The stability of Majorana fermion in correlated quantum wire*

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In this paper, we investigate the effect of the Coulomb interaction between electrons on the stability of Majorana fermion in a heterostructure of s-wave superconductor and quantum wire. In particular, by using the bosonization method and the renormalization group technique, we show that interplay between the so-called umklapp electron–electron scattering process and the superconducting proximity effect plays an extremely important role in determining the phase diagram of the system. We find that, at half-filling, the strong umklapp scattering process suppresses not only the superconducting pairing interaction and hence, destabilizes Majorana fermion in the quantum wire, but aslo results in a Mott insulating state. However, if the proximity effect is sufficiently strong, the topological superconducting phase can still survive and support Majorana fermion in the heterostructure. Furthermore, the existence of a critical Luttinger liquid phase is also found in a narrow region of parameters.

Keywords: Majorana fermion, proximity effects, p-wave pairing heterostructure

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1. Introduction

The concept of Majorana fermion was proposed by Ettore Majorana in 1937.[1] By its definition, the anti-particle of a Majorana fermion is itself. Therefore, the particle must be electrically neutral. Originally, one took a neutrino for such a candidate. However, the anti-neutrino was experimentally observed later and turned out to be a different particle. Up to now, the existence of the Majorana fermion in reality is still a fascinating but unresolved problem.

On the other hand, it is well known in condensed matter physics that the low-lying excitations in solids behave usually like quasi-particles in a short period of time, according to Landau.[2] That made people pondering the possibility that the Majorana fermion may appear in some condensed matter systems as a quasi-particle. For instance, Ivanov proposed that a Majorana fermion-type of quasi-particle can exist in the vortex core of a p-wave superconductor.[3] During the past decade, this speculation has been greatly stimulated by research on topological insulators.[4,5] By the theory of topological insulators, the surface state of a strong topological insulator can be thought of as a single Dirac cone in the simplest case.[6] Therefore, the spin of one moving electron will be coupled to its momentum, forming a helical band structure and reducing the degrees of freedom by one half. Theoretically, this helical band structure can be mapped into the spectrum of a spinless fermion system. If an s-wave superconductor is now put near such a topological insulator, it should lead to superconductivity in the surface of the latter by the so-called proximity effect.[7] However, unlike the previously known examples, the induced superconducting state is of p-wave ordering in this case, as pointed out by Fu and Kane.[8,9] In other words, for a replacement of the genuine exotic p-wave superconductor, we can engineer one by putting a three-dimensional topological insulator and an ordinary s-wave superconductor in close proximity. Consequently, Fu and Kane suggested that the Majorana fermion should appear as the bounded state of zero energy in a heterostructure of topological insulator and s-wave superconductor. This proposal recently generated tremendous interest in search of the Majorana fermion in some artificially manufactured systems such as carbon nanowires,[10,11] graphene,[12] the heterostructure of the magnetic quantum dot and s-wave superconductors,[13] the heterostructure of semiconductor films and s-wave superconductors,[14,15] as well as the heterostructure of semiconductor wires and s-wave superconductors.[16,17] An important common feature of these materials is that they possess a strong spin–orbit interaction which makes the spin and momentum of the electron coupled.

Among the above-mentioned systems, the heterostructure of quantum wire and the s-wave superconductor suggested by the authors of Refs. [16] and [17] seems the most promising one. This model is structurally very simple. It requires only four main ingredients: one quantum wire, the existence of strong electronic spin–orbit coupling in the quantum wire, the presence of an external magnetic field, and an s-wave superconducting film. As those authors argued, under appropriate experimental conditions, Majorana fermion-type of quasi-particles will appear at the ends of the quantum wire. Very recently, several groups carried out experimental verifications on this proposal.[18–23] The initial results showed plausible evidences for the existence of the Majorana fermion in this system.[18]

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However, from the theoretical point of view, there are still some ambiguities which should be clarified. For instance, the authors of Refs. [16] and [17] did not consider the effect of the Coulomb interaction on the stability of Majorana quasi-fermion in their pioneering works. More precisely, they assumed actually that the properties of charged carriers in the quantum wire can be described by Landau's Fermi liquid theory.[2] The defining feature of such a liquid is the appearance of discontinuity in the density of states at the Fermi surface. Obviously, this is an over-simplified picture. As is well known, the renormalization effects by the Coulomb interaction between electrons are extremely strong in a lower-dimensional many-body system. In particular, for a quantum wire, these effects can even change its physical properties qualitatively. As a matter of fact, the electron–electron interaction in a one-dimensional system easily destroys the discontinuity in the density of states at the Fermi surface of the charged carriers and replaces it with a power-law singularity.[24] Furthermore, any quasi-particle in this system is no longer a fermion. They are bosons, i.e., gapless collective sound modes. Such a quantum state is called Luttinger liquid in the literature.[25] Therefore, a natural question that arose is whether the Majorana fermion-type of quasi-particles can survive if the effects of the Coulomb interaction are taken into consideration.

This important issue has been previously addressed by several groups with a standard method, which is called "g-ology" treatment.[26–29] First, one classifies the electron scattering interactions with respect to the transferred momenta by them. Then, one uses the perturbative renormalization group method to determine their relevance to the low-lying excitation spectrum of the system. By applying "g-ology", these authors found that the Majorana fermion-type of quasi-particles at the ends of quantum wire is indeed stable in some region of parameters. Certainly, this result is very encouraging. However, for some technical reasons, they did not include a special type of electronic scattering process, i.e., the so-called umklapp process in their calculations. Since this scattering process competes strongly with the superconducting proximity effect, it can definitely destabilize the topological superconducting phase in the quantum wire and hence, diminish the chance for Majorana fermions to survive. Therefore, one would naturally wonder whether negligence of such a scattering process in calculation is justified. In the present work, we shall remedy this shortcoming.

In order to overcome technical difficulties and simplify our analysis, we shall combine "g-ology" treatment with the bosonization method, which maps all the fermionic degrees of freedom in a one-dimensional system to some bosonic degrees of freedom,[30] in our analysis. It is based on the following fact that some interactions between particles are easier to treat in the bosonic representation than in the original fermionic one. With this approach, we study first the effects of the Coulomb interaction between electrons at commensurate fillings in detail. We find that the umklapp scattering can be indeed neglected in this case. However, the forward scatterings may suppress the formation of topological superconducting state and hence, give rise to a gapless Luttinger liquid phase in the quantum wire. Then, we consider the half-filling case, in which the umklapp scattering process becomes extremely important. We show that it may lead to either a gapless Luttinger liquid phase or a Mott insulating phase. But, by applying the renormalization group technique, we find that, if the proximity effect between the quantum wire and the s-wave superconductor is sufficiently strong, a p-wave superconducting phase can still prevail in the quantum wire. Consequently, the existence of the Majorana fermion at the ends of the quantum wire is stabilized, even if the umklapp scattering process is included.

This paper is organized as follows. In Section 2, we introduce the Hamiltonian of the heterostructure of quantum wire and s-wave superconductor. We derive its energy dispersion and determine the region of parameters in which a non-trivial topological superconducting state exists. In Section 3, we map the Hamiltonian to an effective model. Then, we show that the Majorana fermion-type of quasi-particles occurs at the ends of the quantum wire. In Section 4, we apply the bosonization technique to the effective Hamiltonian and analyze its phase diagram by the renormalization group method at different fillings. Due to the competition between the umklapp scattering interaction and the superconducting proximity effect in the quantum wire, we obtain a very rich phase diagram. It consists of three parts, i.e., a Luttinger liquid phase, a Mott insulating phase, and a topological superconducting phase. The latter exists only in a specific region of parameters. Finally, in Section 5, we summarize our main results in this work.

