Decay of N-qubit GHZ states in Pauli channels*

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An N-qubit Greenberger–Horne–Zeilinger (GHZ) state has many applications in various quantum information tasks and can be realized in different experimental schemes. A GHZ diagonal state evolves to another GHZ diagonal state in independent parallel Pauli channels. We give the explicit expression of the resultant GHZ diagonal state in terms of the initial state and channel parameters. If the initial state is a pure N qubit GHZ state or a three-qubit GHZ diagonal state admits a condition, the full separability criterion of the Pauli noisy state is equivalent to positive partial transpose (PPT) criterion. Thus the fully separable condition follows.

Keywords: full separability, GHZ state, Pauli noise

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1. Introduction

Quantum entanglement plays a fundamental role in quantum information science. Multipartite entanglement provides a critical resource for quantum secret sharing, quantum error-correcting codes and quantum computation.[1,2] Experimentally, multipartite entanglement has been observed in ion traps,[3] photon polarization,[4] superconducting phase and circuit qubit systems,[5] and nitrogen-vacancy centers in diamond.[6] All of these experiments center at preparing graph states, including the GHZ states as special cases. A graph state will become a graph diagonal state in a Pauli environment.[7,8] One of the key issues is to determine whether the GHZ states contaminated by Pauli environment is genuinely entangled or not entangled at all. The action of a general physical decoherece process on an initial density operator ρ can be described by $\Lambda(\rho)$, where Λ is a completely positive trace-preserving map. All such maps can be expressed in a Kraus representation, $\Lambda(\rho)=\sum_\mu p_\mu A_\mu \rho A_\mu^\dagger$. A very important class of decoherent processes is described by the Pauli maps, separable (non-entangling) maps whose Kraus operators $\sqrt{p_\mu}A_\mu$ are given by tensor products of Pauli operators X,Y,Z, and the identity. Examples of these are the collective or individual depolarizing, dephasing or bit-flip channels. Hence arbitrary Pauli maps are important noisy channels. Theoretical researches have concentrated on the full separability and biseparability that characterize the entanglement of the GHZ diagonal states.[9–13] A fully separable state is a probability mixture of product states, thus a fully separable state is not entangled at all in any sense. A biseparable state is a probability mixture of vectors which are products of single qubit pure states and two-qubit pure states. A state can be biseparable, yet inseparable under bipartitions. The criterion for biseparability[9] of N-qubit GHZ diagonal states and full separability[11,12,14] of three-qubit GHZ diagonal states have recently been proved. For a general N-qubit GHZ diagonal state, a sufficient condition has been given for the coincidence of thresholds for full separability and positive partial transpose (PPT).[11] So if this sufficient condition is fulfilled, we can use the PPT criterion to detect the full separability of an N-qubit GHZ diagonal state. Apart from the separability, the bipartite entanglement evolution has been investigated for GHZ (or more general, graph) diagonal states in Pauli channels.[7,8] The entanglement evolution is a topic of intensive interest for different systems.[15,16]

In this paper, we will study the change of an N-qubit GHZ state or a three-qubit GHZ diagonal state passing through independent parallel Pauli channels. The full separability and biseparability of the resultant GHZ diagonal states are analyzed.

2. Preliminary

The three-qubit GHZ diagonal states take the form

$$\rho=\sum_{m=0}^{7} p_m |GHZ_m\rangle\langle GHZ_m|, \tag{1}$$

where the p_m form a probability distribution. The GHZ state basis consists of eight vectors

$$|GHZ_m\rangle=(1/\sqrt{2})(|0x_2x_3\rangle\pm|1\bar{x}_2\bar{x}_3\rangle),$$

with $x_i,\bar{x}_i\in\{0,1\}$ and $x_i\neq\bar{x}_i$. In the binary notation, $m=0x_2x_3$ for the "+" states and $m=1\bar{x}_2\bar{x}_3$ for the "−" states. Let us consider an 8×8 density matrix ρ with entries ρ_{ij} in the basis $|000\rangle,|001\rangle,\ldots,|111\rangle$ which are ordered in the canonical way. For GHZ diagonal states, the only nonzero elements of ρ are ρ_{ii} and $\rho_{i,7-i}$ $(i=0,\ldots,7)$, and further we have $\rho_{ii}=\rho_{7-i,7-i}$. More explicitly,

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$$\rho = \begin{bmatrix} \rho_{00} & & & & & & & \rho_{07} \\ & \rho_{11} & & & & & \rho_{16} & \\ & & \rho_{22} & & & \rho_{25} & & \\ & & & \rho_{33} & \rho_{34} & & & \\ & & & \rho_{34} & \rho_{33} & & & \\ & & \rho_{25} & & & \rho_{22} & & \\ & \rho_{16} & & & & & \rho_{11} & \\ \rho_{07} & & & & & & & \rho_{00} \end{bmatrix},$$

where ρ_{00}, $\rho_{07} = (p_0 \pm p_7)/2$; ρ_{11}, $\rho_{16} = (p_1 \pm p_6)/2$; ρ_{22}, $\rho_{25} = (p_2 \pm p_5)/2$; ρ_{33}, $\rho_{34} = (p_3 \pm p_4)/2$.

