aware, conservation laws are very useful for studying behaviors of wave propagation; therefore, several conservation laws related to the present problem will be deduced in Section 3. The analytic solution of the generalized Boussinesq equation will be given in Section 4. Based on the analytical solution, the dissipation effect will be studied. Section 5 is focused on the solitary waves fission phenomenon from theoretical analysis and numerical simulation. Finally, some conclusions will be placed in Section 6.

*Project supported by the Shandong Provincial Key Laboratory of Marine Ecology and Environment and Disaster Prevention and Mitigation Project, China (Grant No. 2012010), the National Natural Science Foundation of China (Grant Nos. 41205082 and 41476019), the Special Funds for Theoretical Physics of the National Natural Science Foundation of China (Grant No. 11447205), and the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), China.

†Corresponding author. E-mail: hwyang1979@163.com

© 2015 Chinese Physical Society and IOP Publishing Ltd

2. Derivation of the generalized Boussinesq equation

According to Ref. [12], if we take the plane polar coordinate (r,θ), then the rotational potential vorticity conserved equation including turbulent dissipation is, in non-dimensional form, given by

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\\&\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right]+\frac{\beta}{r}\frac{\partial\Psi}{\partial\theta}\\&=-\lambda_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right]+Q,\end{aligned}\tag{1}$$

where $\beta=\beta_0(L^2/U)$ and $\beta_0=(\omega_0/R_0)\cos\phi_0$, in which R_0 is the Earth's radius, ω_0 is the angular frequency of the Earth's rotation, ϕ_0 is the latitude;

$$\lambda_0\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right]$$

denotes the vorticity dissipation which is caused by the Ekman boundary layer, and λ_0 is a dissipative coefficient; Q is the external source, the form of Q will be given in the latter; for the other characters, see Ref. [12].

Assume that the basic field is the rotating fluid with angular velocity $\Omega=\Omega(r)$, then the stream function can be written as

$$\Psi=\int^r(\Omega(s)-c_0+\varepsilon\alpha)s\mathrm{d}s+\varepsilon\psi,\tag{2}$$

where c_0 is a constant, which is regarded as a phase angle velocity; $\varepsilon\psi$ is the disturbance part of stream-function, $\varepsilon\ll1$ is a small parameter; and, $O(\alpha)=1$ is called detuning parameter and is used to denote the proximity of the system to a resonate state.

For the sake of studying the role of nonlinearity, the rotational angular velocity is taken as in the following form:

$$\Omega(r)=\begin{cases}\omega(r), & r_1\le r\le r_2,\\ \omega_0, & r\ge r_2,\end{cases}\tag{3}$$

where ω_0 is constant and $\Omega(r)$ is assumed to be smooth across $r=r_2$.

In the domain of $[r_1,r_2]$, in order to balance nonlinearity and turbulent dissipation, and remove the dissipation caused by basic stream function, we take

$$\lambda_0=\varepsilon^2\lambda,\quad Q=\lambda_0\frac{1}{r}\frac{\partial}{\partial r}[r^2(\omega-c_0+\varepsilon\alpha)].\tag{4}$$

By introducing Eqs. (2)–(4) into Eq. (1), we can obtain the following equation about perturbation stream function ψ:

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\omega-c_0+\varepsilon\alpha)\frac{\partial}{\partial\theta}\right.\\&\left.+\varepsilon\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)+\varepsilon^{\frac{3}{2}}\lambda\right]\\&\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]\\&+\frac{1}{r}\left[\beta-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)\right]\frac{\partial\psi}{\partial\theta}=0.\end{aligned}\tag{5}$$

For the domain $[r_2,\infty)$, assuming $\beta=0$ and neglecting dissipation effect, then we have

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+(\omega_0-c_0+\varepsilon\alpha)\frac{\partial}{\partial\theta}+\varepsilon\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\\&\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]=0.\end{aligned}\tag{6}$$

Because the waves in the fluid have multiple time-space scale features, we adopt the following time-space stretching transformations in Eq. (5):

$$\Theta=\varepsilon\theta,\quad r=r,\quad T=\varepsilon^{\frac{3}{2}}t,\tag{7}$$

and the perturbation stream function is taken as

$$\psi=\psi_0(\Theta,r,T)+\varepsilon^{\frac{1}{2}}\psi_1(\Theta,r,T)+\varepsilon\psi_2(\Theta,r,T)+\cdots.\tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (5) yields

$$\begin{aligned}&\varepsilon^0:\ \Gamma\psi_0=0,\\&\varepsilon^{\frac{1}{2}}:\ \Gamma\psi_1=-\frac{\partial}{\partial T}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right)\right],\\&\varepsilon^1:\ \Gamma\psi_2=-\frac{\partial}{\partial T}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right)\right]\\&\quad-\left(\alpha\frac{\partial}{\partial\Theta}+\frac{1}{r}\frac{\partial\psi_0}{\partial r}\frac{\partial}{\partial\Theta}-\frac{1}{r}\frac{\partial\psi_0}{\partial\Theta}\frac{\partial}{\partial r}+\lambda\right)\\&\quad\times\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_0}{\partial r}\right)\right],\end{aligned}\tag{9}$$

where the operator Γ is defined as

$$\begin{aligned}\Gamma=&\frac{1}{r}\frac{\partial}{\partial\Theta}\left\{(\omega-c_0)\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right.\\&\left.+\left[\beta-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)\right]\right\}.\end{aligned}\tag{10}$$

Assume that there is no perturbation at the boundary $r=r_1$, i.e., the boundary conditions are taken as

$$\psi_0=\psi_1=\cdots=0,\qquad r=r_1.\tag{11}$$

For ε^0, by assuming $\psi_0=A(\Theta,T)\phi_0(r)$, then $\phi_0(r)$ satisfies

$$\begin{aligned}&(\omega-c_0)\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0(r)}{\mathrm{d}r}\right)\right]\\&+\left[\beta-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)\right]\phi_0(r)=0,\end{aligned}\tag{12}$$

which is a variable coefficient eigenvalue problem, and $A(\Theta,T)$ is the unknown amplitude in the order $\mathrm{O}(\varepsilon^0)$, which needs higher-order equations to determine.

