On the ascent of infinite dimensional Hamiltonian operators*

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In this paper, the ascent of 2×2 infinite dimensional Hamiltonian operators and a class of 4×4 infinite dimensional Hamiltonian operators are studied, and the conditions under which the ascent of 2×2 infinite dimensional Hamiltonian operator is 1 and the ascent of a class of 4×4 infinite dimensional Hamiltonian operators that arises in study of elasticity is 2 are obtained. Concrete examples are given to illustrate the effectiveness of criterions.

Keywords: root vector, completeness, infinite dimensional Hamiltonian operator, ascent

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1. Introduction

The traditional method of separation of variable (also called Fourier method) is an effective method for solving partial differential equations that can be transformed into the Strum–Liouville problem after separating variables, such as vibrating string equation and harmonic equation. The theoretical basis for Fourier method is completeness of eigenvectors system of self-adjoint operator, so the completeness of eigenvectors system is paid considerable attention.[1–5] We recall that the eigenvector system $\{e_\alpha\}_{\alpha\in A}$ of a linear operator T is called complete in space X, if $\overline{\text{span}\{e_\alpha\}_{\alpha\in A}}=X$, where A is a certain set of indices. In a separable Hilbert space, the completeness of $\{e_\alpha\}_{\alpha\in A}$ can be expressed by form of series, that is, for each $x\in X$ if there exists a constant series $\{C_\alpha\}_{\alpha\in A}$ such that

$$x=\sum_{\alpha\in A}C_\alpha e_\alpha \tag{1}$$

holds, which may be finite or infinite, convergence in norm (see Tehorem II.6 in Ref. [6]), then $\{e_\alpha\}_{\alpha\in A}$ is said to be complete in space X.

However, a great number of applied problems, like the second-order elliptic equation, cannot be reduced to the Strum–Liouville problem, as a consequence, the Fourier method fails to work. In order to overcome this problem, Zhong[7] using the simulation theory between structural mechanics and optimal control introduced Hamiltonian systems into elasticity and proposed the sympletic Fourier method based on Hamiltonian systems, and which generalizes the traditional method of separation of variables and possesses far-reaching significance. It is worth noting that corresponding operator of a Hamiltonian system is called infinite dimensional Hamiltonian operator and it has the following block representation:

$$H=\begin{bmatrix}A & B\\ C & -A^*\end{bmatrix}: D(H)\subset X\times X\to X\times X, \tag{2}$$

where A is densely defined closed linear operator, B, C are self-adjoint operators (see Refs. [8] and [9] and references therein). An infinite dimensional Hamiltonian operator is generally non-self-adjoint and perhaps has generalized eigenvectors, but its root vectors system (i.e., the eigenvector and the generalized eigenvector) is symplectic-orthogonal, and thus the completeness of symplectic-orthogonal root vectors system is the theoretical basis for symplectic Fourier method. The completeness of symplectic orthogonal eigenvectors system has been considered in Refs. [3], [10], and [11]. But the completeness of the root vectors system has not been completely understood, especially, the existence problem of generalized eigenvectors is one of the urgent things in spectral theory of infinite dimensional Hamiltonian operator. Therefore, the main aim of this note is to study the ascent of 2×2 infinite dimensional Hamiltonian operators and a class of 4×4 infinite dimensional Hamiltonian operators and obtain the sufficient conditions under which the ascent of the infinite dimensional Hamiltonian operator is 1 or 2.

As we know, the existence of generalized eigenvectors has to do with the ascent of linear operator. We recall that the ascent (see for example Taylor[12]) of T at λ (denoted by $P_\lambda(T)$) is the smallest positive integer k such that

$$N(T-\lambda)^k=N(T-\lambda)^{k+1}.$$

If there exists $v\neq 0$ such that $(T-\lambda)v=u$, where $u\in N(T-\lambda)$, then v is called generalized eigenvector of the first

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order, and which implies that the ascent of T at λ is lager than 1 (i.e., $P_\lambda(T) \geq 2$). Similarly, one may define the generalized eigenvector whose order is k. The system of eigenvectors and root vectors (i.e., eigenvectors and generalized eigenvectors) are overlapped when $P_\lambda(T) = 1$ for all $\lambda \in \sigma_p(T)$.