Using Pauli matrices X, Y, Z, and 2×2 identity matrix I, the GHZ diagonal states can be written as

$$\begin{aligned} \rho = & \frac{1}{8}[III + \lambda_1 ZZI + \lambda_2 ZIZ + \lambda_3 IZZ + \lambda_4 XXX \\ & + \lambda_5 YYX + \lambda_6 YXY + \lambda_7 XYY], \end{aligned} \tag{2}$$

where tensor product symbols are omitted. The parameters λ_i are

$$(1, \lambda_2, \lambda_1, \lambda_3) = 4(\rho_{00}, \rho_{11}, \rho_{22}, \rho_{33})H', \tag{3}$$

$$(\lambda_4, -\lambda_6, -\lambda_5, -\lambda_7) = 4(\rho_{07}, \rho_{16}, \rho_{25}, \rho_{34})H', \tag{4}$$

where H' is a 4×4 Hadamard matrix. Denote

$$\kappa = \min_{(0 \le i \le 3)} \rho_{ii}, \tag{5}$$

then $\kappa = (1 + \lambda_-)/8$, with $\lambda_- = \min\{\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 - \lambda_2 - \lambda_3, -\lambda_1 + \lambda_2 - \lambda_3, -\lambda_1 - \lambda_2 + \lambda_3\}$, and λ_- is never positive. So

$$\kappa = (1 - |\lambda_-|)/8. \tag{6}$$

When $\prod_{i=4}^{7} \lambda_i < 0$, it has already been shown in Ref. [11] that the positive partial transpose (PPT) criterion is necessary and sufficient for the full separability of a GHZ diagonal state, namely,

$$1 - |\lambda_-| \ge \sum_{i=4}^{7} |\lambda_i|. \tag{7}$$

Under the condition $\prod_{i=4}^{7} \lambda_i < 0$, we have $\sum_{i=4}^{7} |\lambda_i| = 8 \max_{(0 \le i \le 3)} |\rho_{i,7-i}|$, thus the condition (7) can be written as

$$\min_{(0 \le i \le 3)} \rho_{ii} \ge \max_{(0 \le i \le 3)} |\rho_{i,7-i}|, \tag{8}$$

which has a more clear physical meaning that the minimal diagonal element of the density matrix is no less than the maximal absolute value of the off-diagonal element. When $\prod_{i=4}^{7} \lambda_i \ge 0$, the necessary and sufficient criterion for the full separability of a GHZ diagonal state is also known.[12]

The biseparable criterion for a GHZ diagonal state ρ is[9]

$$|\rho_{07}| \le \rho_{11} + \rho_{22} + \rho_{33}. \tag{9}$$

3. Graph state and Pauli channel

A graph $G = (V, \Gamma)$ is composed of an N vertex set V and an edge set specified by the adjacency matrix Γ, which is an $N \times N$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if vertices a, b are connected and $\Gamma_{ab} = 0$ otherwise. The neighborhood of a vertex a is denoted by $N_a = \{v \in V | \Gamma_{av} = 1\}$, that is, the set of all the vertices that are connected to a. The graph state related to graph G is defined as

$$|G\rangle = \frac{1}{\sqrt{2^N}} \sum_{\mu \in \{0,1\}^{\otimes N}} (-1)^{\mu \Gamma \mu^T/2} |\mu\rangle,$$

where $|\mu\rangle$ is the joint eigenstate of Pauli operators Z_a $(a \in V)$ with eigenvalues $(-1)^{\mu_a}$. It can be shown that the graph state is the joint $+1$ eigenstate of the N vertex stabilizers

$$K_a = X_a Z_{N_a},\ a \in V.$$

The N commutative vertex stabilizers are generators of an Abelian group K. The group is called the stabilizer group of the graph state $|G\rangle$. Every element $K_s \in K$ stabilizes $|G\rangle$ such that $K_s |G\rangle = |G\rangle$.

A qubit Pauli channel is described by Kraus operators $\sqrt{f} I, \sqrt{p_x} X, \sqrt{p_y} Y, \sqrt{p_z} Z$, where $f = 1 - p_x - p_y - p_z$. The output state of the channel is

$$\mathscr{E}(\rho) = f\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z \tag{10}$$

for input state ρ. We characterize the channel by a parameter vector $(x, y, z) = (p_x, p_y, p_z)/f$. Let us consider a local Clifford group which is defined to consist of all local unitary operators mapping the Pauli group to itself under conjugation. There are two basic local Clifford operators, one is the Hadamard gate H, the other is the phase gate P. Up to a global phase factor, any one-qubit Clifford operation is a product of operators chosen from the set $\{H, P\}$, where

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{bmatrix}. \tag{11}$$

For a single qubit unitary gate U, consider the channel $\mathscr{E}_U = U^\dagger \circ \mathscr{E} \circ U$, (here '$\circ$' denotes the successive applications of the gate or channel) we have

$$\mathscr{E}_U(\rho) = f\rho + p_x X'\rho X' + p_y Y'\rho Y' + p_z Z'\rho Z', \tag{12}$$

with $X' = U^\dagger X U$, $Y' = U^\dagger Y U$, $Z' = U^\dagger Z U$. Hence $\mathscr{E}_H(\rho) = f\rho + p_x Z\rho Z + p_y Y\rho Y + p_z X\rho X$, for $HZH = X$. Thus the channel $\mathscr{E}_H$ is characterized by the vector (z, y, x). Similarly, for phase gate P, we have $\mathscr{E}_P(\rho)$ with characteristic vector (y, x, z). The local complementary transform is an essential tool in graph state theory. For a graph G and its vertex a, the local complementary operator is

$$U_a(G) = \exp(-\mathrm{i}\pi X_a/4) \exp(\mathrm{i}\pi Z_{N_a}/4), \tag{13}$$

We have $\mathscr{E}_{U_a}(\rho_a)$ with characteristic vector (x,z,y) for qubit a, and $\mathscr{E}_{U_a}(\rho_b)$ $(b \in N_a)$ with characteristic vector (y,x,z) for the neighborhood of a.