For $\varepsilon^{1/2}$, by analysis, we deduce that

$$\frac{\partial \psi_1}{\partial \Theta} = \frac{\partial A}{\partial T}\phi_1(r),$$

where $\phi_1(r)$ satisfies the following equation:

$$\begin{aligned}&(\omega - c_0)\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_1(r)}{\mathrm{d}r}\right)\right] + \left[\beta - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)\right]\phi_1(r)\\ &= \left[\beta - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)\right]\phi_0(r). \qquad (13)\end{aligned}$$

Let us proceed to consider the equation of order ε. By multiplying the both sides of the order ε equation by

$$\frac{r\phi_0(r)}{\omega - c_0}\frac{\partial}{\partial \Theta}$$

and integrating it with respect to r from r_1 to r_2, we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial \Theta^2} r\left(\phi_0\frac{\partial \psi_2}{\partial r} - \frac{\mathrm{d}\phi_0}{\mathrm{d}r}\psi_2\right)\Big|_{r=r_2}\\ &= -\frac{\partial^2 A}{\partial T^2}\int_{r_1}^{r_2} r\phi_0\left[\frac{\beta - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)}{(\omega - c_0)}\right]\\ &\times\left[\frac{\phi_0}{(\omega - c_0)^2} - \frac{\phi_1}{(\omega - c_0)}\right]\mathrm{d}r\\ &-\frac{\partial^2 A}{\partial \Theta^2}\int_{r_1}^{r_2}\frac{\phi_0}{2(\omega - c_0)}\left(\frac{\partial \phi_0}{\partial r} - \phi_0\frac{\partial}{\partial r}\right)\\ &\times\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)\right]\mathrm{d}r - \left(\lambda\frac{\partial A}{\partial \Theta} + \alpha\frac{\partial^2 A}{\partial \Theta^2}\right)\\ &\times\int_{r_1}^{r_2}\frac{\phi_0}{(\omega - c_0)}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)\mathrm{d}r, \qquad (14)\end{aligned}$$

in which we only solve the boundary conditions on ϕ and ψ_2 to determine the equation governing the amplitude A.

In the domain of $[r_2, +\infty)$, the following transformations are adopted in Eq. (6):

$$\rho = \theta, \quad r = r, \quad T = \varepsilon^{\frac{3}{2}}t, \qquad (15)$$

and $\bar{\psi}$ notes the perturbation function, and then introducing Eq. (15) into Eq. (6), we have the ε^0 order equation as follows:

$$(\omega_0 - c_0)\frac{\partial}{\partial \rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\psi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \bar{\psi}}{\partial \rho^2}\right] = 0. \qquad (16)$$

From Eq. (16), we deduce that

$$\begin{cases}\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \bar{\psi}}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \bar{\psi}}{\partial \rho^2} = 0, & r \geq r_2,\\ \bar{\psi} \to 0, & r \to \infty,\end{cases} \qquad (17)$$

where the integral constant is taken as zero. The solution of Eq. (17) can be written as

$$\bar{\psi}(\rho, r, T, \varepsilon) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2 - r_2^2)\bar{\psi}(\rho', r, T, \varepsilon)}{r_2^2 - 2r_2 r\cos(\rho - \rho') + r^2}\mathrm{d}\rho'. \qquad (18)$$

Taking the derivative with respect to r for both sides of Eq. (18) leads to

$$\begin{aligned}\frac{\partial \bar{\psi}}{\partial r} &= \frac{r_2}{\pi}\int_0^{2\pi}\bar{\psi}(\rho', r, T, \varepsilon)\\ &\times\frac{[2r_2 r - (r^2 + r_2^2)\cos(\rho - \rho')]}{[r_2^2 - 2r_2 r\cos(\rho - \rho') + r^2]^2}\mathrm{d}\rho'. \qquad (19)\end{aligned}$$

Because of the smoothness condition of the solution at $r = r_2$, we obtain

$$\begin{aligned}&\psi_0(\Theta, r_2, T) + \varepsilon^{\frac{1}{2}}\psi_1(\Theta, r_2, T) + \varepsilon\psi_2(\Theta, r_2, T)\\ &= \bar{\psi}(\rho, r_2, T, \varepsilon) + O(\varepsilon^2), \qquad (20)\\ &\frac{\partial \psi_0(\Theta, r_2, T)}{\partial r} + \varepsilon^{\frac{1}{2}}\frac{\partial \psi_1(\Theta, r_2, T)}{\partial r} + \varepsilon\frac{\partial \psi_2(\Theta, r_2, T)}{\partial r}\\ &= \frac{\partial \bar{\psi}(\rho, r_2, T, \varepsilon)}{\partial r} + O(\varepsilon^2). \qquad (21)\end{aligned}$$

From Eq. (20), we can judge

$$\begin{aligned}&A(\Theta, T)\phi_0 = \bar{\psi}(\rho, r_2, T, \varepsilon),\\ &\psi_1(\Theta, r_2, T) = \psi_2(\Theta, r_2, T) = 0. \qquad (22)\end{aligned}$$

