Hamiltonian structure, Darboux transformation for a soliton hierarchy associated with Lie algebra $so(4,\mathbb{C})$*

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(Received 23 December 2014; revised manuscript received 24 March 2015; published online 20 June 2015)

In this paper, we first introduce a Lie algebra of the special orthogonal group, $g = so(4,\mathbb{C})$, whose elements are 4×4 trace-free, skew-symmetric complex matrices. As its application, we obtain a new soliton hierarchy which is reduced to AKNS hierarchy and present its bi-Hamiltonian structure and Liouville integrability. Furthermore, for one of the equations in the resulting hierarchy, we construct a Darboux matrix T depending on the spectral parameter λ.

Keywords: zero curvature equation, recursion operator, Hamiltonian structure, Darboux transformation

PACS: 02.20.Sv, 03.65.Aa **DOI:** 10.1088/1674-1056/24/8/080201

1. Introduction

With the development of soliton theory, Lie algebra shows great promise. As it is well known, professor Tu presented a method which used Lie algebra as a tool to construct systematically the Lax equation. Especially, Tu[1] obtained trace identity-based loop algebra A_1. Later, many kinds of Lie algebras and their subalgebras were constructed, and some hierarchies with physics meanings are given.[2–11] Recently, professor Ma[12] generated a hierarchy of soliton equations associated with the real Lie algebra $so(3,\mathbb{R})$ and shows that each equation has a bi-Hamiltonian structure and integrable in the Liouville sense. The real Lie algebra of the $so(3,\mathbb{R})$ is not isomorphic to the real Lie algebra $sl(2,\mathbb{R})$ over the real field $\mathbb{R}$, and the soliton hierarchy[12] is not gauge-equivalent to the famous AKNS soliton hierarchy over the real field $\mathbb{R}$. In fact, the two Lie algebras are isomorphic to each other over the complex field $\mathbb{C}$, thus hierarchy[12] is equivalent to the AKNS hierarchy over the complex field $\mathbb{C}$.

It is known that Darboux transformation provides us with a purely algebraic and powerful method to find a new solution of a nonlinear equation from a known solution in Refs. [13]–[16]. The classical Darboux transformation[17] mainly deals with a Sturm–Liouville problem, which presented solutions to the Korteweg-de Vries hierarchy of $(1+1)$-dimensional integrable systems with new potentials and new wave functions. Based on the work, proper transformation that applies to $(2+1)$-dimensional integrable systems was found. The method of the Darboux transformation has also been applied to the discrete integrable systems successfully. The present paper aims to review application of the Darboux transformation technique for superintegrable systems. In general, we assume every element of T is polynomial with respect to λ.

In this paper, we first introduce a Lie algebra of the special orthogonal group, $g = so(4,\mathbb{C})$, which is a 4×4 trace-free, skew-symmetric complex matrix set. As its application, we obtain a soliton hierarchy which is reduced to AKNS hierarchy and presented its Hamiltonian structures and Liouville integrability. Furthermore, for one of the equations in the resulting hierarchy, we present the Darboux matrix T, which the element is related to the spectral parameter λ according to the different position and number of spectral parameter λ.

2. Matrix loop algebra and soliton equations

2.1. Matrix loop algebra $\tilde{so}(4,\mathbb{C})$ and new soliton hierarchy

Firstly, let us consider the simple Lie algebra of the special orthogonal group, $G = so(4,\mathbb{C})$, the Lie algebra of 4×4 trace-free, skew-symmetric complex matrices. It has a basis

$$e_1=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$e_3=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad e_4=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix},$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61170183 and 11271007), SDUST Research Fund, China (Grant No. 2014TDJH102), the Fund from the Joint Innovative Center for Safe and Effective Mining Technology and Equipment of Coal Resources, Shandong Province, the Promotive Research Fund for Young and Middle-aged Scientisits of Shandong Province, China (Grant No. BS2013DX012), and the Postdoctoral Fund of China (Grant No. 2014M551934).