For two local Clifford equivalent graph states as inputs to parallel Pauli channels, we obtain one output state from another by a proper permutation of the elements of the characteristic vectors. Thus one only needs to study one of the local Clifford equivalent graph states.

For N parallel Pauli channel array $\mathscr{N} = \mathscr{E}_1 \otimes \mathscr{E}_2 \otimes \cdots \mathscr{E}_N = \bigotimes_{i=1}^N \mathscr{E}_i$ with an input of N-qubit graph state $|G\rangle$, we have the output state

$$\rho = \mathscr{N}(|G\rangle) = \sum_\alpha \eta_\alpha E_\alpha (|G\rangle\langle G|) E_\alpha^\dagger, \tag{14}$$

with $\eta_\alpha = \prod_{i=1}^N f^{l_{i0}} p_x^{l_{i1}} p_y^{l_{i2}} p_z^{l_{i3}}$ for Pauli error operator $E_\alpha = \bigotimes_{i=1}^N X^{l_{i1}} Y^{l_{i2}} Z^{l_{i3}}$, where $l_{ij} = 0,1$, and $\sum_{j=0}^3 l_{ij} = 1$. The subscript α denotes the index of all possible tensor products of Pauli operators X, Y, Z, and identity I. Consider the output state in graph state bases. A graph state basis is

$$\left|G_\mu\right\rangle = Z^\mu |G\rangle, \tag{15}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ is a binary string, $Z^\mu = \prod_{a\in V} Z_a^{\mu_a}$. In graph state bases, we have

$$\begin{aligned} \langle G_\mu | \rho | G_\nu \rangle &= \sum_\alpha \eta_\alpha \langle G_\mu | E_\alpha | G\rangle \langle G | E_\alpha^\dagger | G_\nu\rangle \\ &= \sum_\alpha \eta_\alpha \langle G | Z^\mu E_\alpha | G\rangle \langle G | E_\alpha^\dagger Z^\nu | G\rangle. \end{aligned} \tag{16}$$

According to the orthogonality of graph state bases, $\langle G | Z^\mu E_\alpha | G\rangle = 0$ except $Z^\mu E_\alpha \in K$ up to phase factors $\pm 1, \pm \mathrm{i}$. Hence

$$\langle G_\mu | \rho | G_\nu \rangle = \delta_{\mu\nu} \sum_{\alpha | E_\alpha \in Z^\mu K} \eta_\alpha. \tag{17}$$

4. Three-qubit GHZ state and Pauli channel

The three-qubit GHZ state $|GHZ\rangle = |GHZ_0\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ is locally equivalent to the graph state $|G\rangle = H_1 H_2 |GHZ\rangle$ where H_1 and H_2 are local Haramard transforms on the first and second qubits, respectively. The stabilizer generators are $K_1 = X_1 Z_3$, $K_2 = X_2 Z_3$, $K_3 = X_3 Z_1 Z_2$ for the three-qubit graph state. We may list the error operators in Table 1.

Notice that the number of Pauli operators is 4^3 for a three-qubit state. The operators in Table 1 exhaust all the 64 Pauli error operators. We classify the error operators in Table 1 according to the stabilizer group K. An error operator E_α may belong to the stabilizer group K or its cosets in the Pauli group. The coset heads are specified by Z^μ. We then get the density matrix of the output state according to Eq. (17). Consider the parallel Pauli channels with different parameters, namely $\mathscr{N} = \bigotimes_i \mathscr{E}_i$, with input graph state, the output state is denoted as $\mathscr{N}(|G\rangle) = \sum_{k\in\{0,1\}^3} p'_k |G_k\rangle\langle G_k|$, we have

$$\begin{aligned} p'_{000} &= \sum_{\alpha | E_\alpha \in K} \eta_\alpha = f_1 f_2 f_3 (1 + x_1 z_3 + x_2 z_3 + x_1 x_2) \\ &\quad + f_1 f_2 f_3 (z_1 z_2 x_3 + y_1 z_2 y_3 + z_1 y_2 y_3 + y_1 y_2 x_3), \end{aligned} \tag{18}$$

where $(x_i, y_i, z_i) = (p_{xi}, p_{yi}, p_{zi})/f_i$. $f_i = 1 - p_{xi} - p_{yi} - p_{zi}$ are the channel parameters for the i-th qubit. In a more systematical way, $p'_{000}/F = (1+x_1)(1+x_2)(1+z_3) + (1-x_1)(1-x_2)(1-z_3) + (z_1+y_1)(z_2+y_2)(x_3+y_3) + (z_1-y_1)(z_2-y_2)(x_3-y_3)$, where $F = f_1 f_2 f_3/2$.

Consider the channel $\mathscr{N} = \bigotimes_i \mathscr{E}_i$ with input state $|GHZ\rangle$, the output state is

$$\mathscr{N}(|GHZ\rangle) = H_1 H_2 \mathscr{N}_{H_1 H_2}(|G\rangle) H_1 H_2. \tag{19}$$

Denote $\mathscr{N}_{H_1 H_2}(|G\rangle) = \sum_{k\{0,1\}^3} p_k^H |G_k\rangle\langle G_k|$, then

$$\begin{aligned} \frac{p_{000}^H}{F} &= \prod_{i=1}^3 (1+z_i) + \prod_{i=1}^3 (1-z_i) \\ &\quad + \prod_{i=1}^3 (x_i + y_i) + \prod_{i=1}^3 (x_i + y_i). \end{aligned} \tag{20}$$

Where we have used the fact that Hadamard transformations on the first and second qubits interchange their channel parameters of bit flipping and phase flipping, namely, $x_i \Leftrightarrow z_i$ $(i = 1,2)$. Denote $a_i = f_i + p_{zi}$, $b_i = p_{xi} + p_{yi}$, $c_i = f_i - p_{zi}$, $d_i = p_{xi} - p_{yi}$, $(i = 1,2,3)$, then

$$p_{000}^H = \frac{1}{2}\left(\prod_{i=1}^3 a_i + \prod_{i=1}^3 b_i + \prod_{i=1}^3 c_i + \prod_{i=1}^3 d_i\right). \tag{21}$$

Table 1. Pauli operators classified as stabilizer group and its cosets. In the table, 'St' stands for 'Stabilizers', and 'Co' stands for 'Cosets' of the stabilizer group.