By substituting Eq. (22) into Eq. (19), we have

$$\frac{\partial \bar{\psi}(\rho, r_2, T, \varepsilon)}{\partial r} = \varepsilon\phi_0(r_2)\frac{\partial^2 J(A(\Theta, T))}{\partial \Theta^2}, \qquad (23)$$

where

$$J(A(\Theta, T)) = \frac{r_2}{2\pi}\int_0^{2\pi} A(\Theta', T)\ln\left(\sin\left(\frac{\Theta - \Theta'}{2}\right)\right)\mathrm{d}\Theta'.$$

From Eqs. (21) and (23), we have

$$\phi_0{}'(r_2) = 0, \quad \frac{\partial \psi_2(\Theta, r_2, T)}{\partial r} = \phi_0(r_2)\frac{\partial^2 J(A(\Theta, T))}{\partial \Theta^2}. \qquad (24)$$

Substituting Eqs. (22) and (24) into Eq. (14) yields

$$\begin{aligned}&\frac{\partial^2 A}{\partial T^2} + \alpha a_1\frac{\partial^2 A}{\partial \Theta^2} + a_2\frac{\partial^2 A^2}{\partial \Theta^2}\\ &+ a_3\frac{\partial^4}{\partial \Theta^4}J(A(\Theta, T)) + \lambda a_4\frac{\partial A}{\partial \Theta} = 0, \qquad (25)\end{aligned}$$

where

$$\begin{aligned}a &= \int_{r_1}^{r_2} r\phi_0\left[\frac{\beta - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}r^2\omega}{\mathrm{d}r}\right)}{(\omega - c_0)}\right]\left[\frac{\phi_0}{(\omega - c_0)^2} - \frac{\phi_1}{(\omega - c_0)}\right]\mathrm{d}r,\\ a_1 &= \int_{r_1}^{r_2}\frac{\phi_0}{(\omega - c_0)}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)\mathrm{d}r/a,\\ a_2 &= \int_{r_1}^{r_2}\frac{\phi_0}{2(\omega - c_0)}\left(\frac{\partial \phi_0}{\partial r} - \phi_0\frac{\partial}{\partial r}\right)\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)\right]\mathrm{d}r/a,\\ a_3 &= r_2{\phi_0}^2(r_2)/a,\\ a_4 &= \int_{r_1}^{r_2}\frac{\phi_0}{(\omega - c_0)}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)\mathrm{d}r/a.\end{aligned}$$

Equation (25) is a new integro-differential equation including the dissipation effect and is first derived to describe the

evolution of Rossby solitary waves in a rotating fluid. According to the definition of algebraic Rossby solitary waves,[10] the Rossby solitary waves which are denoted as Eq. (25) are also called algebraic Rossby solitary waves. However, we note that the model is greatly different from the BO model, it has the $\partial^2 A/\partial T^2$ term instead of $\partial A/\partial T$ term. In addition, the integral term $\partial^4 J(A(\Theta,T))/\partial\Theta^4$ is used to express dispersion effect, and its form is different from the Hilbert transformation term in BO equation. Neglecting dissipation effect and replacing the term $\partial^4 J(A(\Theta,T))/\partial\Theta^4$ into $\partial^4 A/\partial\Theta^4$, equation (25) deduces to Boussinesq equation. For convenience, we call Eq. (25) the generalized Boussinesq equation.

3. Conservation laws related to the generalized Boussinesq equation

Conservation laws provide strong insights into how flows are constrained and play an increasingly important role in fluid dynamics. The purpose of this section is to present some conservation laws related to generalized Boussinesq equation and, furthermore, to express some conserved quantities associated with algebraic Rossby solitary waves. Because the generalized Boussinesq equation is new, the study about conserved quantities of algebraic Rossby solitary waves is original, even though lots of research about conservation laws of the soliton equation has been carried out.[13–16] Because the dissipation effect can cause the conserved quantities to be destroyed, the dissipation effect is ignored in this section, i.e., $\lambda = 0$.

Because of particularity of generalized Boussinesq equation, we need to rewrite Eq. (25) as the following system of evolutionary equations:

$$\begin{aligned} A_{\mathrm{T}} &= -B_{\Theta\Theta}, \\ B_{\mathrm{T}} &= \alpha a_1 A + a_2 A^2 + a_3 \frac{\partial^2}{\partial\Theta^2} J(A). \end{aligned} \tag{26}$$

According to the definition of conservation law, if there are functions E and F to satisfy the following equation:

$$D_{\mathrm{T}} E + D_\Theta F = 0, \tag{27}$$

then equation (27) is called the conservation law of system (26), we call E conservation density and F conservation flux.

Because of periodicity condition, we can assume that the values of A, A_Θ, $A_{\Theta\Theta}$, $A_{\Theta\Theta\Theta}$, B, B_Θ at $\Theta = 0$ equal to those at $\Theta = 2\pi$. First, it is not difficult to find the first conservation law, as follows:

$$D_{\mathrm{T}}\left(\int_0^{2\pi} A\,\mathrm{d}\Theta\right) + D_\Theta\left(\int_0^{2\pi} B_\Theta\,\mathrm{d}\Theta\right) = 0. \tag{28}$$

The analogue to the KdV equation, $E_1 = \int_0^{2\pi} A\,\mathrm{d}\Theta$ is used to express the mass of algebraic Rossby solitary waves. Equation (28) shows that the mass of algebraic Rossby solitary waves do not vary with time; i.e., it is a conserved quantity.