2. The heterostructure of quantum wire and s-wave superconductor

The layout of a heterostructure is shown Fig. 1. It consists of a quantum wire in proximity with an s-wave superconductor. To induce p-wave superconductivity in quantum wire, an external magnetic field is also introduced. For definiteness, the s-wave superconductor is laid flat on the xy-plane and the uniform magnetic field is perpendicular to its surface. Moreover, the quantum wire is placed along the direction of the x-axis. With respect to this setup, the Hamiltonian of the system can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm P} + \mathcal{H}_{\rm I} \tag{1}$$

with

$$\mathcal{H}_0 = \int \mathrm{d}x \psi_\sigma^\dagger(x) \left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu \right) \right.$$

$$+\alpha p_x\sigma_y - B\sigma_z\Big]\psi_\sigma(x),$$
$$\mathcal{H}_{\rm P} = \int {\rm d}x(\Delta\psi_\uparrow(x)\psi_\downarrow(x)+{\rm h.c.}),$$
$$\mathcal{H}_{\rm I} = \int {\rm d}xU\rho_\uparrow(x)\rho_\downarrow(x). \qquad (2)$$

In $\mathcal{H}_0$, μ is the chemical potential of an electron in quantum wire. α and B denote the Rashba spin–orbit coupling constant and the external magnetic field, respectively.[1)] In Hamiltonian $\mathcal{H}_{\rm P}$, Δ stands for the pairing amplitude induced by the s-wave superconductor through the proximity effect and σ_x, σ_y, σ_z are the conventional Pauli matrices for the electronic spin degree of freedom. In the interacting Hamiltonian $\mathcal{H}_{\rm I}$, U represents the Coulomb interaction between electrons. For simplicity, we let it be a contact (δ-function) type of interaction. $\psi_\sigma(x)$ ($\psi^\dagger_\sigma(x)$) is the annihilation (creation) operator which annihilates (creates) an electron with spin $\sigma =\uparrow,\downarrow$ at x. Therefore, $\rho_\sigma(x) = \psi^\dagger_\sigma(x)\psi_\sigma(x)$ is understood for the density operator of the electron.

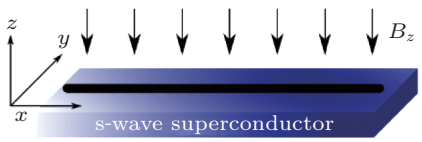


Fig. 1. (color online) Sketch of the heterostructure of quantum wire and s-wave superconductor.

When $B = 0$, the single particle Hamiltonian $\mathcal{H}_0$ can be easily diagonalized. Its energy spectrum is given by $E_\pm(k) = \xi_k \pm \alpha k$ with $\xi_k \equiv k^2/2m - \mu$. Here, k is the one-dimensional momentum of an electron in the quantum wire. As shown in Fig. 2, the Rashba spin–orbital interaction splits the energy spectrum into two branches, which cross each other at momentum $k = 0$. In other words, the single particle energy level is degenerate at this point. However, this degeneracy can be lifted by an external magnetic field. In fact, when $B \neq 0$, the single particle energy spectrum becomes $\mathcal{E}_\pm(k) = \xi_k \pm \sqrt{\alpha^2k^2 + B^2}$. It has an upper band and a lower band, as shown in Fig. 3. More importantly, since the spin of an electron is now coupled to its momentum by the spin–orbit interaction, the quasi-particles should be thought of as spinless fermions. In other words, an infinitesimal magnetic field can polarize the charged carriers in the quantum wire.

In the next section, we shall show that two (spinless) quasi-particles in the lower band can be paired with a p-wave ordering by Hamiltonian $\mathcal{H}_{\rm P}$. Consequently, the Majorana fermion will be induced at both ends of the quantum wire if its length is finite, as the authors of Refs. [16] and [17] showed in their works.

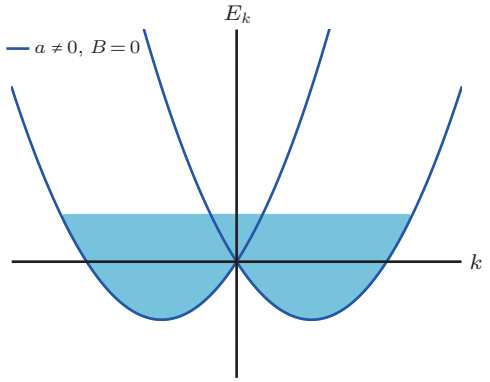


Fig. 2. (color online) Energy spectrum of the spin–orbit interacting quantum wire without magnetic field.

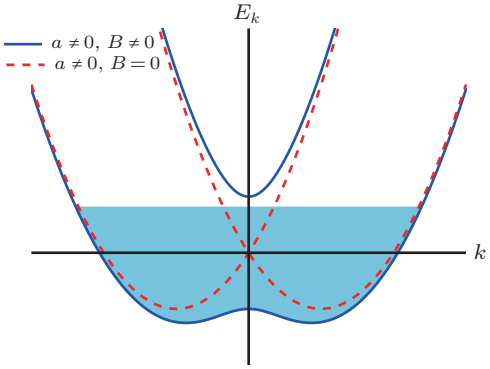


Fig. 3. (color online) Energy spectrum of the spin–orbit interacting quantum wire with magnetic field.

3. The existence of Majorana fermion in quantum wire

In order to make our analysis in the next section more readable, we shall give a brief review on the existence of the Majorana fermion in the above-mentioned heterostructure[16,17] here.

For simplicity, we set the Coulomb interaction U to be zero in Hamiltonian (1). Therefore, electrons in the quantum wire can be treated as ideal fermions. Following Refs. [16] and [17], we write further the Hamiltonian of the heterostructure of quantum wire and s-wave superconductor into the Bogoliubov-de Gennes (BdG) form

$$\mathcal{H} = \frac{1}{2}\int {\rm d}x\Psi(x)^\dagger\mathcal{H}_{\rm BdG}\Psi(x)$$
$$= \frac{1}{2}\int {\rm d}x\Psi(x)^\dagger\begin{pmatrix} h_0 & -{\rm i}\sigma_y\Delta^* \\ {\rm i}\sigma_y\Delta & -h_0\end{pmatrix}\Psi(x). \qquad (3)$$

Here, $\Psi(x) = (\psi_\uparrow, \psi_\downarrow, \psi^\dagger_\uparrow, \psi^\dagger_\downarrow)^{\rm T}$ represents the Nambu spinor. The diagonal sub-matrix h_0 in $\mathcal{H}_{\rm BdG}$ is actually the single particle Hamiltonian $\mathcal{H}_0$ of the quantum wire. In the first quantization form, it can be explicitly written as

$$h_0 = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu\right) - B\sigma_z + \alpha\left(-{\rm i}\frac{\partial}{\partial_x}\right)\sigma_y. \qquad (4)$$

[1)] In fact, parameter B is an abbreviation of the Zeeman interaction $\frac{1}{2}g\mu_{\rm B}H$ with g, $\mu_{\rm B}$, and H being the Landé factor, the Bohr magneton, and the external magnetic field, respectively.