Co \ St	I	K_1	K_2	K_1K_2	K_3	K_1K_3	K_2K_3	$K_1K_2K_3$
I	I	X_1Z_3	X_2Z_3	X_1X_2	$Z_1Z_2X_3$	$Y_1Z_2Y_3$	$Z_1Y_2Y_3$	$Y_1Y_2X_3$
Z_1	Z_1	Y_1Z_3	$Z_1X_2Z_3$	Y_1X_2	Z_2X_3	$X_1Z_2Y_3$	Y_2Y_3	$X_1Y_2X_3$
Z_2	Z_2	$X_1Z_2Z_3$	Y_2Z_3	X_1Y_2	Z_1X_3	Y_1Y_3	$Z_1X_2Y_3$	$Y_1X_2X_3$
Z_1Z_2	Z_1Z_2	$Y_1Z_2Z_3$	$Z_1Y_2Z_3$	Y_1Y_2	X_3	X_1Y_3	X_2Y_3	$X_1X_2X_3$
Z_3	Z_3	X_1	X_2	$X_1X_2Z_3$	$Z_1Z_2Y_3$	$Y_1Z_2X_3$	$Z_1Y_2X_3$	$Y_1Y_2Y_3$
Z_1Z_3	Z_1Z_3	Y_1	Z_1X_2	$Y_1X_2Z_3$	Z_2Y_3	$X_1Z_2X_3$	Y_2X_3	$X_1Y_2Y_3$
Z_2Z_3	Z_2Z_3	X_1Z_2	Y_2	$X_1Y_2Z_3$	Z_1Y_3	Y_1X_3	$Z_1X_2X_3$	$Y_1X_2Y_3$
$Z_1Z_2Z_3$	$Z_1Z_2Z_3$	Y_1Z_2	Z_1Y_2	$Y_1Y_2Z_3$	Y_3	X_1X_3	X_1X_2	$X_1X_2Y_3$

Similarly,

$$p^H_{k_1k_2k_3} = \frac{1}{2}\Big[\sum_{\tau=0}^{1} a_1^{k_1\oplus\tau}a_2^{k_2\oplus\tau}a_3^{\tau}b_1^{k_1\oplus\overline{\tau}}b_2^{k_2\oplus\overline{\tau}}b_3^{\overline{\tau}} + (-1)^{k_3}\sum_{\tau=0}^{1} c_1^{k_1\oplus\tau}c_2^{k_2\oplus\tau}c_3^{\tau}d_1^{k_1\oplus\overline{\tau}}d_2^{k_2\oplus\overline{\tau}}d_3^{\overline{\tau}}\Big],$$

where $\overline{\tau} = 1-\tau$. Since

$$\left|G_{k_1,k_2,k_3}\right\rangle = (-1)^{k_1\cdot k_3}H_1H_2\left|GHZ_{k_3,k_1\oplus k_2\oplus k_3,k_1\oplus k_3}\right\rangle,$$

for the output state

$$\mathcal{N}(|GHZ\rangle) = \sum_{k\{0,1\}^3} p_k\left|GHZ_k\right\rangle\left\langle GHZ_k\right|,$$

we have

$$p_{k_1k_2k_3} = p^H_{k_1\oplus k_3,k_2\oplus k_3,k_1} = \frac{1}{2}\Big[\prod_{i=1}^{3}a_i^{k_i}b_i^{\overline{k_i}} + \prod_{i=1}^{3}b_i^{k_i}a_i^{\overline{k_i}} + (-1)^{k_1}\left(\prod_{i=1}^{3}c_i^{k_i}d_i^{\overline{k_i}} + \prod_{i=1}^{3}d_i^{k_i}c_i^{\overline{k_i}}\right)\Big],$$

where $\overline{k_i} = 1-k_i$. The entries of the density matrix $\mathcal{N}(|GHZ\rangle)$ is

$$\mathcal{N}(|GHZ\rangle)_{mm} = \frac{1}{2}\left(\prod_{i=1}^{3}a_i^{k_i}b_i^{\overline{k_i}} + \prod_{i=1}^{3}b_i^{k_i}a_i^{\overline{k_i}}\right), \quad (22)$$

$$\mathcal{N}(|GHZ\rangle)_{m,7-m} = \frac{1}{2}\left(\prod_{i=1}^{3}c_i^{k_i}d_i^{\overline{k_i}} + \prod_{i=1}^{3}d_i^{k_i}c_i^{\overline{k_i}}\right) \quad (23)$$

for $m = 2k_2+k_3 \le 3$. We write the output state $\mathcal{N}(|GHZ\rangle)$ in the operator form as

$$\frac{1}{8}[III+\Lambda_1ZZI+\Lambda_2ZIZ+\Lambda_3IZZ+\Lambda_4XXX + \Lambda_5YYX+\Lambda_6YXY+\Lambda_7XYY], \quad (24)$$

then we have

$$\Lambda_1 = (a_1-b_1)(a_2-b_2)(a_3+b_3),$$
$$\Lambda_2 = (a_1-b_1)(a_2+b_2)(a_3-b_3),$$
$$\Lambda_3 = (a_1+b_1)(a_2-b_2)(a_3-b_3),$$

and

$$\Lambda_4 = (c_1+d_1)(c_2+d_2)(c_3+d_3), \quad (25)$$
$$\Lambda_5 = -(c_1-d_1)(c_2-d_2)(c_3+d_3), \quad (26)$$
$$\Lambda_6 = -(c_1-d_1)(c_2+d_2)(c_3-d_3), \quad (27)$$
$$\Lambda_7 = -(c_1+d_1)(c_2-d_2)(c_3-d_3). \quad (28)$$