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$$e_5=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix},\quad e_6=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\tag{1}$$

and the structure equations of $so(4,\mathbb{C})$ are

$$\begin{aligned}&[e_1,e_2]=[e_4,e_5]=e_3,\quad [e_2,e_3]=-[e_5,e_6]=e_1,\\&[e_3,e_1]=-[e_6,e_4]=e_2,\\&[e_1,e_4]=[e_2,e_5]=[e_3,e_6]=0,\quad [e_2,e_4]=-[e_1,e_5]=e_6,\\&[e_1,e_6]=[e_3,e_4]=e_5,[e_2,e_6]=[e_3,e_5]=-e_4.\end{aligned}\tag{2}$$

We point out that the subalgebra $G_1=\mathrm{span}\{e_1,e_2,e_3\}$ of $so(4,\mathbb{C})$ isomorphic to $so(3,\mathbb{C})$. We adopt the following matrix loop algebra

$$\widetilde{so}(4,\mathbb{C})=so(4,\mathbb{C})\otimes\mathbb{C}[\lambda,\lambda^{-1}].\tag{3}$$

In the following, we work out a new soliton hierarchy from the matrix loop algebra $\widetilde{so}(4,\mathbb{C})$, and introduce a spectral problem

$$\begin{aligned}\phi_x&=U\phi,\\U&=U(u,\lambda)=\lambda e_1+pe_2+qe_3+re_5+se_6\\&=\begin{pmatrix}0&q&\lambda&r\\-q&0&-p&0\\-\lambda&p&0&s\\-r&0&-s&0\end{pmatrix}.\end{aligned}\tag{4}$$

Setting

$$\begin{aligned}V&=Ae_1+Be_2+Ce_3+Fe_4+De_5+Ge_6\\&=\begin{pmatrix}0&C&A&D\\-C&0&-B&F\\-A&B&0&G\\-D&-F&-G&0\end{pmatrix},\end{aligned}$$

then the stationary zero curvature equation

$$V_x=[U,V]$$

gives rise to a solution for V as follows:

$$A_x=pC+sD-qB-rG,\tag{5a}$$
$$B_x=-\lambda C-sF+qA,\tag{5b}$$
$$C_x=-pA+\lambda B-rF,\tag{5c}$$
$$D_x=-sA+qF+\lambda G,\tag{5d}$$
$$F_x=rC+sB-qD-pG,\tag{5e}$$
$$G_x=rA-\lambda D+pF.\tag{5f}$$

Substituting

$$V=\sum_{i\geq0}V_i\lambda^{-i},\ A=\sum_{i\geq0}A_i\lambda^{-i},\ B=\sum_{i\geq0}B_i\lambda^{-i},\ \cdots$$

into Eqs. (5), we obtain

$$B_{n+1}=C_{nx}+pA_n+rF_n,\tag{6a}$$
$$C_{n+1}=-B_{nx}+qA_n-sF_n,\tag{6b}$$
$$D_{n+1}=-G_{nx}+rA_n+pF_n,\tag{6c}$$
$$G_{n+1}=D_{nx}+sA_n-qF_n,\tag{6d}$$
$$A_n=\partial^{-1}(pC_n+sD_n-qB_n-rG_n),\tag{6e}$$
$$F_n=\partial^{-1}(rC_n+sB_n-qD_n-pG_n).\tag{6f}$$

On taking the initial values

$$A_0=-1,B_0=C_0=D_0=F_0=G_0=0,$$

and determining the sequence of $\{A_i,\ B_i,\ C_i,\ D_i,\ F_i,\ G_i,|i>0\}$. Therefore, the first few sets can be computed as follows:

$$B_1=-p,C_1=-q,D_1=-r,G_1=-s,A_1=F_1=0,\tag{7a}$$
$$B_2=-q_x,C_2=p_x,D_2=s_x,G_2=-r_x,$$
$$A_2=\frac{1}{2}(p^2+q^2+r^2+s^2),F_2=rp-sq,\tag{7b}$$
$$B_3=p_{xx}+\frac{1}{2}p(p^2+q^2+3r^2+s^2)-rsq,$$
$$C_3=q_{xx}+\frac{1}{2}q(p^2+q^2+r^2+3s^2)-rsp,\tag{7c}$$
$$D_3=r_{xx}+\frac{1}{2}r(3p^2+q^2+r^2+s^2)-spq,$$
$$G_3=s_{xx}+\frac{1}{2}s(p^2+3q^2+r^2+s^2)-rpq,\tag{7d}$$
$$A_3=pq_x-qp_x+sr_x-rs_x,F_3=rq_x-qr_x+sp_x-ps_x,\tag{7e}$$
$$B_4=q_{xxx}+\frac{3}{2}(p^2+q^2+r^2+s^2)q_x+3(qs-pr)s_x,\tag{7f}$$
$$C_4=-p_{xxx}-\frac{3}{2}(p^2+q^2+r^2+s^2)p_x-3(pr-qs)r_x,\tag{7g}$$
$$D_4=-s_{xxx}-\frac{3}{2}(p^2+q^2+r^2+s^2)s_x+3(pr-qs)q_x,\tag{7h}$$
$$G_4=r_{xxx}+\frac{3}{2}(p^2+q^2+r^2+s^2)r_x-3(qs-pr)p_x,\tag{7i}$$
$$\begin{aligned}A_4=&-pp_{xx}-qq_{xx}-rr_{xx}-ss_{xx}+\frac{1}{2}(p_x^2+q_x^2+r_x^2+s_x^2)\\&-\frac{3}{8}(p^4+q^4+r^4+s^4)-\frac{9}{4}(p^2r^2+q^2s^2)\\&-\frac{3}{4}(p^2+r^2)(q^2+s^2)+3pqrs.\end{aligned}\tag{7j}$$