Next, by direct calculation, we can also obtain the following equation:

$$D_{\mathrm{T}} E_2 + D_\Theta F_2 = 0, \tag{29}$$

where

$$\begin{aligned} E_2 &= \frac{1}{2}\int_0^{2\pi} (AB_\Theta - A_\Theta B)\,\mathrm{d}\Theta, \\ F_2 &= \frac{1}{2}\int_0^{2\pi} \left[B_\Theta^2 - BB_{\Theta\Theta} - \frac{a_2}{3}A^3 - a_3 AJ(A_{\Theta\Theta})\right]\mathrm{d}\Theta. \end{aligned}$$

In a similar way, by analogue the KdV equation, E_2 is used to express the momentum of algebraic Rossby solitary waves. Equation (29) shows that the momentum of algebraic Rossby solitary waves is conserved.

In what follows, we can prove that system (26) also satisfies the following conservation law:

$$D_{\mathrm{T}} E_3 + D_\Theta F_3 = 0, \tag{30}$$

where

$$E_3 = \frac{1}{2}\int_0^{2\pi} \left[\frac{2}{3}a_2 A^3 + \alpha a_1 A^2 + a_3 AJ(A_{\Theta\Theta}) - B_\Theta^2\right]\mathrm{d}\Theta,$$

and

$$F_3 = \int_0^{2\pi} \left[B_\Theta B_{\mathrm{T}} - \frac{1}{2}a_3 AJ(A_{\mathrm{T}})\right]\mathrm{d}\Theta.$$

Here, E_3 is used to express the energy of algebraic Rossby solitary waves. Equation (30) shows the energy of algebraic Rossby solitary waves is conserved. Please note that in the derivation process of Eqs. (29) and (30), the following relations are used:

$$J(A)_{\Theta\Theta} = J(A_{\Theta\Theta}), \quad \int_0^{2\pi} uJ(v)\,\mathrm{d}\Theta = -\int_0^{2\pi} vJ(u)\,\mathrm{d}\Theta. \tag{31}$$

Finally, let us seek the fourth conservation law of system (26). Multiplying Θ/E_1 for both sides of the first equation of system (26) and integrating with respect to Θ from 0 to 2π yields

$$D_{\mathrm{T}}\left(\frac{\int_0^{2\pi}\Theta A\,\mathrm{d}\Theta}{E_1}\right) + D_\Theta\left(\frac{\int_0^{2\pi}(\Theta B_\Theta - B)\,\mathrm{d}\Theta}{E_1}\right) = 0, \tag{32}$$

where $E_4 = \int_0^{2\pi}\Theta A\,\mathrm{d}\Theta / \int_0^{2\pi} A\,\mathrm{d}\Theta$. According to Ono,[7] E_4 denotes the velocity of the center of gravity for the ensemble of solitary waves. Equation (32) shows that the velocity of the center of gravity of algebraic solitary waves is conserved.

In this section, four conservation relations of the generalized Boussinesq equation without dissipation effect are derived and we draw the conclusion that the mass, the momentum, the energy as well as the velocity of the center of gravity of algebraic Rossby solitary waves are conserved. In fact, besides the above four conservation laws that we have obtained, we need to ask whether there are other conservation laws of generalized Boussinesq equation? We also need to ask whether there is no limit to the KdV equation? These questions remain to be answered in the future.

4. Analytical solution of generalized Boussinesq equation and dissipation effect

In this section, we will derive the analytical solution of generalized Boussinesq equation and analyze the dissipation effect. First, we will look for the exact solution of generalized Boussinesq equation without the dissipation effect. Based on the exact solution, the approximate analytical solution of generalized Boussinesq equation with dissipation effect will be derived. By employing the approximate analytical solution, the dissipation effect will be considered.

Without the dissipation effect, equation (25) can be rewritten as

$$\frac{\partial^2 A}{\partial T^2}+\alpha a_1\frac{\partial^2 A}{\partial \Theta^2}+a_2\frac{\partial^2 A^2}{\partial \Theta^2}+a_3\frac{\partial^3}{\partial \Theta^3}\zeta(A(\Theta,T))=0, \quad (33)$$

where

$$\zeta(A(\Theta,T))=\frac{\partial J(A(\Theta,T))}{\partial \Theta}.$$

By assuming that A is the function of $\eta=\Theta-\mu T$, then equation (33) can be simplified into

$$\mu^2\frac{\partial^2 A}{\partial \eta^2}+\alpha a_1\frac{\partial^2 A}{\partial \eta^2}+a_2\frac{\partial^2 A^2}{\partial \eta^2}+a_3\frac{\partial^3}{\partial \eta^3}\zeta(A(\Theta,T))=0. \quad (34)$$

By integrating Eq. (34) twice with respect to the variable η and taking the integration constants as zero, we have

$$(\mu^2+\alpha a_1)A+a_2A^2+a_3\frac{\mathrm{d}}{\mathrm{d}\eta}\zeta(A)=0. \quad (35)$$

We can prove that equation (35) has the following solution:

$$A=\frac{2A_0\sigma}{A_0-\sigma}\Big/\left(\cos\eta+\frac{A_0+\sigma}{A_0-\sigma}\right), \quad (36)$$

where $\mu^2=-(4\sigma^2a_2^2+r_2^2a_3^2)/(8\sigma a_2)-\alpha a_1$ and $\sigma=r_2^2a_3^2/(4A_0a_2^2)$. Here, we note that μ shows the propagation speed of algebraic Rossby solitary waves and can be taken as positive or negative. When μ is positive, the solitary waves propagate towards the right; when it is negative, they propagate towards the left. This is a special feature of the new algebraic Rossby solitary waves and is different from the common algebraic Rossby solitary waves, which propagate towards one direction.