Hence

$$\prod_{i=4}^{7}\Lambda_i = -\prod_{j=1}^{3}(c_j^2-d_j^2)^2 \le 0. \quad (29)$$

For any GHZ diagonal state with $\prod_{i=4}^{7}\Lambda_i \le 0$, the necessary and sufficient criterion of fully separability is determined by the PPT condition.[11,12] Thus the fully separability criterion is the PPT criterion for the output state of the GHZ state passing through any parallel Pauli channel.

Consider the separability of a noisy GHZ state. For an initial GHZ state

$$|GHZ\rangle = (|000\rangle+|111\rangle)/\sqrt{2},$$

after passing through the parallel Pauli channels the output state $\mathcal{N}(|GHZ\rangle)$ is expressed in expression (24). The fully separable criterion is the PPT criterion (8), which is

$$\min_{(0\le i\le 3)}\mathcal{N}(|GHZ\rangle)_{ii} \ge \max_{(0\le i\le 3)}\left|\mathcal{N}(|GHZ\rangle)_{i,7-i}\right|.$$

For identical Pauli channels with $p_{xi}=p_x, p_{yi}=p_y, p_{zi}=p_z$, $(i=1,2,3)$, the fully separable criterion reduces to $ab \ge \left|c^3+d^3\right|$, where $a=f+p_z$, $b=p_x+p_y$, $c=f-p_z$, $d=p_x-p_y$, and $f=1-p_x-p_y-p_z$. Namely

$$p_z \le \frac{1}{2}(1-p_+-\sqrt[3]{(1-p_+)p_+-p_-^3}), \quad (30)$$

where $p_\pm = p_x \pm p_y$. The border of the regions of full separability and entanglement is shown in Fig. 1.

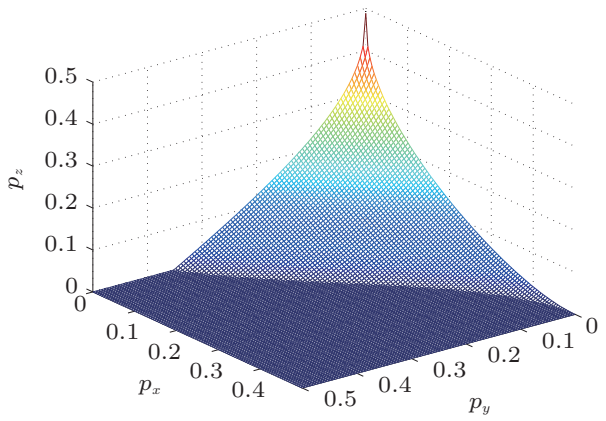


Fig. 1. (color online) The border surface of full separability and entanglement.

For identical depolarizing channels we have $p_x = p_y = p_z = p$, the full separable criterion reduces to

$$p \ge 0.110827. \quad (31)$$

The biseparable criterion (9) now reduces to $3ab \ge \left|c^3+d^3\right|$, namely

$$p_z \le \frac{1}{2}(1-p_+-\sqrt[3]{3(1-p_+)p_+-p_-^3}). \quad (32)$$

The border of the regions of biseparability and genuine entanglement is shown in Fig. 2.

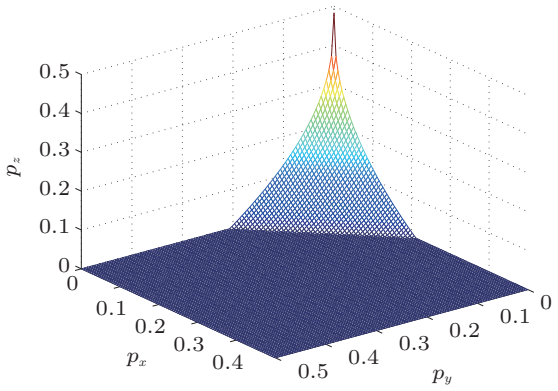


Fig. 2. (color online) The border surface of biseparability and genuine entanglement.

For identical depolarizing channels, the biseparable criterion reduces to

$$p \geq 0.071149. \tag{33}$$

5. Three-qubit GHZ diagonal as input for Pauli channel

Consider the input state of GHZ diagonal state $\rho = \sum_m q_m |GHZ_m\rangle\langle GHZ_m| = H_1H_2 \sum_m q_m |G_k\rangle\langle G_k| H_1H_2$, where the binary string $m = (m_1, m_2, m_3) \in \{0,1\}^3$, the binary string $k = (m_1 \oplus m_3, m_2 \oplus m_3, m_1)$. The output state of parallel Pauli channel $\mathcal{N} = \bigotimes_i \mathcal{E}_i$ is

$$\begin{aligned}\mathcal{N}(\rho) &= \sum_m q_m \mathcal{N}(|GHZ_m\rangle) \\ &= \sum_m q_m H_1H_2 \mathcal{N}_{H_1H_2}(|G_k\rangle) H_1H_2. \end{aligned} \tag{34}$$