Here, we set $\hbar = 1$. Hamiltonian (3) can be put into a more concise form

$$\mathcal{H}_{\rm BdG} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\tau_z - B\sigma_z\tau_z + \alpha\alpha\left(-{\rm i}\frac{\partial}{\partial_x}\right)\sigma_y\tau_z + \Delta\sigma_y\tau_y \quad (5)$$

by introducing the Pauli matrices τ_x, τ_y, and τ_z for the particle–hole degree of freedom in the system. By applying the standard Bogoliubov canonical transformation, we diagonalize $\mathcal{H}_{\rm BdG}$ and obtain its energy spectrum as

$$E_\pm(k) = \pm\sqrt{\xi_k^2 + \alpha^2k^2 + B^2 + \Delta^2 \pm 2\sqrt{\xi_k^2B^2 + \xi_k^2\alpha^2k^2 + B^2\Delta^2}}, \quad (6)$$

where k is the one-dimensional momentum of the electron in the quantum wire. This energy spectrum is plotted in Fig. 4. From the graph, we see that two energy gaps open up at points $k = 0$ and $k = k_{\rm F}$ by the superconducting parameter Δ, which is induced by the proximity effect between the quantum wire and the s-wave superconductor. We shall denote these gaps by $\Delta E(0)$ and $\Delta E(k_{\rm F})$, respectively. It is easy to see that, while $\Delta E(k_{\rm F})$ does not vanish for any external magnetic field B, $\Delta E(0) = \left|\sqrt{\mu^2+\Delta^2} - B\right|$ becomes zero at $B_c = \sqrt{\mu^2+\Delta^2}$, as shown in Fig. 5. The existence of $B_{\rm c}$ manifests a quantum phase transition from a topological superconducting phase with $B > \sqrt{\mu^2+\Delta^2}$ to a trivial superconducting phase with $B < \sqrt{\mu^2+\Delta^2}$.[16,17]

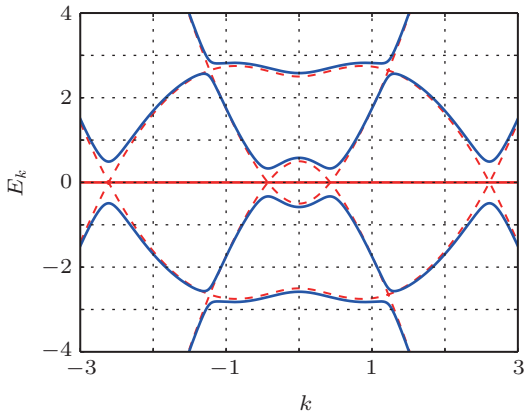


Fig. 4. (color online) Energy spectrum of the spin–orbit interacting quantum wire with both magnetic field and proximity interaction.

In the following, we shall concentrate on the case of large B. In other words, we assume that both conditions $B > \sqrt{\mu^2+\Delta^2}$ and $B \gg \Delta$ are simultaneously satisfied. It implies that the system is in the topological superconducting phase. For convenience, we shall go back to the original Hamiltonian defined in Eq. (1). As shown in Section 2, the energy spectrum of $\mathcal{H}_0$ has an upper branch and a lower branch when $B \neq 0$. Therefore, it can be re-written as

$$\mathcal{H}_0 = \sum_k \left(\mathcal{E}_+(k)\psi_+^\dagger(k)\psi_+(k) + \mathcal{E}_-(k)\psi_-^\dagger(k)\psi_-(k)\right), \quad (7)$$

in which $\mathcal{E}_\pm(k) = k^2/2m - \mu \pm \sqrt{B^2+\alpha^2k^2}$. Correspondingly, $\psi_+^\dagger(k)$ ($\psi_+(k)$) and $\psi_-^\dagger(k)$ ($\psi_-(k)$) are the (quasi)electron creation (annihilation) operators, which are related to the original electron operators $\psi_\uparrow^\dagger(k)$ ($\psi_\uparrow(k)$) and $\psi_\downarrow^\dagger(k)$ ($\psi_\downarrow(k)$) by a Bogoliubov-type of unitary transformation, at momentum k.

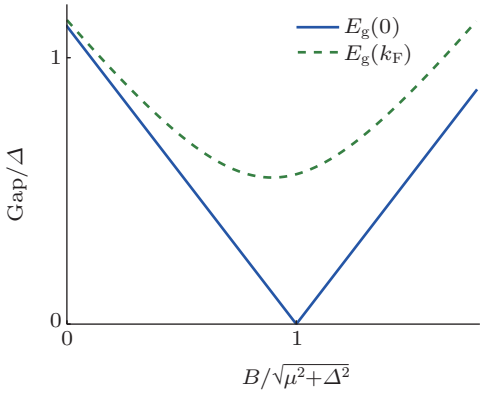


Fig. 5. (color online) Variation of gaps $\Delta E_g(0)$ (the solid line) and $\Delta E_g(k_{\rm F})$ (the dashed line) as magnetic field B changes.

Next, we substitute this unitary transformation of fermion operators into Hamiltonian $\mathcal{H}_{\rm P}$. In terms of the new set of operators $\psi_+^\dagger(k), \psi_+(k), \psi_-^\dagger(k)$, and $\psi_-(k)$, we have

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\rm P} = \sum_p (\mathcal{E}_+(k)\psi_+^\dagger(k)\psi_+(k) + \mathcal{E}_-(k)\psi_-^\dagger(k)\psi_-(k) \\ &\quad + \Delta_{\rm p}(k)\psi_+(k)\psi_+(-k) - \Delta_{\rm p}(k)\psi_-(k)\psi_-(-k) \\ &\quad + \Delta_{\rm s}(k)\psi_+(k)\psi_-(-k) + {\rm h.c.}) \end{aligned} \quad (8)$$

with

$$\Delta_{\rm s}(k) = \frac{{\rm i}B\Delta}{\sqrt{B^2+\alpha^2k^2}} \quad \text{and} \quad \Delta_{\rm p}(k) = \frac{{\rm i}\alpha k}{2\sqrt{B^2+\alpha^2k^2}}\Delta.$$

Interestingly, $\Delta_{\rm s}(k)$ and $\Delta_{\rm p}(k)$ can be thought of as the superconducting pairing amplitudes of quasi-particles. In particular, $\Delta_{\rm p}(k)$ yields a p-wave pairing in the quantum wire. In other words, a p-wave superconductor required for the existence of the Majorana fermion[3] may be realized in such a heterostructure of quantum wire and s-wave superconductor. Indeed, in the large-B limit, one can ignore the upper band in the spectrum of quasi-particles and keep only the contribution from the lower band. Consequently, the low-energy effective Hamiltonian of the quantum wire is given by

$$\begin{aligned}\mathcal{H}_{\rm eff} &\approx \sum_k \left[E_-(k)\psi_-^\dagger(k)\psi_-(k) + \Delta_{\rm p}(k)\psi_+(k)\psi_+(-k) - \Delta_{\rm p}(k)\psi_-(k)\psi_-(-k)\right] \\ &\approx \sum_k \left[\left(\frac{k^2}{2m} - \mu - B - \frac{\alpha^2k^2}{2B}\right)\psi_-^\dagger(k)\psi_-(k) + \Delta_{\rm p}\psi_+(k)\psi_+(-k) - \Delta_{\rm p}\psi_-(k)\psi_-(-k)\right] \\ &= \sum_k \left[\left(\frac{k^2}{2m^*} - \mu_{\rm eff}\right)\psi_-^\dagger(k)\psi_-(k)\right.\end{aligned}$$

$$+\Delta_{\rm p}\psi_{+}(k)\psi_{+}(-k)-\Delta_{\rm p}\psi_{-}(k)\psi_{-}(-k)\Big], \qquad (9)$$

where $\mu_{\rm eff}=\mu+B$ and $m^{*}=1(1/m)-(\alpha^{2}/B)$ are the effective chemical potential and mass of quasi-particles, respectively. Furthermore, the p-wave superconducting pairing amplitude is reduced to $\Delta_{\rm p}\approx {\rm i}\alpha k\Delta/2B$ in the large-B limit. Now, it is easy to see that $\mathcal{H}_{\rm eff}$ is the Hamiltonian of a p-wave pairing spinless fermion system.