In the following, we consider the dissipation effect and look for the approximate analytical solution of Eq. (25). Taking a new space coordinate

$$\rho=\Theta-\int_0^T\sqrt{-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\,\mathrm{d}T, \quad (37)$$

and assuming $A_0=A_0(\lambda T)$, then we obtain

$$\frac{\partial^2 A}{\partial T^2}-\sqrt{-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\frac{\partial^2 A}{\partial T\partial\rho}$$
$$-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}\frac{\partial^2 A}{\partial\rho^2}+a_2\frac{\partial^2 A^2}{\partial\rho^2}+a_3\frac{\partial^4}{\partial\rho^4}J(A)$$
$$+\lambda a_4\frac{\partial A}{\partial\rho}=0. \quad (38)$$

By taking two time scales as follows:

$$\tau=T,\quad \vartheta=\lambda T, \quad (39)$$

and expanding the solution as follows:

$$A(\rho,T)=A_1(\rho,\tau,\vartheta)+\lambda A_2(\rho,\tau)+\cdots, \quad (40)$$

the following approximate equations can be obtained:

$$\begin{aligned}\lambda^0\ :\ &\frac{\partial^2 A_1}{\partial\tau^2}-\sqrt{-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\frac{\partial^2 A_1}{\partial\tau\partial\rho}\\&-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}\frac{\partial^2 A_1}{\partial\rho^2}+a_2\frac{\partial^2 {A_1}^2}{\partial\rho^2}+a_3\frac{\partial^4}{\partial\rho^4}J(A_1)=0,\\ \lambda^1\ :\ &\frac{\partial^2 A_2}{\partial\tau^2}-\sqrt{-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\frac{\partial^2 A_2}{\partial\tau\partial\rho}\\&-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}\frac{\partial^2 A_2}{\partial\rho^2}+2a_2\frac{\partial^2 A_1A_2}{\partial\rho^2}+a_3\frac{\partial^4}{\partial\rho^4}J(A_2)\\&=-a_4\frac{\partial A_1}{\partial\rho}-\frac{\partial^2 A_1}{\partial\tau\partial\vartheta}.\end{aligned} \quad (41)$$

Setting

$$\zeta=\rho+\sqrt{-\frac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\,\tau,$$

and introducing it into Eq. (41) gives

$$\lambda^0\ :\ \frac{\partial^2 A_1}{\partial\tau^2}+\alpha a_1\frac{\partial^2 A_1}{\partial\zeta^2}+a_2\frac{\partial^2 {A_1}^2}{\partial\zeta^2}+a_3\frac{\partial^4}{\partial\zeta^4}J(A_1)=0, \quad (42)$$

$$\begin{aligned}\lambda^1\ :\ &\frac{\partial^2 A_2}{\partial\tau^2}+\alpha a_1\frac{\partial^2 A_2}{\partial\zeta^2}+2a_2\frac{\partial^2 A_1A_2}{\partial\zeta^2}+a_3\frac{\partial^4}{\partial\zeta^4}J(A_2)\\&=-a_4\frac{\partial A_1}{\partial\zeta}-\frac{\partial^2 A_1}{\partial\tau\partial\vartheta}.\end{aligned} \quad (43)$$

The solution of Eq. (42) has been given in Eq. (36) in the following form:

$$A_1=\frac{\dfrac{2A_0\sigma}{A_0-\sigma}}{\cos(\zeta-\mu\tau)+\dfrac{A_0+\sigma}{A_0-\sigma}}. \quad (44)$$

By employing the definition of ζ as well as Eq. (37), it is easy to find the approximate analytical solution of Eq. (25), as follows:

$$A=\frac{\dfrac{2A_0\sigma}{A_0-\sigma}}{\cos\left(\Theta-\int_0^T\sqrt{-\dfrac{4\sigma^2a_2^2+r_2^2a_3^2}{8\sigma a_2}-\alpha a_1}\,\mathrm{d}T\right)+\dfrac{A_0+\sigma}{A_0-\sigma}}. \quad (45)$$

Next, we need to determine the expression of $A_0(\lambda T)$, equation (43) will be discussed.

Suppose

$$A_2 = B(\gamma), \quad \gamma = \zeta - \mu\tau, \tag{46}$$

and substitute it to Eq. (43), equation (43) can be rewritten as

$$\mu^2 \frac{\mathrm{d}^2 B}{\mathrm{d}\gamma^2} + \frac{\partial^2}{\partial\gamma^2}\left[\alpha a_1 B + 2a_2 A_1 B + a_3 J\left(\frac{\partial^2 B}{\partial\gamma^2}\right)\right] = L(A_1), \tag{47}$$

where

$$L(A_1) = -\frac{a_4}{\mu}\frac{\partial A_1}{\partial\tau} - \frac{\partial^2 A_1}{\partial\tau\partial\vartheta}.$$

Only when the following condition is satisfied:

$$\int_0^{2\pi} K(\gamma) L(A_1)\,\mathrm{d}\gamma = 0, \tag{48}$$

then equation (47) can be solved. Here, $K(\gamma)$ is the solution of the following equation:

$$\mu^2 \frac{\mathrm{d}^2 K}{\mathrm{d}\gamma^2} + \alpha a_1 \frac{\partial^2}{\partial\gamma^2} K + 2a_2 A_1 \frac{\partial^2}{\partial\gamma^2} K - a_3 J\left(\frac{\partial^4 K}{\partial\gamma^4}\right) = 0. \tag{49}$$