Now, taking

$$V^{(n)}=\sum_{m=0}^{n}V_m\lambda^{n-m},\quad n\geq0,\tag{8}$$

and reading a hierarchy of soliton equations:

$$u_{t_n}=K_n=\begin{pmatrix}-C_{n+1}\\B_{n+1}\\G_{n+1}\\-D_{n+1}\end{pmatrix},\tag{9}$$

from the zero curvature equations, we have

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0,\quad n\geq0.\tag{10}$$

2.2. Hamiltonian structures and Liouville integrability for Eq. (9)

It is direct to compute that

$$\frac{\partial U}{\partial \lambda}=e_1,\ \frac{\partial U}{\partial p}=e_2,\ \frac{\partial U}{\partial q}=e_3,\ \frac{\partial U}{\partial r}=e_5,\ \frac{\partial U}{\partial s}=e_6,$$

and read

$$\mathrm{Tr}\left(V\frac{\partial U}{\partial \lambda}\right)=-2A,\ \mathrm{Tr}\left(V\frac{\partial U}{\partial p}\right)=-2B,\ \mathrm{Tr}\left(V\frac{\partial U}{\partial q}\right)=-2C,$$

$$\mathrm{Tr}\left(V\frac{\partial U}{\partial r}\right)=-2D,\ \mathrm{Tr}\left(V\frac{\partial U}{\partial s}\right)=-2G.$$

Now the corresponding trace identity becomes

$$\frac{\delta}{\delta u}\int A\,\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}B\\C\\D\\G\end{pmatrix}.$$

Balancing coefficients of each power of λ in the above equality we obtain

$$\frac{\delta}{\delta u}\int A_{n+1}\,\mathrm{d}x=(\gamma-n)\begin{pmatrix}B_n\\C_n\\D_n\\G_n\end{pmatrix},\quad n\geq 0.$$

The case of $m=1$ tells $\gamma=0$, and thus we have

$$\frac{\delta}{\delta u}\int -\frac{A_{n+1}}{n}\,\mathrm{d}x=\begin{pmatrix}B_n\\C_n\\D_n\\G_n\end{pmatrix},\quad n\geq 1.$$

Consequently, we obtain the following Hamiltonian structure for the soliton hierarchy (9)

$$u_{t_n}=K_n=\begin{pmatrix}-C_{n+1}\\B_{n+1}\\G_{n+1}\\-D_{n+1}\end{pmatrix}=J\frac{\delta H_n}{\delta u},\quad (n\geq 0)\tag{11}$$

with the Hamiltonian operator

$$J=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\tag{12}$$

and the Hamiltonian functionals

$$H_n=-\int\frac{A_{n+2}}{n+1}\,\mathrm{d}x,\quad n\geq 0.\tag{13}$$

The recursion operator ϕ can be determined by the recursion relations (6a)–(6f)

$$\phi=\begin{pmatrix}r\partial^{-1}s-p\partial^{-1}q & \partial+r\partial^{-1}r+p\partial^{-1}p & -r\partial^{-1}q+p\partial^{-1}s & -r\partial^{-1}p-p\partial^{-1}r\\ -\partial-s\partial^{-1}s-q\partial^{-1}q & -s\partial^{-1}r+q\partial^{-1}p & s\partial^{-1}q+q\partial^{-1}s & s\partial^{-1}p-q\partial^{-1}r\\ -r\partial^{-1}q+p\partial^{-1}s & r\partial^{-1}p+p\partial^{-1}r & r\partial^{-1}s-p\partial^{-1}q & -\partial-r\partial^{-1}r-p\partial^{-1}p\\ -s\partial^{-1}q-q\partial^{-1}s & s\partial^{-1}p-q\partial^{-1}r & \partial+s\partial^{-1}s+q\partial^{-1}q & -s\partial^{-1}r+q\partial^{-1}p\end{pmatrix}.$$