When the length of the quantum wire is finite, the momentum of quasi-particle is no longer a conserved quantity. In this case, it is more convenient to re-write the effect Hamiltonian into a first quantization form. By dropping the lower band index of fermion operators, we obtain

$$\mathcal{H}_{\rm eff} = \int {\rm d}x\Big[\psi^{\dagger}(x)\left(-\frac{1}{2m^{*}}\frac{\partial^{2}}{\partial x^{2}}-\mu_{\rm eff}\right)\psi(x) + \frac{\alpha\Delta}{2B}\left(\psi(x)\frac{\partial\psi(x)}{\partial x}+{\rm h.c.}\right)\Big]. \qquad (10)$$

It is actually the continuum version of the Kitaev chain Hamiltonian studied in Ref. [31]. This model can be exactly solved. One found that it possesses bounded states of zero energy at both ends of the quantum wire. These states behave like Majorana fermions. In other words, a nonzero p-wave pairing amplitude can induce Majorana fermion-type of quasi-particles in a one-dimensional spinless fermion system of finite length. Therefore, we conclude that Majorana fermions can survive in the quantum wire if the renormalized p-wave pairing amplitude by the additional Coulomb interaction between electrons is still nonzero. Otherwise, they are destabilized. It is this criterion by which we shall determine the stability of Majorana fermion in the following analysis.

4. Majorana fermion in the correlated quantum wire

With the above preparations, we are now ready to investigate the stability of Majorana fermion in the quantum wire when the correlation effects between electrons are taken into consideration. Naturally, parameter U is nonzero in this case and hence, Hamiltonian $\mathcal{H}_{\rm I}$ should be added to $\mathcal{H}$. As mentioned in the Introduction, we shall apply both the bosonization method and the renormalization group analysis to deal with the effects caused by this interaction term.

Since the quantum wire is a one-dimensional system, it makes application of the bosonization method possible. The motivation behind this approach is based on the fact that any low-energy excitation mode in a one-dimensional many-body fermion system can be taken as a superposition of collective bosonic sound modes.[24] In other words, a low-energy fermion state is equivalent to a coherent boson state in one dimension. Therefore, one expects that the fermion operator $\psi(x)$ should be proportional to a coherent operator $\exp[{\rm i}\varphi(x)]$, with $\varphi(x)$ being a boson operator.[30] We would like to emphasize that this useful equivalence relation fails in higher dimensional space.

First, we apply the bosonization method to Hamiltonian (10) and re-write it with respect to the bosonic degrees of freedom. For this purpose, we need to linearize the Hamiltonian around the two Fermi points $k_{\rm F}$ and $-k_{\rm F}$. Following the standard procedure given in Ref. [30], we expand the spinless fermion operator as

$$\psi(x)={\rm e}^{{\rm i}k_{\rm F}x}\psi_{\rm R}(x)+{\rm e}^{-{\rm i}k_{\rm F}x}\psi_{\rm L}(x) \qquad (11)$$

and then substitute it into Hamiltonian (10). It yields

$$\mathcal{H}_{\rm eff} = -{\rm i}v_{\rm F}\psi_{\rm R}^{\dagger}(x)\frac{\partial\psi_{\rm R}(x)}{\partial x}+{\rm i}v_{\rm F}\psi_{\rm L}^{\dagger}(x)\frac{\partial\psi_{\rm L}(x)}{\partial x} +\Delta_{\rm eff}\left(\psi_{\rm L}(x)\psi_{\rm R}(x)-\psi_{\rm R}^{\dagger}(x)\psi_{\rm L}^{\dagger}(x)\right). \qquad (12)$$

In this Hamiltonian, $v_{\rm F}=k_{\rm F}/m^{*}$ is the effective Fermi velocity and $\Delta_{\rm eff}={\rm i}\alpha k_{\rm F}\Delta/B$. The fermions represented by $\psi_{\rm R}^{\dagger}(x)$ and $\psi_{\rm L}^{\dagger}(x)$ are called the right and the left movers, respectively. It should be pointed out that any term which contains either factor ${\rm e}^{{\rm i}k_{\rm F}x}$ or ${\rm e}^{-{\rm i}k_{\rm F}x}$ is dropped out from the Hamiltonian because they are fast oscillating. Therefore, as long as the low-energy excitations are concerned, these terms are immaterial.

Similarly, one can also linearize the interaction Hamiltonian $\mathcal{H}_{\rm I}$. Without further ado, we write it into the following form of "g-ology"

$$\mathcal{H}_{\rm I} = g_{2}\psi_{\rm R}^{\dagger}(x)\psi_{\rm L}^{\dagger}(x)\psi_{\rm L}(x)\psi_{\rm R}(x) +\frac{g_{4}}{2}\left[\left(\psi_{\rm R}^{\dagger}(x)\psi_{\rm R}(x)\right)^{2}+\left(\psi_{\rm L}^{\dagger}(x)\psi_{\rm L}(x)\right)^{2}\right] +g_{\rm u}\left({\rm e}^{-4{\rm i}k_{\rm F}x}\psi_{\rm R}^{\dagger}(x)\frac{\partial\psi_{\rm R}^{\dagger}(x)}{\partial x}\psi_{\rm L}(x)\frac{\partial\psi_{\rm L}(x)}{\partial x}+{\rm h.c.}\right). \qquad (13)$$

Here, parameters $g_{2}=U\alpha^{2}k_{\rm F}^{2}/4B$ and $g_{4}=U\alpha^{2}k_{\rm F}^{2}/4B$ represent the amplitudes of forward scattering processes while

$$g_{\rm u}=-\frac{U\alpha^{2}k_{\rm F}^{2}}{4B}$$

stands for the umklapp scattering process. The existence of factor ${\rm e}^{\pm 4{\rm i}k_{\rm F}x}$ in the $g_{\rm u}$-term makes it only important for the half-filling case, in which $k_{\rm F}=\pi/2$.

Next, as argued at the beginning of this section, we write the fermion operators in the form of bosonic coherent operators. For spinless fermion, the formula is well known.[30] We simply have

$$\psi_{\rm R}(x)=\frac{F_{\rm R}}{\sqrt{2\pi a}}\exp({\rm i}\phi_{\rm R}(x)),$$
$$\psi_{\rm L}(x)=\frac{F_{\rm L}}{\sqrt{2\pi a}}\exp(-{\rm i}\phi_{\rm L}(x)), \qquad (14)$$

where $\phi_{\rm R,L}(x)$ are the corresponding boson field operators. Operators $F_{\rm R,L}$ are called the Klein factors. They satisfy the

fermion anti-commutation relations. Constant a denotes the microscopic distance cut-off, which is introduced to make each step of our calculations finite. In the final step, we shall let it tend to zero. Substituting Eq. (14) into Eqs. (12) and (13), we obtain the bosonized effective Hamiltonian of the interacting quantum wire as

$$\mathcal{H} = \frac{1}{2\pi}\left[uK\left(\frac{\partial\theta(x)}{\partial x}\right)^2 + \frac{u}{K}\left(\frac{\partial\phi(x)}{\partial x}\right)^2\right] + \frac{\Delta_{\rm eff}}{\pi a}\cos 2\theta(x) + \frac{g_{\rm u}}{2\pi^2 a^2}\cos\left(4\phi(x) - 4k_{\rm F}x\right), \quad (15)$$

in which the new boson field operators are defined by

$$\phi(x) \equiv -\frac{1}{2}\left[\phi_{\rm R}(x) + \phi_{\rm L}(x)\right], \quad \theta(x) \equiv \frac{1}{2}\left[\phi_{\rm R}(x) - \phi_{\rm L}(x)\right]. \quad (16)$$

These operators satisfy the commutation relation

$$\left[\phi(x), \frac{1}{\pi}\frac{\partial\theta(x')}{\partial x'}\right] = {\rm i}\delta(x - x'). \quad (17)$$

Therefore, operator $\Pi \equiv (1/\pi)\partial_x\theta$ is defined to be the canonical momentum of $\phi(x)$. Moreover, parameter

$$u = v_{\rm F}\sqrt{\left(1 + \frac{g_4}{2\pi v_{\rm F}}\right)^2 - \left(\frac{g_2}{2\pi v_{\rm F}}\right)^2}$$

is the renormalized Fermi velocity and the dimensionless coefficient

$$K = \sqrt{\left(1 + \frac{g_4 - g_2}{2\pi v_{\rm F}}\right)\Big/\left(1 + \frac{g_4 + g_2}{2\pi v_{\rm F}}\right)} = \frac{1}{\sqrt{1 + \dfrac{U}{4\pi v_{\rm F}}\dfrac{\alpha^2 k_{\rm F}^2}{B^2}}}$$

describes the interaction strength between quasi-particles. In particular, when the interaction is attractive, $K > 1$. Similarly, for the repulsive interactions, $K < 1$.