Throughout this paper, X always denotes the separable infinite dimensional complex Hilbert space and consider the orthogonal direct sum $X \times X$ which is a Hilbert space whose inner product is also denoted by $(\cdot,\cdot)$. The symbols $\mathcal{C}, \mathcal{R}, \mathrm{i}\mathcal{R}$ denote the set of complex numbers, the set of real numbers, the set of pure imaginary numbers, respectively. $[\,x\ y\,]^{\mathrm{T}}$ denotes the transform of row vector $[\,x\ y\,]$. $N(T) = \{x \in D(T) : Tx = 0\}$ and $\sigma_p(T)$ denote the kernel and the point spectra (i.e. the set of all eigenvalues) of operator T, respectively.

2. Preliminaries

We start with the following lemma.

Lemma 1 Let T be a densely defined closed operator in a Hilbert space X, and for any $u \in N(T-\lambda)$ there exists $v \in N(T^* - \overline{\lambda})^k$ such that $(u,v) \neq 0$, then the ascent of T at λ is smaller than $k+1$. If $k = 1$ then the ascent of T at λ is 1 (i.e., $P_\lambda(T) = 1$).

Proof Suppose $P_\lambda(T) = k+1$, then there exists $u_0 \in X$ such that

$$(T-\lambda)^{k+1}u_0 = 0, \quad (T-\lambda)^k u_0 \neq 0,$$

and which implies $(T-\lambda)^k u_0 \in N(T-\lambda)$. By given condition, there exists $v \in N(T^* - \overline{\lambda})^k$ such that $((T-\lambda)^k u_0, v) \neq 0$, that say, $(u_0, (T^* - \overline{\lambda})^k v) \neq 0$, which contradicts to $(T^* - \overline{\lambda})^k v = 0$ and thus $P_\lambda(T) \leq k$.

Remark 1 For any $\lambda \in \sigma_p(T)$, if $P_\lambda(T) = 1$ then we can see that the system of eigenvector and root vectors are overlapped, i.e., the generalized eigenvector does not appear, in this case we need only to consider the completeness of eigenvectors system. On the contrary, if $P_\lambda(T) \geq 2$, then there exist generalized eigenvectors and we may consider the completeness of root vectors system, and which implies that the symplectic Fourier method generalizes the traditional Fourier method, since the self-adjoint operator has no generalized eigenvectors. Therefore, it is of interest to find conditions under which the ascent of T at λ is 1, 2 or others.

3. The ascent of 2×2 infinite dimensional Hamiltonian operator

Theorem 1 Let $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : D(H) \subset X \times X \to X \times X$ be an infinite dimensional Hamiltonian operator with $\sigma_p(H)\backslash\{0\} \neq \emptyset$, $D(B) \subseteq D(A^*)$ and B is nonnegative and invertible (i.e. $0 \in \rho(B)$).

(i) If $B^{-1}A$ is a symmetric operator (i.e., $(B^{-1}Ax, x) \in R, \forall x \in D(A)$). Then for any $0 \neq \lambda \in \sigma_p(H)$ we have $P_\lambda(H) = 1$.

(ii) If $C + (A^* + \lambda)B^{-1}(A - \lambda)$ is surjective, $\lambda \in \sigma_p(H)$. Then $P_\lambda(H) \geq 2$.

Proof

(P-1) Let $u = [\,f\ g\,]^{\mathrm{T}} \in N(H-\lambda)$, then in view of

$$Af + Bg = \lambda f,$$

$$Cf - A^* g = \lambda g,$$

we obtain

$$\lambda^2(B^{-1}f, f) + \lambda(f, B^{-1}Af) - \lambda(B^{-1}Af, f) - (Cf, f) - (B^{-1}Af, Af) = 0.$$

Since $(B^{-1}Af, f) \in R$, so we have $\lambda(f, B^{-1}Af) - \lambda(B^{-1}Af, f) = 0$ and

$$\lambda^2(B^{-1}f, f) = (Cf, f) + (B^{-1}Af, Af) \in \mathcal{R},$$

and thus $\sigma_p(H) \subset \mathcal{R} \cup \mathrm{i}\mathcal{R}$.