For every input of graph basis state $|G_k\rangle$, the output of channel $\mathcal{N}_{H_1H_2}$ is $\mathcal{N}_{H_1H_2}(|G_k\rangle) = \sum_{n\in\{0,1\}^3} p_n^H |G_{k\oplus n}\rangle\langle G_{k\oplus n}| = \sum_n p_{n\oplus k}^H |G_n\rangle\langle G_n|$. Hence

$$\begin{aligned} & H_1H_2 \mathcal{N}_{H_1H_2}(|G_k\rangle) H_1H_2 \\ &= \sum_{n\in\{0,1\}^3} p_{n\oplus k}^H H_1H_2 |G_n\rangle\langle G_n| H_1H_2 \\ &= \sum_{l\in\{0,1\}^3} p^H_{l_1+l_3+k_1, l_2+l_3+k_2, l_1+k_3} |GHZ_l\rangle\langle GHZ_l| \\ &= \sum_l p_{l_1+k_3, l_2+k_1+k_2+k_3, l_3+k_1+k_3} |GHZ_l\rangle\langle GHZ_l| \\ &= \sum_l p_{l\oplus m} |GHZ_l\rangle\langle GHZ_l|. \end{aligned}$$

Thus

$$\begin{aligned}\mathcal{N}(\rho) &= \sum_m q_m \sum_l p_{l\oplus m} |GHZ_l\rangle\langle GHZ_l| \\ &= \sum_l P_l |GHZ_l\rangle\langle GHZ_l|. \end{aligned} \tag{35}$$

Where $P_l = \sum_m q_m p_{m\oplus l}$. In computational basis, we have $\rho' = \mathcal{N}(\rho) = \sum_n (\rho'_{nn} |n\rangle\langle n| + \rho'_{n,7-n} |n\rangle\langle 7-n|)$, with $\rho'_{nn} = \frac{1}{2}(P_n + P_{7-n})$ for $n = 0, \ldots, 7$, and

$$\rho'_{n,7-n} = \rho'_{7-n,n} = \frac{1}{2}(P_n - P_{7-n}),$$

for $n = 0, \ldots, 3$. Define

$$(\lambda'_4, -\lambda'_6, -\lambda'_5, -\lambda'_7) = 4(\rho'_{07}, \rho'_{16}, \rho'_{25}, \rho'_{34})H',$$

then

$$\begin{aligned}\lambda'_4 &= \sum_{i=0}^{3} P_i - \sum_{i=4}^{7} P_i = \sum_l (-1)^{l_1} P_l \\ &= \sum_m \sum_l q_m p_{m\oplus l} (-1)^{l_1} \\ &= \left(\sum_{i=0}^{3} q_i - \sum_{i=4}^{7} q_i\right)\left(\sum_{i=0}^{3} p_i - \sum_{i=4}^{7} p_i\right) \\ &= \lambda_4 \Lambda_4. \end{aligned}$$

Similarly, we have $\lambda'_i = -\Lambda_i \lambda_i$ for $i = 5, 6, 7$. Hence we have

$$\prod_{i=4}^{7} \lambda'_i = -\prod_{i=4}^{7} \Lambda_i \prod_{j=4}^{7} \lambda_j. \tag{36}$$

We have proved that $-\prod_{j=4}^{7} \Lambda_j \geq 0$ for a Pauli channel, thus $\prod_{i=4}^{7} \lambda'_i$ and $\prod_{i=4}^{7} \lambda_i$ have the same sign.

Define $(1, \lambda'_2, \lambda'_1, \lambda'_3) = 4(\rho'_{00}, \rho'_{11}, \rho'_{22}, \rho'_{33})H'$, then we have $\lambda'_i = \Lambda_i \lambda_i$ for $i = 1, 2, 3$. In the operator representation, the output state of the Pauli channel is

$$\begin{aligned}\rho' &= \frac{1}{8}[III + \lambda'_1 ZZI + \lambda'_2 ZIZ + \lambda'_3 IZZ + \lambda'_4 XXX \\ &\quad + \lambda'_5 YYX + \lambda'_6 YXY + \lambda'_7 XYY], \end{aligned}$$

for the input state of a GHZ diagonal state (2) , with relationships of

$$\lambda'_i = \Lambda_i \lambda_i \text{ for } i = 1, \ldots 4, \tag{37}$$

$$\lambda'_i = -\Lambda_i \lambda_i \text{ for } i = 5, \ldots, 7. \tag{38}$$

In the operator representation of the input and output states, the effect of the Pauli channel on GHZ diagonal state is simply characterized by the decay of the parameters.

6. *N*-qubit GHZ state and Pauli channel

In graph state basis, an N-qubit GHZ state $|G\rangle$ is stabilized by the following mutual commutating stabilizer operators

$$K_j = \begin{cases} X_N \prod_{i=1}^{N-1} Z_i, & j = N, \\ Z_N X_j, \end{cases}$$

that is, $K_j |G\rangle = |G\rangle$, the GHZ state $|G\rangle$ is the common eigenstate of all these stabilizer operators K_j with eigenvalues being $+1$. Here X_j, Z_j are the Pauli X and Z matrices applied to the j-th qubit. The N-qubit GHZ state is

$$|G\rangle = \frac{1}{\sqrt{2}}[|+\rangle^{\otimes(N-1)} |0\rangle + |-\rangle^{\otimes(N-1)} |1\rangle],$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. All states $|G_\mu\rangle = Z_\mu |G\rangle$ for $\mu \in \{0,1\}^{\otimes N}$ constitute an orthnormal basis set, where $Z_\mu =$