Equation (9) has bi-Hamiltonian structure

$$u_{t_n}=K_n=\begin{pmatrix}-C_{n+1}\\B_{n+1}\\G_{n+1}\\-D_{n+1}\end{pmatrix}=J\frac{\delta H_n}{\delta u}=M\frac{\delta H_{n-1}}{\delta u},\quad n\geq 1,\tag{14}$$

with the Hamiltonian operator

$$M=J\phi=\begin{pmatrix}\partial+s\partial^{-1}s+q\partial^{-1}q & s\partial^{-1}r-q\partial^{-1}p & -s\partial^{-1}q-q\partial^{-1}s & -s\partial^{-1}p+q\partial^{-1}r\\ r\partial^{-1}s-p\partial^{-1}q & \partial+r\partial^{-1}r+p\partial^{-1}p & -r\partial^{-1}q+p\partial^{-1}s & -r\partial^{-1}p-p\partial^{-1}r\\ -s\partial^{-1}q-q\partial^{-1}s & s\partial^{-1}p-q\partial^{-1}r & \partial+s\partial^{-1}s+q\partial^{-1}q & -s\partial^{-1}r+q\partial^{-1}p\\ r\partial^{-1}q-p\partial^{-1}s & -r\partial^{-1}p-p\partial^{-1}r & -r\partial^{-1}s+p\partial^{-1}q & \partial+r\partial^{-1}r+p\partial^{-1}p\end{pmatrix}.\tag{15}$$

2.3. The reductions of Eq. (14)

Case 1 $r=s=0$

$$\begin{aligned}u_{1t_n}&=K_{1n}=\begin{pmatrix}-C_{n+1}\\B_{n+1}\end{pmatrix}\\&=J_1\frac{\delta H_n}{\delta u}=M_1\frac{\delta H_{n-1}}{\delta u},\quad (n\geq 1).\end{aligned}\tag{16}$$

Case 2 $p=q=0$

$$\begin{aligned}u_{2t_n}&=K_{2n}=\begin{pmatrix}G_{n+1}\\-D_{n+1}\end{pmatrix}\\&=J_2\frac{\delta H_n}{\delta u}=M_2\frac{\delta H_{n-1}}{\delta u},\quad (n\geq 1),\end{aligned}\tag{17}$$

where

$$J_1=-J_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

$$M_1=\begin{pmatrix}\partial+q\partial^{-1}q & -q\partial^{-1}p\\ -p\partial^{-1}q & \partial+p\partial^{-1}p\end{pmatrix},$$

$$M_2=\begin{pmatrix}\partial+s\partial^{-1}s & -s\partial^{-1}r\\ -r\partial^{-1}s & \partial+r\partial^{-1}r\end{pmatrix},$$

and H_n, H_{n-1} are the same as in Eq. (13). From the above derivations, we know that the two cases have bi-Hamiltonian structure and are gauge-equivalent to the AKNS hierarchy over the complex field $\mathbb{C}$, respectively.

2.4. A nonlinear example

The first nonlinear integrable system in the hierarchy (9) reads

$$\begin{aligned} u_{t_2} &= \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_{t_2} = \begin{pmatrix} -C_3 \\ B_3 \\ G_3 \\ -D_3 \end{pmatrix} \\ &= \begin{pmatrix} -q_{xx} - \frac{1}{2}q(p^2+q^2+r^2+3s^2) + rsp \\ p_{xx} + \frac{1}{2}p(p^2+q^2+3r^2+s^2) - rsq \\ s_{xx} + \frac{1}{2}s(p^2+3q^2+r^2+s^2) - rpq \\ -r_{xx} - \frac{1}{2}r(3p^2+q^2+r^2+s^2) + spq \end{pmatrix}. \end{aligned} \tag{18}$$