To determine the phase diagram of Hamiltonian (15), we apply the renormalization group analysis.[32,33] The basic idea of this method is to integrate out the higher-energy degrees of freedom of a many-body system but keep the effects of virtual transition processes intact.[34] Following the standard procedure, we expand first the partition function of the system as

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int \mathcal{D}\phi\, {\rm e}^{-\hat{\mathcal{H}}_{\rm eff}}\, {\rm e}^{-\hat{\mathcal{H}}_{\rm I}} \equiv \langle {\rm e}^{-\hat{\mathcal{H}}_{\rm I}}\rangle_0 \approx 1 - \langle\hat{\mathcal{H}}_{\rm I}\rangle_0 + \frac{1}{2}\langle\hat{\mathcal{H}}_{\rm I}^2\rangle_0. \quad (18)$$

Here, operators $\hat{\mathcal{H}}_{\rm eff}$ and $\hat{\mathcal{H}}_{\rm I}$ are integrals of the Hamiltonian densities $\mathcal{H}_{\rm eff}(x)$ and $\mathcal{H}_{\rm I}(x)$ over the quantum wire, respectively. Z_0 is the partition function of $\hat{\mathcal{H}}_{\rm eff}$ alone. Then, we apply the so-called operator production expansion (OPE) formulas to each product in the third term $\frac{1}{2}\langle\hat{\mathcal{H}}_{\rm I}^2\rangle_0$ and keep only the singular parts of the averaged value. By integrating out variable x from a to $a{\rm e}^{{\rm d}l}$, we get the one-loop contribution of $\frac{1}{2}\langle\hat{\mathcal{H}}_{\rm I}^2\rangle_0$. Finally, by combining the result with the second term $\langle\hat{\mathcal{H}}_{\rm I}\rangle_0$ in the same equation, we establish the flow equations satisfied by parameters K, $\Delta_{\rm eff}$, and $g_{\rm u}$ with respect to scaling $a \to a{\rm e}^{{\rm d}l}$. The solutions of these equations yield the phase diagram of the system. A more detailed explanation of this procedure will be given in Appendix A of this paper. One can also find some of its concrete applications in our previous works[35,36] on the phase diagrams of some frustrated localized spin ladders.

After a quite complicated but otherwise straightforward calculation, we obtain the following flow equations for parameters K, $\Delta_{\rm eff}$, and $g_{\rm u}$

$$\frac{{\rm d}y_\Delta}{{\rm d}l} = \left(2 - \frac{1}{K}\right)y_\Delta, \quad \frac{{\rm d}y_{g_{\rm u}}}{{\rm d}l} = (2 - 4K)y_{g_{\rm u}}, \quad \frac{{\rm d}K}{{\rm d}l} = y_\Delta^2 - K^2 y_{g_{\rm u}}^2, \quad (19)$$

where $y_\Delta \equiv 2\Delta_{\rm eff}/\sqrt{\pi}$ and $y_{g_{\rm u}} \equiv 2g_{\rm u}/\sqrt{\pi^3}$. To find solutions of this equation, let us first consider the case in which the electron filling factor of the quantum wire is different from half-filling.

As argued above, the umklapp scattering interaction represented by $g_{\rm u}$ can be neglected in this case since $k_{\rm F} \neq \pi/2$. Therefore, by setting $g_{\rm u} = 0$, the above equations are reduced to

$$\frac{{\rm d}y_\Delta}{{\rm d}l} = \left(2 - \frac{1}{K}\right)y_\Delta, \quad \frac{{\rm d}K}{{\rm d}l} = y_\Delta^2. \quad (20)$$

These equations have a trivial solution $(K = \frac{1}{2}, y_\Delta = 0)$. It represents a fixed point with $\Delta_{\rm eff} = 0$, which indicates the absence of p-wave pairing in the system. On the other hand, since the renormalization group flow equations hold only in the sense of perturbation, the region of parameters around this fixed point deserve more careful scrutiny. For convenience, we define $K = 1/2 + y_K/2$. Substituting it into the reduced flow equations, we obtain

$$\frac{{\rm d}y_\Delta}{{\rm d}l} = 2y_K y_\Delta, \quad \frac{{\rm d}y_K}{{\rm d}l} = 2y_\Delta^2. \quad (21)$$

Multiplying the first equation by y_Δ and the second one by y_K, we can re-write these equations into

$$\frac{\rm d}{{\rm d}l}\left(y_\Delta^2 - y_K^2\right) = 0. \quad (22)$$

Therefore, we have solution $y_\Delta^2 - y_K^2 \equiv C$, where C is a constant. Obviously, for different values of C, the flow curve of parameters varies. Consequently, a whole set of flows are generated, as shown in Fig. 6. We notice that these flow curves are separated by two lines denoted by $y_\Delta = y_K$ and $y_\Delta = -y_K$, respectively. Moreover, since the flow curves are symmetric about the line of $y_\Delta = 0$, we need only to consider the cases of $y_\Delta > 0$ in the following analysis.

When the Coulomb repulsion between electrons is strong, i.e., $K < 1/2$, $y_K = 2K - 1 < 0$ holds. In this case, there exist

two possibilities. First, if the induced superconducting pairing interaction is weak such that parameter $y_\Delta = 2\Delta_{\rm eff}/\sqrt{\pi}$ satisfies relation $C = y_\Delta^2 - y_K^2 < 0$, then we see from Fig. 6 that the renormalized parameters will flow to a fixed point ($\widetilde{y}_\Delta = 0$, $\widetilde{K} < 1/2$), which describes a Luttinger liquid phase. However, if the induced superconducting pairing interaction is sufficiently strong and the integral constant C becomes positive, we find that both parameters y_Δ and K tend to infinity. It characterizes a topological superconducting phase. Since the low energy physics of the quantum wire is governed by these fixed points, we conclude that, although a Coulomb repulsion between electrons is hostile for the existence of topological superconducting phase and may lead to the occurrence of a gapless Luttinger liquid phase in the quantum wire, a sufficiently strong proximity effect will eventually win over the competition and produce a topological superconducting phase.

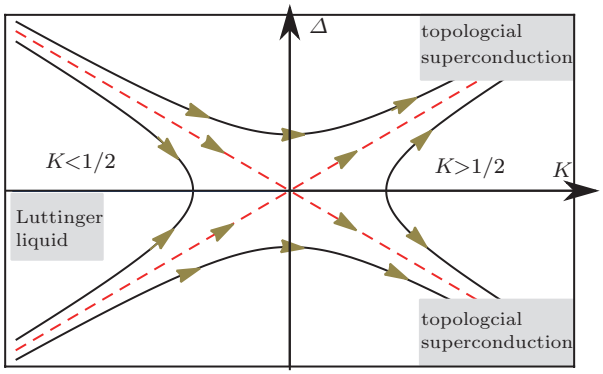


Fig. 6. (color online) The RG flow trajectories of parameters in the case of non-half-filling. The phase diagram of the quantum wire consists of a Luttinger liquid phase and a topological superconducting phase, as obtained from Eq. (20). The origin point stands for $(K = 1/2, \Delta = 0)$. The blue line represents the attracting fixed points of $\Delta = 0$, $K < 1/2$, which control the gapless Tomonaga–Luttinger liquid phase. The topological superconducting phase is characterized by the points, which flow eventually to $\Delta = \infty$ or $\Delta = -\infty$.