If $\sigma_p(H) \subset \mathcal{R}$, for $\lambda \in \sigma_p(H)\backslash\{0\}$ taking $v = [\,-\lambda B^{-1}f - B^{-1}Af\ \ -f\,]^{\mathrm{T}}$, then in view of $v \in D(JHJ)$ and $H^*|_{D(JHJ)} = JHJ$, we have

$$(H^* - \overline{\lambda})v = (H^* - \lambda)v = J(H+\lambda)Jv = 0,$$

and

$$\begin{aligned}(u,v) &= \left(\begin{bmatrix} f \\ \lambda B^{-1}f - B^{-1}Af \end{bmatrix}, \begin{bmatrix} -\lambda B^{-1}f - B^{-1}Af \\ -f \end{bmatrix} \right) \\ &= -2\lambda(B^{-1}f, f) \neq 0,\end{aligned}$$

hence by Lemma 1, $P_\lambda(H) = 1$.

If $\sigma_p(H)\backslash\{0\} \subset \mathrm{i}\mathcal{R}$, for $\lambda \in \sigma_p(H)\backslash\{0\}$ taking $v = \left[\,-\lambda B^{-1}f + B^{-1}Af\ \ f\,\right]^{\mathrm{T}}$, then also $v \in D(H^*)$ and

$$(H^* - \overline{\lambda})v = (H^* + \lambda)v = J(H-\lambda)Jv = 0,$$

and

$$\begin{aligned}(u,v) &= \left(\begin{bmatrix} f \\ \lambda B^{-1}f - B^{-1}Af \end{bmatrix}, \begin{bmatrix} -\lambda B^{-1}f - B^{-1}Af \\ -f \end{bmatrix} \right) \\ &= -2\lambda(B^{-1}f, f) \neq 0,\end{aligned}$$

hence by Lemma 1, we also have $P_\lambda(H) = 1$.

P-2 Let $u = [\,f\ g\,]^{\mathrm{T}} \in N(H-\lambda)$, then $g = \lambda B^{-1}f - B^{-1}Af$. Since operator $C + (A^* + \lambda)B^{-1}(A-\lambda)$ is surjective, so there exists $v_1 \in D(A) \cap D(C)$ such that

$$(C + (A^* + \lambda)B^{-1}(A-\lambda))v_1 = A^*B^{-1}f - B^{-1}Af + 2\lambda B^{-1}f.$$

Therefore, taking $v = \left[\,v_1\ \ B^{-1}f - B^{-1}(A-\lambda)v_1\,\right]^{\mathrm{T}}$ then we have

$$(H-\lambda)v = u,$$

and which implies $P_\lambda(H) \geq 2$. The proof is complete.

Remark 2 When $B^{-1}A$ is symmetric and $0 \in \sigma_p(H)$, then the ascent of H at $\lambda = 0$ is larger than 1. In fact, let

$u=[\,f\ \ g\,]^{\mathrm{T}}\in N(H)$, and taking $v=[\,f\ \ -B^{-1}Af+B^{-1}f\,]^{\mathrm{T}}$, then in view of $D(B)\subseteq D(A^*)$, $g=-B^{-1}Af=A^*B^{-1}f$, and $Cf+A^*B^{-1}Af=0$ we have $v\in D(H), v\neq 0$, and

$$Hv=u,$$

which implies that the ascent of H at 0 is larger than 1 (i.e., exist generalized eigenvectors).

Remark 3 When $D(C)\subset D(A)$, C is positive and $C^{-1}A^*$ is a symmetric operator then we also obtain a similar conclusion.

Next we shall give an example to illustrate the effectiveness of Theorem 1.

Example 1 Consider the following partial differential equation

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2},\\ u(0,t)=u(1,t)=0,\\ u(x,0)=\varphi(y), u_t(x,0)=\phi(y),\end{cases}$$

and transform it into Hamiltonian system then we obtain the following Hamiltonian regular equation

$$\frac{\partial}{\partial t}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}0 & I\\ \dfrac{\partial^2}{\partial x^2} & 0\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix},$$

where $\partial v/\partial t=\partial^2 u/\partial x^2$. Let $X=L^2[0,1]$, then we obtain the following Hamiltonian operator

$$H=\begin{bmatrix}A & B\\ C & -A^*\end{bmatrix}=\begin{bmatrix}0 & I\\ \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} & 0\end{bmatrix},$$

where $D(C)=\{x\in X: x' \text{ absolutely continuous }, x', x''\in X, x(0)=x(1)=0\}$. It is easy to check that $0\in\rho(B), D(B)\subset D(A^*)$ and $B^{-1}A=0$ is symmetric, so by Theorem 1, we have $P_\lambda(H)=1$ for any $0\neq\lambda\in\sigma_p(H)$ (i.e., the operator H has no generalized eigenvectors).