It possesses the following bi-Hamiltonian structure

$$u_{t_2} = J\frac{\delta H_2}{\delta u} = M\frac{\delta H_1}{\delta u}, \tag{19}$$

where the Hamiltonian pair $\{J, M\}$ is defined by Eqs. (12) and (15), and the Hamiltonian functionals, $\{H_1, H_2\}$, are given by

$$\begin{aligned} H_1 &= -\frac{1}{2}\int (pq_x - qp_x + sr_x - rs_x)\,\mathrm{d}x, \\ H_2 &= -\frac{1}{3}\int [-pp_{xx} - qq_{xx} - rr_{xx} - ss_{xx} \\ &\quad + \frac{1}{2}(p_x^2 + q_x^2 + r_x^2 + s_x^2) \\ &\quad - \frac{3}{8}(p^4+q^4+r^4+s^4) - \frac{9}{4}(p^2r^2+q^2s^2) \\ &\quad - \frac{3}{4}(p^2+r^2)(q^2+s^2) + 3pqrs]\,\mathrm{d}x. \end{aligned} \tag{20}$$

3. Darboux transformations of the spectral problem (18)

We consider the spectral problem (4) by introducing the temporal part

$$\begin{aligned} \phi_t &= N\phi, \\ N &= \sum_{m=0}^{2} V_m \lambda^{2-m} \\ &= \begin{pmatrix} 0 & -\lambda q + p_x & \alpha & -\lambda r + s_x \\ \lambda q - p_x & 0 & \lambda p + q_x & rp - sq \\ -\alpha & -\lambda p - q_x & 0 & -\lambda s - r_x \\ \lambda r - s_x & -rp + sq & \lambda s + r_x & 0 \end{pmatrix}, \end{aligned} \tag{21}$$

where $\alpha = -\lambda^2 + \frac{1}{2}(p^2+q^2+r^2+s^2)$.

The compatibility condition $\phi_{xt} = \phi_{tx}$ yields a zero curvature equation

$$U_t - N_x + [U, N] = 0. \tag{22}$$

Now we consider a Darboux transformation

$$\overline{\phi} = T\phi, \tag{23}$$

where T is defined by

$$T_x + TU = \overline{U}T, \tag{24}$$

and

$$T_t + TN = \overline{N}T. \tag{25}$$

A new spectral problem reads

$$\overline{\phi}_x = \overline{U\phi}, \quad \overline{\phi}_t = \overline{N\phi}, \tag{26}$$

where $\overline{U}$ and $\overline{N}$ has the same form as U and N, except replacing p, q, r, and s with $\overline{p}$, $\overline{q}$, $\overline{r}$, and $\overline{s}$.

Now we consider the basic forms of T. First of all, we assume

$$T = \lambda(a_{ij})_{4\times4} + (b_{ij})_{4\times4}, \tag{27}$$

where a_{ij} and b_{ij} $(i, j = 1, 2, 3, 4)$ are functions of x and t. By inserting Eq. (27) into Eq. (24), we read that the coefficients matrix of λ^2 are

$$\begin{pmatrix} a_{31}+a_{13} & a_{32} & -a_{11}+a_{33} & a_{34} \\ a_{23} & 0 & -a_{21} & 0 \\ -a_{11}+a_{33} & -a_{12} & -a_{13}-a_{31} & -a_{14} \\ a_{43} & 0 & -a_{41} & 0 \end{pmatrix}. \tag{28}$$

In the following, we consider the simplest non-trivial forms of T

$$a_{22} \neq 0, \text{ and } a_{i,j} = 0 \text{ in otherwise.} \tag{29}$$

3.1. Spatial scales of the Darboux transformation of Eq. (18)

Under the condition equation (29), we insert Eq. (27) into Eq. (24) again, compare the coefficients of λ^j $(j = 2, 1, 0)$, and know it is obviously trivial for the case of $j = 2$. We then have

$$\begin{aligned} &a_{22x} = 0, \quad b_{13} = -b_{31}, \quad b_{11} = b_{33}, \\ &b_{14} = b_{41} = b_{34} = b_{43} = 0, \quad b_{32} = -a_{22}\overline{q}, \\ &b_{23} = -a_{22}q, \quad b_{21} = a_{22}p, \quad b_{12} = a_{22}\overline{p}, \end{aligned} \tag{30}$$

for the coefficients matrix of λ $(j = 1)$

$$\begin{pmatrix} -b_{13}-b_{31} & -b_{32}-a_{22}\overline{q} & b_{11}-b_{33} & -b_{34} \\ -b_{23}-a_{22}q & a_{22x} & b_{21}-a_{22}p & 0 \\ b_{11}-b_{33} & b_{12}-a_{22}\overline{p} & b_{31}+b_{13} & b_{14} \\ -b_{43} & 0 & b_{41} & 0 \end{pmatrix} = 0. \tag{31}$$