Next, we consider the case of $K > 1/2$. It indicates that the Coulomb repulsion between electrons is weak. Since parameter $y_K = 2K - 1$ is now positive, the induced superconducting pairing interaction is relevant by the first equation in Eq. (21). Therefore, the renormalized parameters will flow to fixed points $(y_\Delta = \pm\infty, K = \infty)$, which represent the existence of a topological superconducting phase.

Finally, we study the half-filled case, in which the umklapp scattering process between electrons plays an important role. Now, we need to solve the complete set of flow equations (19). These equations have two trivial solutions $(K = 1/2, y_\Delta = y_{g_u}/2)$ and $(K = 1/2, y_\Delta = -y_{g_u}/2)$, which represent two fixed points (or lines), respectively. As done above, we shall introduce new variables

$$K = 1/2 + y_K/2 \equiv 1/2 + y_3, \quad y_1 \equiv 2y_\Delta - y_{g_u}, \\ y_2 \equiv 2y_\Delta + y_{g_u} \tag{23}$$

around these fixed points (lines) and re-write the flow equations as

$$\frac{{\rm d}y_1}{{\rm d}l} = y_2 y_3, \quad \frac{{\rm d}y_2}{{\rm d}l} = y_3 y_1, \quad \frac{{\rm d}y_3}{{\rm d}l} = y_1 y_2. \tag{24}$$

These equations can be easily integrated. For instance, if we multiply the first equation by y_1 and the second one by y_2, then their difference gives

$$y_1 \frac{{\rm d}y_1}{{\rm d}l} - y_2 \frac{{\rm d}y_2}{{\rm d}l} = \frac{1}{2}\frac{\rm d}{{\rm d}l}\left(y_1^2 - y_2^2\right) = 0. \tag{25}$$

The integration of this equation is a constant. More precisely, we have

$$y_{12} \equiv y_1^2 - y_2^2 = (2y_\Delta - y_{g_u})^2 - (2y_\Delta + y_{g_u})^2 = A. \tag{26}$$

Similarly, we obtain also

$$y_{23} \equiv y_2^2 - y_3^2 = (2y_\Delta + y_{g_u})^2 - (2y_K)^2 = B, \\ y_{13} \equiv y_1^2 - y_3^2 = (2y_\Delta - y_{g_u})^2 - (2y_K)^2 = C, \tag{27}$$

where B and C are integral constants. By varying constants A, B, and C, we can draw the renormalization group flow curves of y_K, y_Δ, and y_{g_u} in the three-dimensional parameter space. However, it is a cumbersome task. Therefore, we shall directly derive some important conclusions from these solutions without explicitly showing their diagrams.

First, we find the existence of two critical surfaces determined by equations $B = 0$ and $C = 0$, which can be put into a unified form

$$(2y_K - |2y_\Delta| + |y_{g_u}|)(2y_K + |2y_\Delta| - |y_{g_u}|) = 0. \tag{28}$$

Consequently, the three-dimensional space of parameters is divided into two regions by a critical plane whose equation is

$$2y_K + |2y_\Delta| - |y_{g_u}| = 0. \tag{29}$$

Since the flow curves of parameters do not cross the critical plane, we should separately study the properties of the system in these regions.

(i) In the region of $2y_K + |2y_\Delta| - |y_{g_u}| < 0$, we need only to follow the flow curves with $y_K < 0$. This is due to the fact that the flow equations (24) are established around the fixed lines $(K = 1/2, y_\Delta = y_{g_u}/2)$ and $(K = 1/2, y_\Delta = -y_{g_u}/2)$, on which $|2y_\Delta| - |y_{g_u}| = 0$ holds. Therefore, each flow curve must pass a point with $y_K < 0$ in this region. Following these curves, we see that, as $y_K \to -\infty$, parameters $|y_{g_u}| \to \infty$ and $|y_\Delta| \to 0$. It implies that the superconducting pairing amplitude will be suppressed as the strength of the Coulomb repulsion between electrons gets stronger. Therefore, the quantum wire is a Mott insulator and Majorana fermion quasi-particles disappear at its ends in this region.

(ii) For the region in which $2y_K + |2y_\Delta| - |y_{g_u}| > 0$, we follow the flow curves with $y_K > 0$. As y_K goes to infinity, we see that $|y_{g_u}| \to 0$ and $|y_\Delta| \to \infty$. It tells us that, as the

Coulomb repulsion gets weaker, the superconducting pairing interaction wins over. In the meantime, the umklapp scattering process becomes less important and can eventually be neglected. Therefore, the quantum wire is in a topological superconducting state, which supports Majorana fermions.

Special attention should be paid to the case of $K = 1/2$. By its definition, the initial value of parameter y_K is now equal to zero, i.e., $y_K(0) = 0$. Substituting it into Eq. (19), we find that both parameters y_Δ and y_{g_u} become marginal. Therefore, if relation $|2y_\Delta(0)| - |y_{g_u}(0)| = 0$ is initially satisfied, the flow curve will be completely contained in the critical plane and the system is in a gapless Luttinger liquid state. However, if the initial values of y_Δ and y_{g_u} satisfy $|2y_\Delta(0)| > |y_{g_u}(0)|$, then $2y_K + |2y_\Delta| - |y_{g_u}| > 0$ holds. According to conclusion (i), it implies that, the superconducting pairing interaction is marginally relevant while the umklapp scattering process is marginally irrelevant under renormalization. Similarly, by conclusion (ii), if condition $|2y_\Delta(0)| < |y_{g_u}(0)|$ holds, the superconducting pairing interaction should become marginally irrelevant while the umklapp scattering process is marginally relevant under renormalization .

In summary, by renormalization group analysis, we find that, for $K > 1/2$, the umklapp scattering process can indeed be neglected. Therefore, the correlated quantum wire is still in a topological superconducting state and supports the Majorana fermion type of quasi-particles at its ends. However, when $K < 1/2$ holds, the superconducting pairing interaction becomes irrelevant and the umklapp scattering process will eventually prevail. Unlike the gapless Luttinger liquid state in the non-half filling case, it leads to a Mott insulating ground state, and hence, destabilizes the existence of the Majorana fermion in the quantum wire. Finally, the situation is rather complicated if $K = 1/2$. In this case, both the superconducting pairing interaction and the umklapp scattering process are marginal. Therefore, the quantum wire can be in either a gapless Luttinger liquid state, a topological superconducting state or a Mott insulating state, depending on the initial values of parameters y_Δ and y_{g_u}.

5. Summary

In this paper, we studied the effect of the Coulomb interaction between electrons on the stability of the Majorana fermion type of quasi-particles in a heterostructure of quantum wire and s-wave superconductor. In particular, we consider the competition between the so-called umklapp electron–electron scattering process, which was neglected in some previous investigations, and the superconducting proximity effect in such a system. By applying the bosonization method and the real-space renormalization technique, we established first the flow equations satisfied by various coupling constants. Then, by solving these equations, we find that a strong umklapp scattering process can indeed subdue the superconducting pairing interaction in the quantum wire and lead to a Mott insulating state, which is quite different from the gapless Tomonaga–Luttinger liquid phase in the non-half-filled case. Naturally, that makes the Majorana fermion at the ends of the quantum wire vanish. However, if the proximity effect between the quantum wire and the s-wave superconductor is sufficiently strong, the topological superconducting ordering can still survive. Therefore, Majorana fermion quasi-particles are stabilized in the heterostructure. Moreover, we find also the possible existence of a gapless Luttinger liquid state in a narrow region of parameters for the system.