On the other hand, through direct calculation we can obtain that the eigenvector system and the root vectors system of the infinite dimensional Hamiltonian operator H are overlapped and are equal to

$$u_n=\begin{bmatrix}\sin k\pi x\\ k\pi\sin k\pi x\end{bmatrix},\ \ k=\pm1,\pm2,\ldots,$$

corresponding eigenvalues are $\lambda_k=k\pi, k=\pm1,\pm2,\ldots$, which tallies with the above theoretical result.

4. The ascent of 4×4 infinite dimensional Hamiltonian operator

When $P_\lambda(H)\geq 2$, the operator exists generalized eigenvectors, hence one may consider the completeness of the root vectors system (i.e. the eigenvector and the generalized eigenvector). The main aim of this section is to obtain the sufficient conditions under which the ascent of a class of infinite dimensional Hamiltonian operator is 2 (i.e., except eigenvectors, the infinite dimensional Hamiltonian operator only has the first-order generalized eigenvectors). But, for general infinite dimensional Hamiltonian operators, it is not so easy to obtain the conditions of the ascent being 2, and because lots of infinite dimensional Hamiltonian operators which appear in the study of elasticity are of the form

$$H=\begin{bmatrix}0 & A_1 & B_1 & 0\\ A_2 & 0 & 0 & B_2\\ C_1 & 0 & 0 & -A_2^*\\ 0 & C_2 & -A_1^* & 0\end{bmatrix},\tag{3}$$

see the following examples.

Example 2 The basic equation of a plate strip is as follows:[13]

$$\frac{\partial^2 M_x}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=0.\tag{4}$$

Let

$$X=L^2[0,b]\times L^2[0,b]\times L^2[0,b]\times L^2[0,b],$$

then the corresponding infinite dimensional Hamiltonian operator of the system (4) is

$$H=\begin{bmatrix}0 & v\dfrac{\mathrm{d}}{\mathrm{d}y} & D(1-v^2) & 0\\ -\dfrac{\mathrm{d}}{\mathrm{d}y} & 0 & 0 & 2D(1-v)\\ 0 & 0 & 0 & -\dfrac{\mathrm{d}}{\mathrm{d}y}\\ 0 & -\dfrac{\mathrm{d}^2}{D\mathrm{d}y^2} & v\dfrac{\mathrm{d}}{\mathrm{d}y} & 0\end{bmatrix},$$

where $D>0$ and

$$D(H)=\left\{\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{bmatrix}\in X:\ \begin{array}{l}\psi_1(0)=\psi_1(b)=0, \dfrac{\mathrm{d}\psi_2(0)}{D\mathrm{d}x}-v\psi_3(0)=0,\\ \dfrac{\mathrm{d}\psi_2(b)}{D\mathrm{d}x}-v\psi_3(b)=0,\\ \psi_i(x) \text{ are absolutely continuous and } \psi_i'(x)\in L^2[0,b], i=1,3,4,\\ \psi_2'(x) \text{ are absolutely continuous and } \psi_2''(x)\in L^2[0,b]\end{array}\right\}.$$

Example 3[14] The free vibration of rectangular thin plates with two opposite edges simply supported is given by

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)W-\rho\omega W=0, \tag{5}$$

where $W(x,y)$ is the vibration model, ρ is the mass density, D is the bending stiffness, and ω is the natural frequency. The boundary conditions for simply supported edges $y=0$ and $y=1$ are

$$W|_{y=0,y=1}=0, \left.\frac{\partial^2 W}{\partial y^2}\right|_{y=0,y=1}=0.$$

Let $X=L^2[0,1]\times L^2[0,1]\times L^2[0,1]\times L^2[0,1]$, then the corresponding infinite dimensional Hamiltonian operator of system (5) is

$$H=\begin{bmatrix} 0 & I & 0 & 0 \\ -\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}-\sqrt{\rho\omega^2/D} & 0 & 0 & -I \\ 0 & 0 & 0 & \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}-\sqrt{\rho\omega^2/D} \\ 0 & 0 & -I & 0 \end{bmatrix},$$

where

$$D(H)=\left\{\begin{bmatrix} W \\ \theta \\ u \\ v \end{bmatrix}\in X:\ \begin{array}{l} W(0)=W(1)=0, \\ v(0)=v(1)=0, \\ W',v' \text{ are absolutely continuous and } W''(x),v''\in L^2[0,1] \end{array}\right\}.$$

Consequently, in this section the ascent of infinite dimensional Hamiltonian operators of form (3) is considered, and the sufficient condition of $P_\lambda(H)=2$ is given.