The solution of Eq. (49) is

$$K(\gamma) = \frac{2A_0\sigma}{A_0 - \sigma} \Big/ \left[\cos(\gamma) + \frac{A_0 + \sigma}{A_0 - \sigma}\right].$$

By employing Eq. (48) and the expression of $K(\gamma)$, then we have

$$A_0 = \overline{A}_0 \exp\left(-a_4\sqrt{-(4\bar{\sigma}^2 a_2^2 + r_2^2 a_3^2/8\bar{\sigma}a_2 - \alpha a_1}\lambda T\right),$$
$$\bar{\sigma} = r_2^2 a_3^2/4\overline{A}_0 a_2^2, \tag{50}$$

where $\overline{A}_0$ is the initial amplitude. Therefore, the approximate analytical solutions of generalized Boussinesq equation is

$$A = \frac{2A_0 r_2^2 a_3^2/(4A_0^2 a_2^2 - r_2^2 a_3^2)}{\cos\left(\Theta \pm \int_0^T \sqrt{-\dfrac{r_2^2 a_3^2 + 4a_2^2 A_0^2}{8A_0 a_2} - \alpha a_1}\,\mathrm{d}T\right) + (4A_0^2 a_2^2 + r_2^2 a_3^2)/(4A_0^2 a_2^2 - r_2^2 a_3^2)}. \tag{51}$$

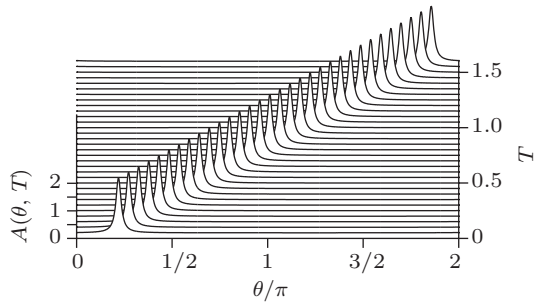


Fig. 1. Solitary waves evolution in the absence of dissipation ($\lambda = 0$).

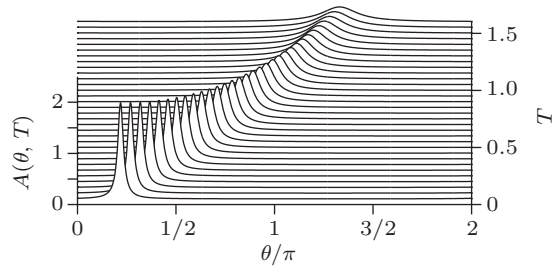


Fig. 2. Solitary waves evolution in the presence of dissipation ($\lambda = 1.2$).

Figures 1–3 give the evolutions of amplitude and vertex of algebraic solitary wave with and without the dissipation effect, respectively. From Fig. 1, we can find that the amplitude of algebraic solitary waves remains invariant during propagation process without the dissipation effect. While in Fig. 2, due to dissipation effect, the amplitude decreases with the time evolution. By comparing the two conditions ($\lambda = 0$ and $\lambda = 1.2$) in Fig. 3, we can conclude that dissipation effect also has an important impact on the propagation speed of algebraic solitary waves because it can cause the propagation speed to decrease.

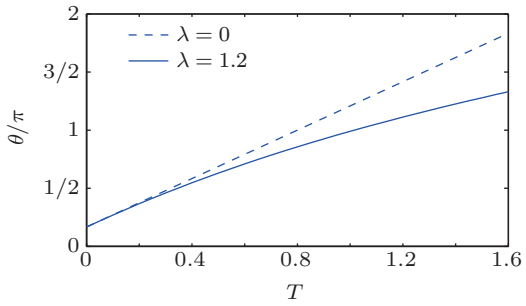


Fig. 3. (color online) The trace of the vertex of solitary waves.

In this section, we adopt an integrable ODE method[17] to obtain traveling type solution of Eq. (33). Furthermore, by employing the perturbation method,[18] the approximate analytical solution of Eq. (25) is also derived. With the help of the solution, the evolution of algebraic solitary waves under the influence of dissipation effect is analyzed. In fact, there are other methods to solve the above equation, such as multiple exp-function method,[19,20] bilinear method,[21] Darboux transformation method,[22,23] Riccati equation expansion method,[24] similarity reduction,[25] and so on. They are of interest in analytical geophysical fluid dynamics and deserve to be studied in the future.

5. Fission of algebraic solitary waves

In Section 4, we study the steady progressing wave solution of Eq. (25) due to the equilibrium between nonlinearity and dispersion. In the present section, we investigate unsteady behaviors of algebraic Rossby solitary waves. In fact, the algebraic Rossby solitary waves that we derive in this paper cannot be called the common "soliton", because in the nonlinear process, an algebraic Rossby solitary wave at initial time can split into a series of algebraic Rossby solitary waves with the passage of time, which is greatly different from the classic solitary waves. Next, we will describe the fission process of algebraic Rossby solitary waves and discuss the impact of the detuning parameter α on the fission process.

Here, we take the initial value of algebraic solitary wave as follows:

$$A(\theta,0)=\frac{\dfrac{2A_0\sigma_0}{A_0-\sigma_0}}{\cos(\theta+\theta_0)+\dfrac{A_0+\sigma_0}{A_0-\sigma_0}},\quad(\sigma_0>0). \tag{52}$$

Without loss of generality, suppose that the initial algebraic solitary wave asymptotically splits into two algebraic Rossby solitary waves with different amplitudes and speeds

$$A(\theta,T)=\sum_{i=1}^{2}\frac{\dfrac{2A_i\sigma_i}{A_i-\sigma_i}}{\cos(\theta-\mu_iT+\theta_0)+\dfrac{A_i+\sigma_i}{A_i-\sigma_i}}, \tag{53}$$

where $\sigma_i=r_2^2a_3^2/(4A_ia_2^2)$, $\mu_i^2=-\alpha a_1-(r_2^2a_3^2+4a_2^2\sigma_i^2)/(8\sigma_ia_2)$, $q_0=r_2^2a_3^2/a_2^2$, $i=1,2$.