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We would like to thank Dr. Chen Yu and Mr. An Wei for helpful discussions on several important issues.

Appendix A

In this appendix, we shall describe the renormalization group analysis used above in a more detailed manner. Unlike most of cases in which the renormalization group calculations are done in momentum space (one can find a concrete example in Ref. [37]), we shall carry out our analysis in real space[30,33] here. This approach is more convenient for our purpose because the quantum wire in the heterostructure has a finite length. That makes the momentum of electron no longer a good quantum number.

As we have already shown in Section 4, the partition function of a general quantum many-body system can be written as

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int \mathcal{D}\phi\, \mathrm{e}^{-\hat{\mathcal{H}}_{\mathrm{eff}}}\mathrm{e}^{-\hat{\mathcal{H}}_{\mathrm{I}}} \equiv \langle \mathrm{e}^{-\hat{\mathcal{H}}_{\mathrm{I}}}\rangle_0$$
$$\approx 1 - \langle\hat{\mathcal{H}}_{\mathrm{I}}\rangle_0 + \frac{1}{2}\langle\hat{\mathcal{H}}_{\mathrm{I}}^2\rangle_0. \tag{A1}$$

Now, we assume that operator $\hat{\mathcal{H}}_{\mathrm{I}}$ has the following form

$$\hat{\mathcal{H}}_{\mathrm{I}} = \int \mathrm{d}^d x \sum_n \frac{g_n}{a^{d-\Delta_n}}\hat{O}_n(x), \tag{A2}$$

where $\hat{O}_n(x)$ stands for a local operator and g_n is the corresponding coupling constant. We let d denote the space–time dimension of the system and a be a microscopic distance cutoff. Furthermore, Δ_n is the scaling dimension of operator $\hat{O}_n$. Its introduction makes the coupling constant g_n dimensionless. Then, the above expansion can be re-written into

$$\frac{Z}{Z_0} = 1 - \int \mathrm{d}^d x \sum_n \frac{g_n}{a^{d-\Delta_n}}\langle\hat{O}_n(x)\rangle_0 + \frac{1}{2}\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2$$
$$\times \sum_{n,m} \frac{g_n g_m}{a^{2d-\Delta_n-\Delta_m}}\langle\hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0. \tag{A3}$$

To evaluate $\langle\hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0$ in this expansion, we apply the so-called operator product expansion formula developed in quantum field theory. Take two operators $\hat{O}_i(x)$ and $\hat{O}_j(y)$. If the distance between x and y is sufficiently small comparing

with all the length scales of relevant physical processes, then their product can be expanded as

$$\hat{O}_i(x)\hat{O}_j(y) \sim \sum_k \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} \hat{O}_k\left(\frac{x+y}{2}\right). \tag{A4}$$

Constants $\{C_{ijk}\}$ are called the OPE coefficients. This equation should be understood in the following sense that, for another local operator $\Phi(z)$, correlation equation

$$\begin{aligned} &\langle \hat{O}_i(x)\hat{O}_j(x)\Phi(z)\rangle_0 \\ &= \sum_k \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} \left\langle \hat{O}_k\left(\frac{x+y}{2}\right)\Phi(z)\right\rangle_0 \end{aligned} \tag{A5}$$

holds if the distances from z to either x or y is much larger than $|x-y|$.

Now, we are ready to derive the flow equations satisfied by the coupling constants $\{g_n\}$. In the first step, we change the microscopic cut-off from a to $a\,\mathrm{e}^{\delta l}$ with $\delta l > 0$ being an infinitely small quantity. Under this scaling transformation, the factor $g_n/a^{d-\Delta_n}$ in the integrals of Z/Z_0 is renormalized to $g_n'/a^{d-\Delta_n}\,\mathrm{e}^{(d-\Delta_n)\delta l}$. Since the choice of microscopic length cut-off should not affect the physical results, we require that

$$\begin{aligned} \frac{Z}{Z_0} &= 1 - \int \mathrm{d}^d x \sum_n \frac{g_n}{a^{d-\Delta_n}} \langle \hat{O}_n(x)\rangle_0 + \frac{1}{2}\int_{|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \\ &\quad \times \sum_{n,m} \frac{g_n g_m}{a^{2d-\Delta_n-\Delta_m}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \\ &= 1 - \int \mathrm{d}^d x \sum_n \frac{g_n'}{a^{d-\Delta_n}\,\mathrm{e}^{(d-\Delta_n)\delta l}} \langle \hat{O}_n(x)\rangle_0 \\ &\quad + \frac{1}{2}\int_{|x_1-x_2|>a\,\mathrm{e}^{\delta l}} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \\ &\quad \times \sum_{n,m} \frac{g_n' g_m'}{a^{2d-\Delta_n-\Delta_m}\,\mathrm{e}^{(2d-\Delta_n-\Delta_m)\delta l}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \end{aligned} \tag{A6}$$

hold.

Let us consider first the second integral on the right-hand side of the above equation. We have

$$\begin{aligned} &\frac{1}{2}\int_{|x_1-x_2|>a\,\mathrm{e}^{\delta l}} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \sum_{n,m} \frac{g_n' g_m'}{a^{2d-\Delta_n-\Delta_m}\,\mathrm{e}^{(2d-\Delta_n-\Delta_m)\delta l}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \\ &= \frac{1}{2}\sum_{n,m}\int_{|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \\ &\quad - \frac{1}{2}\sum_{n,m}\int_{a\,\mathrm{e}^{\delta l}>|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \\ &= \frac{1}{2}\sum_{n,m}\int_{|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0 \\ &\quad - \frac{1}{2}\sum_{n,m}\int_{a\,\mathrm{e}^{\delta l}>|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \sum_k \frac{C_{mnk}}{|x_1-x_2|^{\Delta_n+\Delta_m-\Delta_k}} \left\langle \hat{O}_k\left(\frac{x_1+x_2}{2}\right)\right\rangle_0. \end{aligned} \tag{A7}$$

In the last step of our derivation, the formula of operator product expansion is used. Next, we introduce new variables $x = (x_1+x_2)/2$ and $x' = x_1 - x_2$ to simplify the second integral on the right-hand side of Eq. (A7). It is reduced to

$$\begin{aligned} &-\frac{1}{2}\sum_{n,m}\int_{a\,\mathrm{e}^{\delta l}>|x|>a} \mathrm{d}^d x\,\mathrm{d}^d x' \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \sum_k \frac{C_{mnk}}{|x|^{\Delta_n+\Delta_m-\Delta_k}} \langle \hat{O}_k(x)\rangle_0 \\ &\approx -\frac{1}{2}\sum_{n,m}\int_{a\,\mathrm{e}^{\delta l}>|x|>a} \mathrm{d}^d x\,\mathrm{d}^d x' \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \sum_k \frac{C_{mnk}}{a^{\Delta_n+\Delta_m-\Delta_k}} \langle \hat{O}_k(x)\rangle_0 \\ &\approx -\frac{\delta l\,\Omega_d}{2}\sum_{n,m}\sum_k \int \mathrm{d}^d x \frac{C_{mnk}\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{d-\Delta_k}} \langle \hat{O}_k(x)\rangle_0, \end{aligned} \tag{A8}$$

where Ω_d is the solid angle of a d-dimensional sphere.