$\prod_{i=1}^{N} Z_i^{\mu_i}$. The projector onto the GHZ state has a direct representation in terms of the corresponding stabilizer operators,[17]

$$|G\rangle\langle G| = \frac{1}{2^N}\sum_{\mu\in\{0,1\}^{\otimes N}} K_\mu,$$

where $K_\mu = \prod_{i=1}^{N} K_i^{\mu_i}$. A state $\rho = \sum_\mu q_\mu |G_\mu\rangle\langle G_\mu|$ with $q_\mu \geq 0$ and $\sum_\mu q_\mu = 1$ is called a GHZ diagonal state, we have

$$\rho = \frac{1}{2^N}\sum_\mu s_\mu K_\mu \tag{39}$$

in terms of stabilizer operators, where

$$s_\mu = \sum_\nu (-1)^{\mu\cdot\nu} q_\nu \tag{40}$$

and $\mu,\nu \in \{0,1\}^{\otimes N}$. Inversely,

$$q_\mu = \sum_\nu (-1)^{\mu\cdot\nu} s_\nu / 2^N.$$

Clearly, we have the trace of the state to be $s_0 = \sum_\mu q_\mu = 1$ and the minimal eigenvalue of ρ should be non-negative, so

$$\min_{\mu\in\{0,1\}^{\otimes N}} q_\mu = \min_{\mu\in\{0,1\}^{\otimes N}} \sum_\nu (-1)^{\mu\cdot\nu} s_\nu / 2^N \geq 0.$$

Consider the application of the Pauli channel $\mathcal{N} = \bigotimes_{i=1}^{N} \mathcal{E}_i$ to the initial GHZ state $|G\rangle$. The state will evolve to a mixed state in the form of Eq. (39) after it passes through the channel. The problem is to determine all the parameters s_μ in Eq. (39) in terms of the channel parameters. The qubit channel $\mathcal{E}_i$ is described by the parameters p_{xi}, p_{yi}, p_{zi}, and $f_i = 1 - p_{xi} - p_{yi} - p_{zi}$. Denote $(x_i, y_i, z_i) = (p_{xi}, p_{yi}, p_{zi})/f_i$ as above. We may write $K_\mu = \prod_{i=1}^{N} K_i^{\mu_i}$ in terms of X and Z operators according to the generators of the stabilizer,

$$\begin{aligned} K_\mu &= \left(X_N \prod_{j=1}^{N-1} Z_j\right)^{\mu_N} \prod_{i=1}^{N-1} (Z_N X_i)^{\mu_i} \\ &= \begin{cases} X_1^{\mu_1}\cdots X_{N-1}^{\mu_{N-1}} Z_N^{\mu_1+\cdots+\mu_{N-1}}, & \mu_N = 0, \\ Z_1 X_1^{\mu_1}\cdots Z_{N-1} X_{N-1}^{\mu_{N-1}} X_N Z_N^{\mu_1+\cdots+\mu_{N-1}}, & \mu_N = 1. \end{cases} \end{aligned}$$

For the error operator $E_\alpha \in K$, the input state $|G\rangle$ is not changed by the error E_α since K is the stabilizer of $|G\rangle$. The probability of $|G\rangle$ after the channel application is the probability of error operator E_α belonging to the stabilizer K. We have

$$q_{\mathbf{0}} = \sum_{\alpha | E_\alpha \in K} \eta_\alpha,$$

where $\mathbf{0} = (0,\ldots,0)$ is the binary string with all elements nullified. We should summarize the corresponding η_α for $E_\alpha \in K$. For the stabilizer operator K_μ with $\mu_N = 0$, the contribution to the probability $q_{\mathbf{0}}$ is $F x_1^{\mu_1}\cdots x_{N-1}^{\mu_{N-1}} z_N^{\mathrm{mod}(\sum_{i=1}^{N-1}\mu_i, 2)}$, namely,

$$\frac{F}{2}\left[(1+z_N)\prod_{i=1}^{N-1}(1+x_i) + (1-z_N)\prod_{i=1}^{N-1}(1-x_i)\right],$$

where $F = \prod_{i=1}^{N} f_i$. For the stabilizer operator K_μ with $\mu_N = 1$, the contribution to the probability $q_{\mathbf{0}}$ is

$$\frac{F}{2}\left[(x_N+y_N)\prod_{i=1}^{N-1}(z_i+y_i) + (x_N-y_N)\prod_{i=1}^{N-1}(z_i-y_i)\right].$$

In general, the probability q_μ of obtaining the state $|G_\mu\rangle$ when sending the N-qubit GHZ state $|G\rangle$ through a parallel Pauli channel is

$$\begin{aligned} q_\mu = \frac{F}{2}\Bigg[&(1+z_N)\prod_{i=1}^{N-1}(1+x_i)^{1-\mu_i}(z_i+y_i)^{\mu_i} \\ &+ (-1)^{\mu_N}(1-z_N)\prod_{i=1}^{N-1}(1-x_i)^{1-\mu_i}(z_i-y_i)^{\mu_i} \\ &+ (x_N+y_N)\prod_{i=1}^{N-1}(1+x_i)^{\mu_i}(z_i+y_i)^{1-\mu_i} \\ &+ (-1)^{\mu_N}(x_N-y_N)\prod_{i=1}^{N-1}(1-x_i)^{\mu_i}(z_i-y_i)^{1-\mu_i}\Bigg]. \end{aligned}$$

After some algebra, and denoting

$$\xi = (\mu_1,\ldots,\mu_{N-1}) \in \{0,1\}^{\otimes(N-1)},$$

we have s_μ for $\mu_N = 0$ and $\mu_N = 1$,

$$\begin{aligned} s_{\xi 0} &= \left(a_N + (-1)^{\sum_{j=1}^{N-1}\mu_j} b_N\right)\prod_{i=1}^{N-1}(a_i + (-1)^{\mu_i} b_i), \\ s_{\xi 1} &= \left(c_N + (-1)^{\sum_{j=1}^{N-1}\mu_j} d_N\right)\prod_{i=1}^{N-1}(c_i + (-1)^{\mu_i} d_i), \end{aligned}$$

respectively, where $a_N, c_N = f_N \pm p_{zN}$; $b_N, d_N = p_{xN} \pm p_{yN}$; $a_i, c_i = f_i \pm p_{xi}$; $b_i, d_i = p_{zi} \pm p_{yi}$, $(i = 1,\ldots,N-1)$.