Theorem 2 Let

$$H=\begin{bmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & -A_2^* \\ 0 & C_2 & -A_1^* & 0 \end{bmatrix}$$

be infinite dimensional Hamiltonian operator with $B_i>M, C_i\geq 0, D(B_i)\subset D(A_i^*), i=1,2$, or $B_i<-M, C_i\leq 0, D(B_i)\subset D(A_i^*), i=1,2, M>0$. Let $\lambda\in\sigma_p(H)$ and $U=[\,f_1\ f_2\ g_1\ g_2\,]^{\mathrm{T}}$ be a corresponding eigenvector, if there exists $\alpha^2>1/4$ such that

$$(A_2^*B_2^{-1}-B_1^{-1}A_1)f_2=\frac{2\lambda}{2\alpha-1}B_1^{-1}f_1, \tag{6}$$

$$(A_1^*B_1^{-1}-B_2^{-1}A_2)f_1=\frac{2\lambda}{-2\alpha-1}B_2^{-1}f_2 \tag{7}$$

hold then the accent of H at λ is 2.

Proof Suppose $0\neq\lambda\in\sigma_p(H)$ and $U=[\,f_1\ f_2\ g_1\ g_2\,]$ be a corresponding eigenvector. In view of $(H-\lambda)U=0$ and the invertibility of $B_i, i=1,2$, we have

$$g_1=\lambda B_1^{-1}f_1-B_1^{-1}A_1f_2 \text{ and } g_2=\lambda B_2^{-1}f_2-B_2^{-1}A_2f_1,$$

and thus

$$(C_1-\lambda^2B_1^{-1}+A_2^*B_2^{-1}A_2)f_1=\lambda(A_2^*B_2^{-1}-B_1^{-1}A_1)f_2, \tag{8}$$

$$(C_2-\lambda^2B_2^{-1}+A_1^*B_1^{-1}A_1)f_2=\lambda(A_1^*B_1^{-1}-B_2^{-1}A_2)f_1. \tag{9}$$

Taking

$$V=\begin{bmatrix} \dfrac{\alpha}{\lambda}f_1 \\ -\dfrac{\alpha}{\lambda}f_2 \\ B_1^{-1}f_1+\alpha B_1^{-1}f_1+\dfrac{\alpha}{\lambda}B_1^{-1}A_1f_2 \\ B_2^{-1}f_2-\alpha B_2^{-1}f_2-\dfrac{\alpha}{\lambda}B_2^{-1}A_2f_1 \end{bmatrix},$$

then on account of Eqs. (6)–(9) we have

$$(H-\lambda)V=U,$$

that it to say, the accent of λ satisfies $P_\lambda(H)\geq 2$.

On the other hand, taking

$$\widetilde{U}=[\,-g_1\ g_2\ -f_1\ f_2\,]^{\mathrm{T}}$$

and

$$\widetilde{V}=\begin{bmatrix} -B_1^{-1}f_1-\alpha B_1^{-1}f_1-\dfrac{\alpha}{\lambda}B_1^{-1}A_1f_2 \\ B_2^{-1}f_2-\alpha B_2^{-1}f_2-\dfrac{\alpha}{\lambda}B_2^{-1}A_2f_1 \\ -\dfrac{\alpha}{\lambda}f_1 \\ -\dfrac{\alpha}{\lambda}f_2 \end{bmatrix},$$

then on account of Eqs. (6) and (7) and $H^*|_{D(JHJ)}=JHJ$, we have

$$(H^*-\overline{\lambda})\widetilde{U}=J(H+\lambda)J\widetilde{U}=0,$$

and

$$(H^*-\overline{\lambda})\widetilde{V}=J(H+\lambda)J\widetilde{V}=\widetilde{U},$$

that it to say, $\widetilde{V}\in N((H^*-\overline{\lambda})^2)$. By considering Eqs. (6)–(9), we also have

$$(U,\widetilde{V})=(-2\alpha-1)(B_1^{-1}f_1,f_1)+(1-2\alpha)(B_2^{-1}f_2,f_2).$$

Since $\alpha^2>1/4$, so $(-2\alpha-1)(1-2\alpha)>0$, as a consequence $(U,\widetilde{V})\neq 0$ and by Lemma 1 we have $P_\lambda(H)\leq 2$, and thus $P_\lambda(H)=2$.