In Section 3, we have derive four conservation laws of algebraic Rossby solitary waves without dissipation, we will seek the two amplitudes of algebraic Rossby solitary waves after fission. The conserved quantities $E_i(i=1,2)$ at $T=0$ can be calculated from the initial value (52) with the help of system (26) and Eq. (35) to give

$$E_1=2\pi\sqrt{A_0\sigma_0},\quad E_2=\pi\sqrt{A_0\sigma_0}(A_0+\sigma_0). \tag{54}$$

On the other hand, the conserved quantities can be considered as the sum of those associated with each emergent algebraic solitary wave after fission; it follows that:

$$E_1=2\pi\sqrt{q_0},\quad E_2=\frac{\pi\sqrt{q_0}}{2}\left(A_1+A_2+\frac{q_0}{4A_1}+\frac{q_0}{4A_2}\right). \tag{55}$$

By equating corresponding quantities in Eqs. (54) and (55), we obtain

$$\begin{aligned}&A_0\sigma_0=q_0,\\&\sqrt{A_0\sigma_0}(A_0+\sigma_0)=\frac{\sqrt{q_0}}{2}\left(A_1+A_2+\frac{q_0}{4A_1}+\frac{q_0}{4A_2}\right).\end{aligned} \tag{56}$$

Assuming that $A_1>A_2$, $q_0>0$, and $\sigma_0=0.05$, without loss of generality, we obtain

$$A_1=10(2+\sqrt{3})q_0,\ \ A_2=10(2-\sqrt{3})q_0,\ \ A_0=20q_0. \tag{57}$$

Once the amplitudes A_1 and A_2 are obtained, the corresponding propagation speeds μ_1 and μ_2 are easy to be obtained. The two algebraic solitary waves which generate due to fission can be described clearly. Here, we need emphasis that the conditions for the initial disturbance (52) with $A_0=20q_0$ and $\sigma_0=0.05$ is necessary to split into two solitary waves alone.

The theoretical analysis for the fission phenomenon of algebraic Rossby solitary waves has been carried out, and we will verify the fission phenomenon of algebraic solitary waves by numerical simulation and discuss the effect of detuning parameter α on the fission process of solitary waves in the latter. Figs. 4–6 describe the fission process of the algebraic solitary waves when $\alpha=0$, 10, -1.3, respectively.

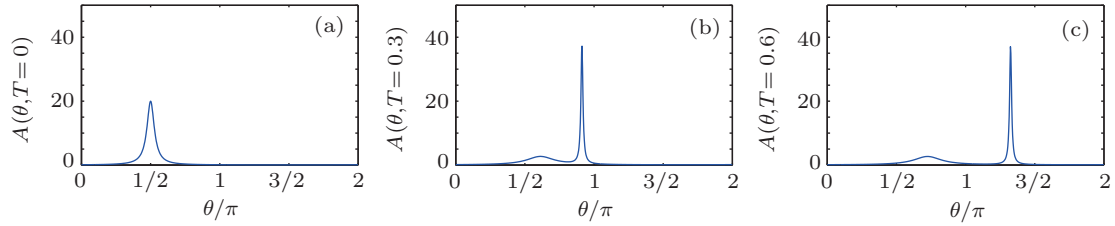


Fig. 4. (color online) $\alpha=0$, $a_1=-1$, $a_2=-1$, $a_3=1$, $r_2=1$, $\lambda=0$, $\theta_0=\pi/2$. (a) $T=0$, (b) $T=0.3$, (c) $T=0.6$.

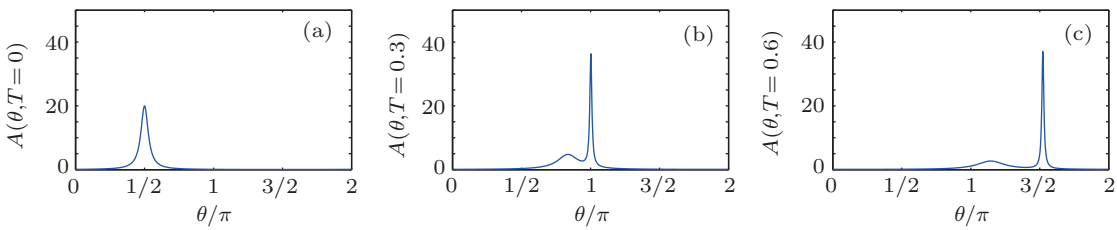


Fig. 5. (color online) $\alpha=10$, $a_1=-1$, $a_2=-1$, $a_3=1$, $r_2=1$, $\lambda=0$, $\theta_0=\pi/2$. (a) $T=0$, (b) $T=0.3$, (c) $T=0.6$.