The case of $j = 0$ leads to the following equations

$$b_{11x} - b_{21}\overline{q} - b_{41}\overline{r} - b_{12}q - b_{14}r = 0, \tag{32}$$

$$b_{12x} - b_{22}\overline{q} - b_{42}\overline{r} + b_{11}q + b_{13}p = 0, \tag{33}$$

$$b_{13x} - b_{23}\overline{q} - b_{43}\overline{r} - b_{12}p - b_{14}s = 0, \tag{34}$$

$$b_{14x} - b_{24}\overline{q} - b_{44}\overline{r} + b_{11}r + b_{13}s = 0, \tag{35}$$

$$b_{21x}+b_{11}\overline{q}+b_{31}\overline{p}-b_{22}q-b_{24}r=0, \tag{36}$$
$$b_{22x}+b_{12}\overline{q}+b_{32}\overline{p}+b_{21}q+b_{23}p=0, \tag{37}$$
$$b_{23x}+b_{13}\overline{q}+b_{33}\overline{p}-b_{22}p-b_{24}s=0, \tag{38}$$
$$b_{24x}+b_{14}\overline{q}+b_{34}\overline{p}+b_{21}r+b_{23}s=0, \tag{39}$$
$$b_{31x}-b_{21}\overline{p}-b_{41}\overline{s}-b_{32}q-b_{34}r=0, \tag{40}$$
$$b_{32x}-b_{22}\overline{p}-b_{42}\overline{s}+b_{31}q+b_{33}p=0, \tag{41}$$
$$b_{33x}-b_{23}\overline{p}-b_{43}\overline{s}-b_{32}p-b_{34}s=0, \tag{42}$$
$$b_{34x}-b_{24}\overline{p}-b_{44}\overline{s}+b_{31}r+b_{33}s=0, \tag{43}$$
$$b_{41x}+b_{11}\overline{r}+b_{31}\overline{s}-b_{42}q-b_{44}r=0, \tag{44}$$
$$b_{42x}+b_{12}\overline{r}+b_{32}\overline{s}+b_{41}q+b_{43}p=0, \tag{45}$$
$$b_{43x}+b_{13}\overline{r}+b_{33}\overline{s}-b_{42}p-b_{44}s=0, \tag{46}$$
$$b_{44x}+b_{14}\overline{r}+b_{34}\overline{s}+b_{41}r+b_{43}s=0. \tag{47}$$

On the other hand, the solutions $\Phi,\overline{\Phi}$ of Eqs. (4) and (26) are two 4×4 matrices, thus $\det(\Phi)=\det(\overline{\Phi})=$ const. from $\mathrm{Tr}(M)=\mathrm{Tr}(\overline{M})=0$, which means that there is a constant $\lambda=\lambda_1$ and a solution $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$ of Eq. (4) which satisfy

$$b_{11}\phi_1+b_{12}\phi_2+b_{13}\phi_3+b_{14}\phi_4=0, \tag{48}$$
$$b_{21}\phi_1+(a_{22}\lambda_1+b_{22})\phi_2+b_{23}\phi_3+b_{24}\phi_4=0, \tag{49}$$
$$b_{31}\phi_1+b_{32}\phi_2+b_{33}\phi_3+b_{34}\phi_4=0, \tag{50}$$
$$b_{41}\phi_1+b_{42}\phi_2+b_{43}\phi_3+b_{44}\phi_4=0. \tag{51}$$