When $0 \in \sigma_p(H)$, let $U = [\, f_1 \;\; f_2 \;\; g_1 \;\; g_2 \,]^{\mathrm{T}}$ be a corresponding eigenvector, taking

$$V = \begin{bmatrix} f_1 \\ f_2 \\ B_1^{-1} f_1 - B_1^{-1} A_1 f_2 \\ B_2^{-1} f_2 - B_2^{-1} A_2 f_1 \end{bmatrix},$$

then we have

$$HV = U,$$

that it to say, the accent of $\lambda = 0$ satisfies $P_0(H) \geq 2$.

On the other hand, taking

$$\widetilde{V} = \begin{bmatrix} -B_1^{-1} f_1 + B_1^{-1} A_1 f_2 \\ -B_2^{-1} f_2 + B_2^{-1} A_2 f_1 \\ f_1 \\ f_2 \end{bmatrix},$$

then we have

$$(H^*)^2 \widetilde{V} = 0,$$

and

$$(U, \widetilde{V}) = -(B_1^{-1} f_1, f_1) - (B_2^{-1} f_2, f_2) \neq 0.$$

By Lemma 1 we have $P_0(H) \leq 2$, and thus $P_0(H) = 2$.

Remark 4 In Example 2, it is easy to check that $B_i > M, C_i \geq 0, D(B_i) \subset D(A_i^*), i = 1, 2$, furthermore, the eigenvectors and eigenvalues of operator H are

$$U_k = \begin{bmatrix} \dfrac{D(1-v)}{\lambda_k} \sin \lambda_k y \\ \dfrac{D(1-v)}{\lambda_k} \cos \lambda_k y \\ \sin \lambda_k y \\ \cos \lambda_k y \end{bmatrix}, \quad \lambda_k = \frac{k\pi}{b}, k = \pm 1, \pm 2, \ldots.$$

Taking $\alpha = 3 + v/2(v-1)$, then $\alpha^2 > 1/4$ (since the Poisson ratio v satisfies $0 \leq v \leq 0.5$) and

$$(A_2^* B_2^{-1} - B_1^{-1} A_1) f_2 = \frac{v-1}{2(1+v)} \sin \lambda_k y = \frac{2\lambda_k}{2\alpha - 1} B_1^{-1} f_1,$$

$$(A_1^* B_1^{-1} - B_2^{-1} A_2) f_1 = \frac{1-v}{2(v+1)} \cos \lambda_k y = \frac{2\lambda_k}{-2\alpha - 1} B_2^{-1} f_2.$$

By Theorem 2, the accent of $\lambda \in \sigma_p(H)$ is 2.

On the other hand, through direct calculation we can see (see also Ref. [13]) that the operator H has the first-order generalized eigenvectors (i.e., $P_{\lambda_k}(H) = 2$)

$$V_k = \begin{bmatrix} -\dfrac{D(3+v)}{2\lambda_k^2} \sin \lambda_k y \\ \dfrac{D(3+v)}{2\lambda_k^2} \cos \lambda_k y \\ -\dfrac{1}{2\lambda_k} \sin \lambda_k y \\ \dfrac{1}{2\lambda_k} \cos \lambda_k y \end{bmatrix}, \quad k = \pm 1, \pm 2, \ldots,$$

and which tallies with the above theoretical result.

5. Conclusions

Fourier method is based on the completeness of eigenfunctions system with self-adjoint operator. Since the ascent of the self-adjoint operator is 1, so the eigenvector system and the root vector system are overlapped. However, the symplectic Fourier method based on infinite dimensional Hamiltonian operator whose ascent is maybe lager than 1, therefore, it is of interest to find conditions under which the ascent of infinite dimensional Hamiltonian operator is 1 or 2. Hence, in this paper we obtain the conditions under which the ascent of 2×2 infinite dimensional Hamiltonian operator is 1 and the ascent of a class of 4×4 infinite dimensional Hamiltonian operators is 2. Our results are helpful to provide theoretical basis for symplectic Fourier method.

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