The sufficient condition for the coincidence of the full separability and the PPT criterion of the GHZ diagonal state ρ is that there is $\delta \in \{0,1\}^{\otimes(N-1)}$ such that $\forall\xi \in \{0,1\}^{\otimes(N-1)}$,[11]

$$\mathrm{sign}(s_{0\cdots01}) s_{\xi 1} (-1)^{\xi\delta} \geq 0. \tag{41}$$

We will prove that inequality (41) is true for the N-qubit GHZ diagonal state ρ obtained by passing N-qubit GHZ state $|G\rangle$ through the parallel Pauli channel. When $\xi = 0\cdots0$, the left side of inequality (41) is $\mathrm{sign}(s_{0\cdots01}) s_{0\cdots01}$, which is no less than 0. For any ξ with weight 1 (the number of 1 in the $N-1$ bit string), say $\xi_i = 1$ and all the other bits are 0, it is always possible to choose δ_i such that

$$\mathrm{sign}(s_{0\cdots01}) s(i) (-1)^{\delta_i} \geq 0, \tag{42}$$

where $s(i) = s_{0\cdots010\cdots01}$, the subscript is an N-bit string with i-th bit and N-th bit being 1 and all the other bits being 0.

$$s(i) = (c_N - d_N)(c_i - d_i)\prod_{j\neq i, j=1}^{N-1}(c_j + d_j).$$

For $\xi = 1\cdots10\cdots0$, the weight of ξ is i, the first i bits are 1, the rest bits are 0. When i is odd, we have

$$\begin{aligned} & s_{\xi 1}s(1)s(2)\cdots s(i) \\ =& \prod_{j=1}^{i}(c_j-d_j)^2(c_j+d_j)^{i-1} \\ & \cdot \prod_{j=i+1}^{N-1}(c_j+d_j)^{i+1}(c_N-d_N)^{i+1} \geq 0. \end{aligned} \tag{43}$$

Then

$$\begin{aligned} & \mathrm{sign}(s_{0\cdots01})s_{\xi 1}(-1)^{\delta_1+\delta_2+\cdots+\delta_i}\prod_{j=1}^{i} \\ & \times[\mathrm{sign}(s_{0\cdots01})s(i)(-1)^{\delta_i}] \\ =& [\mathrm{sign}(s_{0\cdots01})]^{i+1}s_{\xi 1}s(1)s(2)\cdots s(i) \geq 0, \end{aligned}$$

using inequality (42), we arrive at

$$\mathrm{sign}(s_{0\cdots01})s_{\xi 1}(-1)^{\delta_1+\delta_2+\cdots+\delta_i} \geq 0. \tag{44}$$

When i is even, we have

$$\begin{aligned} & s_{\xi 1}s_{0\cdots01}s(1)s(2)\cdots s(i) \\ =& \prod_{j=1}^{i}(c_j-d_j)^2(c_j+d_j)^{i}(c_N-d_N)^{i}(c_N+d_N)^2 \\ & \cdot \prod_{j=i+1}^{N-1}(c_j+d_j)^{i+2} \geq 0. \end{aligned} \tag{45}$$

We also arrive at expression (44) by using inequalities (42) and (45).

By interchanging the qubits, we can prove inequality (41) for any string ξ of weight i and length $N-1$. By varying i from 2 to $N-1$, the inequality (41) can be proved. Hence the fully separable condition for an N-qubit GHZ state damped by Pauli noise coincides with the PPT criterion. The fully separable condition is[11] (which can also be obtained from the PPT criterion by choosing the proper bipartition)

$$\min_{\delta}\sum_{\xi\in\{0,1\}^{\otimes(N-1)}} s_{\xi 0}(-1)^{\delta\cdot\xi} \geq \sum_{\xi\in\{0,1\}^{\otimes(N-1)}} \left|s_{\xi 1}\right|. \tag{46}$$

For an identical depolarizing channel, $p_{xi}=p_{yi}=p_{zi}=p$, the condition (46) is reduced to

$$2[2p(1-2p)]^{N/2} \geq (1-4p)^N, \tag{47}$$

for even N, and

$$[2p(1-2p)]^{(N-1)/2} \geq (1-4p)^N, \tag{48}$$

for odd N.

7. Conclusion

The evolution of an N-qubit GHZ state in the Pauli environment is studied. For a three-qubit GHZ state, the Pauli noise contaminated GHZ state is a genuine-entangled state when the Pauli noise is weak (the noise parameter $p < 0.071149$ for depolarizing channel). It becomes biseparable when the Pauli noise is strong ($0.110827 > p \geq 0.071149$ for depolarizing channel). It becomes fully separable when the Pauli noise is more strong ($p \geq 0.110827$ for depolarizing channel). The full separability criterion of a Pauli noisy three-qubit GHZ state is proved to be equivalent to PPT criterion. It is determined by comparing the minimum diagonal element with the maximal anti-diagonal element of the density matrix in computational basis. We have also proved that the equivalence of the full separability criterion and PPT criterion for an N-qubit GHZ state contaminated by any parallel Pauli noise. The condition for full separability of a Pauli noisy N-qubit GHZ state is obtained.

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