By substituting Eqs. (32), (42), (34), (40), (37), and (47) into Eq. (30), we get

$$b_{11x}=b_{33x}=b_{13x}=b_{31x}=b_{22x}=b_{44x}=0. \tag{52}$$

Combining Eq. (30) with Eqs. (48)–(52), we get

$$\begin{aligned}
&b_{14}=b_{41}=b_{34}=b_{43}=0,\quad b_{23}=-q,\quad b_{21}=p,\\
&a_{22}=b_{22}=b_{44}=b_{13}=-b_{31}=b_{11}=b_{33}=1,\\
&b_{12}=-\frac{1}{\phi_2}(\phi_1+\phi_3),\quad b_{32}=\frac{1}{\phi_2}(\phi_1-\phi_3),\\
&b_{24}=-\frac{1}{\phi_4}(p\phi_1+\lambda_1\phi_2+\phi_2-q\phi_3),\quad b_{42}=-\frac{\phi_4}{\phi_2}.
\end{aligned} \tag{53}$$

The unknown functions $\overline{p},\overline{q}$ can be defined by Eqs. (36) and (38):

$$\begin{aligned}
\overline{p} &= \frac{1}{2}(q_x+p_x)+\frac{r-s}{2\phi_4}(p\phi_1+\lambda_1\phi_2+\phi_2-q\phi_3)\\
&\quad+\frac{1}{2}(p-q),
\end{aligned} \tag{54}$$
$$\begin{aligned}
\overline{q} &= \frac{1}{2}(q_x-p_x)-\frac{r+s}{2\phi_4}(p\phi_1+\lambda_1\phi_2+\phi_2-q\phi_3)\\
&\quad+\frac{1}{2}(p+q),
\end{aligned} \tag{55}$$

and unknown functions $\overline{r},\overline{s}$ can be defined by Eqs. (44) and (46)

$$\overline{r}=\frac{1}{2}(r+s)-\frac{\phi_4}{2\phi_2}(p+q), \tag{56}$$

$$\overline{s}=\frac{1}{2}(s-r)+\frac{\phi_4}{2\phi_2}(q-p). \tag{57}$$

The others of Eqs. (32)–(47) can be proved to be satisfied automatically (under the condition $\phi_x=U\phi$).

3.2. Temporal scales of the Darboux transformation

If one can prove that the transformation equation (27) maps Eq. (21) into $\overline{\phi}_t=\overline{N}\overline{\phi}$, where $\overline{N}$ has the same form as N in Eq. (21) except changing p, q, r, and s with $\overline{p}$, $\overline{q}$, $\overline{r}$, and $\overline{s}$, then the compatibility condition $\overline{\phi}_{xt}=\overline{\phi}_{tx}$ holds, i.e.,

$$\overline{U}_t-\overline{N}_x+[\overline{U},\overline{N}]=0, \tag{58}$$

so that $(\overline{p},\overline{q},\overline{r},\overline{s})$ is a new solution of Eq. (21).

The main idea of the proof is following: here we only need to prove equation $\overline{\phi}_t=\overline{N\phi}$.

Let us compare the coefficient of λ^j $(j=3,2,1,0)$. It is trivial for the cases of $j=3$. For the case of $j=2$, the coefficients matrix is

$$\begin{pmatrix}
-b_{13}-b_{31} & -b_{32}-a_{22}\overline{q} & b_{11}-b_{33} & -b_{34}\\
-b_{23}-a_{22}q & 0 & b_{21}-a_{22}p & 0\\
b_{11}-b_{33} & b_{12}-a_{22}\overline{p} & b_{31}+b_{13} & b_{14}\\
-b_{43} & 0 & b_{41} & 0
\end{pmatrix}=0.$$

We have the same form as Eq. (31) when $a_{22x}=0$.