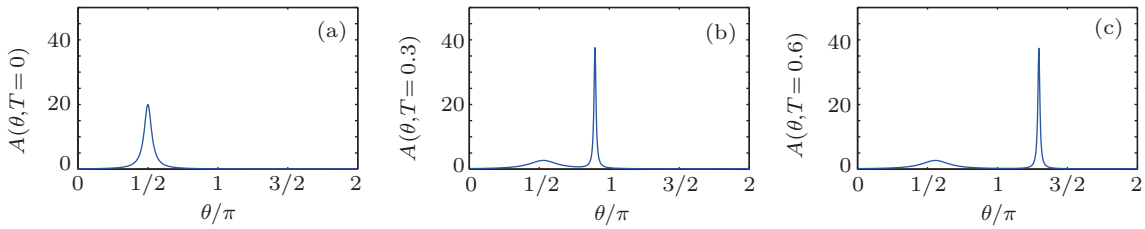


Fig. 6. (color online) $\alpha=-1.3$, $a_1=-1$, $a_2=-1$, $a_3=1$, $r_2=1$, $\lambda=0$, $\theta_0=\pi/2$. (a) $T=0$, (b) $T=0.3$, (c) $T=0.6$.

The three time points $T=0$, $T=0.3$, and $T=0.6$ are taken in Figs. 4–6 to show how the algebraic solitary waves evolve with time. By comparing the three states of algebraic solitary waves at different time points, we can draw the following conclusions: the algebraic solitary wave splits into two solitary waves with different amplitudes, they propagate toward the same direction, the solitary wave with large amplitude runs faster than one with a small amplitude, and the distance between the two solitary waves becomes larger and larger.

Meanwhile, we contrast Figs. 4–6 which owe different detuning parameter α. It is easy to find that the decrease of detuning parameter α can accelerate the occurrence of solitary wave fission phenomenon.

6. Conclusion

In this paper, with the help of perturbation expansions and stretching transformations of time and space method, a new model for algebraic solitary waves in a rotating fluid is obtained in the context of a generalized Boussinesq equation with a dissipation effect. This is greatly different from the classic Rossby solitary waves models such as KdV model, mKdV model, and is also different from the common algebraic Rossby solitary waves models such as BO model and ILW model. By rewriting the generalized Boussinesq equation as a suitable system of evolutionary equations, the conservation laws related to generalized Boussinesq equation without dissipation are obtained, which show that the mass, the momentum, the energy, the velocity of the center of gravity of algebraic Rossby solitary waves are conserved without dissipation effect. The analytical solution of generalized Boussinesq equation without dissipation is also obtained. In particular, the approximate analytical solution of generalized Boussinesq equation with dissipation is derived and used to analyze the impact of dissipation effect on the amplitude and propagation speed of algebraic Rossby solitary waves. Finally, we study the fission feature of algebraic solitary waves from theoretical analysis and numerical simulation. The main results are as follows.

(i) The generalized Boussinesq equation with dissipation effect can be used to describe the evolution of Rossby solitary waves. Because it includes the term $\partial^2 A/\partial T^2$ instead of the term $\partial A/\partial T$, which is new and worth being studied.

(ii) By discussing the approximate analytical solution of generalized Boussinesq equation, we can conclude that dissipation effect can cause the amplitude and the propagation speed of solitary waves decrease.

(iii) The algebraic solitary waves can split into several solitary waves with different amplitudes, they propagate toward the same direction, the solitary wave with large amplitude runs faster than the one with a small amplitude, and the distance between these solitary waves becomes larger and larger. Meanwhile, we also find that the decrease of detuning parameter α can accelerate the occurrence of solitary waves fission phenomenon.

References

[1] Russell J S 1837 *British Association for the Advancement of Science* 41
[2] Long R R, Andrushkiw R I and Huang X H 1964 *J. Atmos. Sci.* **21** 197
[3] Redekopp L G 1977 *J. Fluid. Meth.* **82** 725
[4] Song J and Yang L G 2009 *Chin. Phys. B* **18** 2873
[5] Wadati M 1973 *J. Phys. Soc. Jpn.* **34** 1289
[6] Yang H W, Yin B S, Yang D Z and Xu Z H 2011 *Chin. Phys. B* **20** 120203
[7] Ono H 1975 *J. Phys. Soc. Jpn.* **39** 1082
[8] Ono H 1981 *J. Phys. Soc. Jpn.* **50** 2757
[9] Gao S T 1988 *Adv. Atmos. Sci.* **5** 301
[10] Luo D H 1989 *Sci. Chin. Ser. B* **32** 1233
[11] Yang H W, Yin B S, Dong H H and Shi Y L 2012 *Commun. Theor. Phys.* **58** 425
[12] Chao J P and Huang R X 1980 *Sci. Chin.* **7** 696
[13] Wang X, Chen Y and Dong Z Z 2014 *Chin. Phys. B* **23** 010201
[14] Yu F J 2012 *Chin. Phys. B* **21** 110202
[15] Johnpillai A G and Khalique C M. 2011 *Commun. Nonlinear Sci. Numer. Simulat.* **16** 3081
[16] Gandarias M L, Bruzón M S and Rosa M 2013 *Commun. Nonlinear Sci. Numer. Simulat.* **18** 1600
[17] Ma W X and Fuchssteiner B 1996 *Inter. J. Non-Linear Mech.* **31** 329
[18] Du Z J, Lin W T and Mo J Q 2012 *Chin. Phys. B* **21** 090201
[19] Ma W X and Zhu Z N 2012 *Appl. Math. Comput.* **218** 11871
[20] Zhao Q and Liu S K 2006 *Chin. J. Geop.* **49** 965 (in Chinese)
[21] Ma W X and Chen M 2006 *J. Phys. A* **39** 10787
[22] Zheng H C, Ma W X and Gu X 2013 *Appl. Math. Comput.* **220** 226
[23] Zhaqilao and Qiao Z J 2011 *J. Math. Anal. Appl.* **380** 794
[24] Chen Y, Li B and Zhang H Q 2003 *Chin. Phys.* **12** 940
[25] Zhang Y F and Zhang H Q 2002 *Chin. Phys.* **11** 319