Substituting both Eqs. (A7) and (A8) into the right-hand side of Eq. (A6), we finally obtain

$$\begin{aligned} \frac{Z}{Z_0} &= 1 - \sum_k \int \frac{\mathrm{d}^d x}{a^{d-\Delta_k}} \left[g_k'\,\mathrm{e}^{-(d-\Delta_k)\delta l} + \frac{\delta l\,\Omega_d}{2}\sum_{n,m} C_{mnk}\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)\right]\langle \hat{O}_k(x)\rangle_0 \\ &\quad + \frac{1}{2}\sum_{n,m}\int_{|x_1-x_2|>a} \mathrm{d}^d x_1\,\mathrm{d}^d x_2 \frac{\left(g_n'\,\mathrm{e}^{-(d-\Delta_n)\delta l}\right)\left(g_m'\,\mathrm{e}^{-(d-\Delta_m)\delta l}\right)}{a^{2d-\Delta_n-\Delta_m}} \langle \hat{O}_n(x_1)\hat{O}_m(x_2)\rangle_0. \end{aligned} \tag{A9}$$

If we let

$$g'_k \mathrm{e}^{-(d-\Delta_k)\delta l} + \frac{\delta l \Omega_d}{2} \sum_{n,m} C_{mnk} \left(g'_n \mathrm{e}^{-(d-\Delta_n)\delta l} \right) \times \left(g'_m \mathrm{e}^{-(d-\Delta_m)\delta l} \right) = g_k \tag{A10}$$

and replace factors $g'_n \mathrm{e}^{-(d-\Delta_n)\delta l}$ and $g'_m \mathrm{e}^{-(d-\Delta_m)\delta l}$ in the double integral with g_n and g_m, respectively, then the left-hand side of Eq. (A6) is recovered. By solving Eq. (A10) for g'_k, we are able to derive renormalized values of the coupling constants up to the one-loop level. It reads

$$g'_k = \left[g_k - \frac{\delta l \Omega_d}{2} \sum_{n,m} C_{mnk} \left(g'_n \mathrm{e}^{-(d-\Delta_n)\delta l} \right) \times \left(g'_m \mathrm{e}^{-(d-\Delta_m)\delta l} \right) \right] \mathrm{e}^{(d-\Delta_k)\delta l}. \tag{A11}$$

The errors caused by such a procedure are attributed to higher-order (two-loop) corrections and hence, can be neglected.

Equation (A11) can be re-written as a differential equation, which is called the flow equation for coupling constant g_k. For this purpose, we expand each factor on the right-hand side of the equation to the first order of δl. We obtain

$$\begin{aligned} g'_k &= \left(g_k - \frac{\delta l \Omega_d}{2} \sum_{n,m} C_{mnk} g_n g_m \right) (1 + (d-\Delta_k)\delta l) + O\left((\delta l)^2\right) \\ &= g_k + g_k (d-\Delta_k)\delta l - \frac{\delta l \Omega_d}{2} \sum_{n,m} C_{mnk} g_n g_m + O\left((\delta l)^2\right). \end{aligned} \tag{A12}$$

By shifting g_k from the right-hand side to the left-hand side of Eq. (A12) and dividing the difference by δl, we reach finally the flow equation

$$\begin{aligned} \frac{\mathrm{d}g_k}{\mathrm{d}l} &= (d-\Delta_k) g_k - \frac{\Omega_d}{2} \sum_{n,m} C_{nmk} g_n g_m \\ &= (d-\Delta_k) g_k - \sum_{n,m} \widetilde{C}_{nmk} g_n g_m, \end{aligned} \tag{A13}$$

in limit $\delta l \to 0$. The solutions of all of these equations determine the phase diagram of the corresponding quantum many-body system.

For the operators involved in the interacting Hamiltonian $\hat{\mathcal{H}}_{\mathrm{I}}$ of Eq. (15), their treatment can be found in Chapter 4 of Ref. [33]. Therefore, we are able to write down flow equation (19) for the correlated quantum wire without further ado.

References

[1] Majorana E 1937 *Nuovo Cimento* **14** 171
[2] Abrikosov A A, Gorkov L P and Dzyaloshinski I E 1963 *Methods of Quantum Field Theory in Statistical Physics* (New York: Dover Publications)
[3] Ivanov D A *Phys. Rev. Lett.* 2001 **86** 268
[4] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[5] Qi X L and S C Zhang 2011 *Rev. Mod. Phys.* **83** 1057
[6] Fu L, Kane C L and Mele E J 2007 *Phys. Rev. Lett.* **98** 106803
[7] Abrikosov A A 1988 *Fundamentals of the Theory of Metals* (Amsterdam: North-Holland) p. 483
[8] Fu L and Kane C L 2008 *Phys. Rev. Lett.* **100** 096407
[9] Fu L and Kane C L 2009 *Phys. Rev. Lett.* **102** 216403
[10] Sau J D and Tewari S 2013 *Phys. Rev. B* **88** 054503
[11] Klinovaja J, Gangadharaiah S and Loss D 2012 *Phys. Rev. Lett.* **108** 196804
[12] Klinovaja J, Ferreira G J and Loss D 2012 *Phys. Rev. B* **86** 235416
[13] Nadj-Perge S, Drozdov I K, Bernevig B A and Yazdani A 2013 *Phys. Rev. B* **88** 020407
[14] Sau J D, Lutchyn R M, Tewari S and Das Sarma S 2010 *Phys. Rev. Lett.* **104** 040502
[15] Sau J D, Tewari S, Lutchyn R M, Stanescu T D and Das Sarma S 2010 *Phys. Rev. B* **82** 214509
[16] Lutchyn R M, Sau J D and Das Sarma S 2010 *Phys. Rev. Lett.* **105** 077001
[17] Oreg Y, Refael G and von Oppen F 2010 *Phys. Rev. Lett.* **105** 177002
[18] Mourik V, Zuo K, Frolov S M, Plissard S R, Bakkers E P A M and Kouwenhoven L P 2012 *Science* **336** 1003
[19] Das A, Ronen Y, Most Y, Oreg Y, Heiblum M and Shtrikman H 2012 *Nat. Phys.* **8** 887
[20] Rokhinson L P, Liu X Y and Furdyna J K 2012 *Nat. Phys.* **8** 795
[21] Deng M T, Yu C L, Huang G Y, Larsson M, Caroff P and Xu H Q 2012 *Nano Lett.* **12** 6414
[22] Churchill H O H, Fatemi V, Grove-Rasmussen K, Deng M T, Caroff P, Xu H Q and Marcus C M 2013 *Phys. Rev. B* **87** 241401
[23] Finck A D K, Van Harlingen D J, Mohseni P K, Jung K and Li X 2013 *Phys. Rev. Lett.* **110** 126406
[24] Lieb E and Mattis D 1965 *Math. Phys.* **6** 304
[25] Giamarchi T 2004 *Quantum Physics in One Dimension* (Oxford: Oxford University Press)
[26] Lutchyn R M and Fisher M P A 2011 *Phys. Rev. B* **84** 214528
[27] Gangadharaiah S, Braunecker B, Simon P and Loss D 2011 *Phys. Rev. Lett.* **107** 036801
[28] Sela E, Altland A and Rosch A 2011 *Phys. Rev. B* **84** 085114
[29] Stoudenmire E M, Alicea J, Starykh O A and Fisher M P A 2011 *Phys. Rev. B* **84** 014503
[30] Gogolin A O, Nersesyan A A and Tsvelik A M 1998 *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
[31] Kitaev A Y 2001 *Phys. Usp.* **44** 131
[32] Amit D J 1984 *Field Theory, the Renormalization Group and Critical Phenomena* (Singpore: World Scientific)
[33] Fradkin E 2013 *Field Theories of Condensed Matter Physics* (Cambridge: Cambridge University Press)
[34] Anderson P W 1984 *Basic Notions of Condensed Matter Physics* (California: Benjamin Cummings)
[35] Shi D, Wen R, Liu G H and Tian G S 2010 *Chin. Phys. B* **19** 077503
[36] Wen R, Liu G H and Tian G S 2011 *Commun. Theor. Phys.* **55** 1102
[37] He P S, Oh S J, Chen Y and Tian G S 2010 *Commun. Theor. Phys.* **54** 897