The case of $j=1$ leads to the following equations

$$b_{21}\overline{q}+b_{41}\overline{r}+b_{12}q+b_{14}r=0, \tag{59}$$
$$b_{22}\overline{q}+b_{42}\overline{r}-b_{11}q-b_{13}p-a_{22}\overline{p}_x=0, \tag{60}$$
$$b_{23}\overline{q}+b_{43}\overline{r}+b_{12}p+b_{14}s=0, \tag{61}$$
$$b_{24}\overline{q}+b_{44}\overline{r}-b_{11}r-b_{13}s=0, \tag{62}$$
$$-b_{11}\overline{q}-b_{31}\overline{p}+b_{22}q+b_{24}r-a_{22}p_x=0, \tag{63}$$
$$-b_{12}\overline{q}-b_{32}\overline{p}-b_{21}q-b_{23}p+a_{22t}=0, \tag{64}$$
$$-b_{13}\overline{q}-b_{33}\overline{p}+b_{22}p+b_{24}s+a_{22}q_x=0, \tag{65}$$
$$-b_{14}\overline{q}-b_{34}\overline{p}-b_{21}r-b_{23}s+a_{22}(rp-sq)=0, \tag{66}$$
$$b_{21}\overline{p}+b_{41}\overline{s}+b_{32}q+b_{34}r=0, \tag{67}$$
$$b_{22}\overline{p}+b_{42}\overline{s}-b_{31}q-b_{33}p+a_{22}\overline{q}_x=0, \tag{68}$$
$$b_{23}\overline{p}+b_{43}\overline{s}+b_{32}p+b_{34}s=0, \tag{69}$$
$$b_{24}\overline{p}+b_{44}\overline{s}-b_{31}r-b_{33}s=0, \tag{70}$$
$$-b_{11}\overline{r}-b_{31}\overline{s}+b_{42}q+b_{44}r=0, \tag{71}$$
$$-b_{12}\overline{r}-b_{32}\overline{s}-b_{41}q-b_{43}p+a_{22}(\overline{rp}-\overline{sq})=0, \tag{72}$$
$$-b_{13}\overline{r}-b_{33}\overline{s}+b_{42}p+b_{44}s=0, \tag{73}$$
$$-b_{14}\overline{r}-b_{34}\overline{s}-b_{41}r-b_{43}s=0. \tag{74}$$

After substituting Eqs. (32)–(38), (40)–(44), (46), (47) into Eqs. (59)–(65), (67)–(71), (73), (74) respectively, and replacing Eqs. (66) and (72) with Eq. (30), then we find equations (59)–(74) are valid.

Also the following equations are presented from the case of $j=0$

$$
\begin{aligned}
&b_{11t}-b_{21}\overline{p}_x-b_{41}\overline{s}_x-b_{12}p_x-b_{14}s_x\\
&-\frac{1}{2}b_{31}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)\\
&-\frac{1}{2}b_{13}(p^2+q^2+r^2+s^2)=0, \qquad (75)\\
&b_{12t}-b_{22}\overline{p}_x-b_{42}\overline{s}_x+b_{11}p_x-b_{13}q_x\\
&-\frac{1}{2}b_{32}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)+b_{14}(sq-rp)=0, \qquad (76)\\
&b_{13t}-b_{23}\overline{p}_x-b_{43}\overline{s}_x+b_{12}q_x+b_{14}r_x\\
&-\frac{1}{2}b_{33}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)\\
&+\frac{1}{2}b_{11}(p^2+q^2+r^2+s^2)=0, \qquad (77)\\
&b_{14t}-b_{24}\overline{p}_x-b_{44}\overline{s}_x+b_{11}s_x-b_{13}r_x\\
&-\frac{1}{2}b_{34}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)+b_{12}(rp-sq)=0, \qquad (78)\\
&b_{21t}+b_{11}\overline{p}_x-b_{31}\overline{q}_x-b_{22}p_x-b_{24}s_x\\
&-b_{41}(\overline{rp}-\overline{sq})-\frac{1}{2}b_{23}(p^2+q^2+r^2+s^2)=0, \qquad (79)\\
&b_{22t}+b_{12}\overline{p}_x-b_{32}\overline{q}_x+b_{21}p_x-b_{23}q_x\\
&-b_{42}(\overline{rp}-\overline{sq})-b_{24}(rp-sq)=0, \qquad (80)\\
&b_{23t}+b_{13}\overline{p}_x-b_{33}\overline{q}_x+b_{22}q_x+b_{24}s_x\\
&-b_{43}(\overline{rp}-\overline{sq})+\frac{1}{2}b_{21}(p^2+q^2+r^2+s^2)=0, \qquad (81)\\
&b_{24t}+b_{14}\overline{p}_x-b_{34}\overline{q}_x+b_{21}s_x-b_{23}r_x\\
&-b_{44}(\overline{rp}-\overline{sq})+b_{22}(rp-sq)=0, \qquad (82)\\
&b_{31t}+b_{21}\overline{q}_x+b_{41}\overline{r}_x-b_{32}p_x-b_{34}s_x\\
&+\frac{1}{2}b_{11}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)\\
&-\frac{1}{2}b_{33}(p^2+q^2+r^2+s^2)=0, \qquad (83)\\
&b_{32t}+b_{22}\overline{q}_x+b_{42}\overline{r}_x+b_{31}p_x-b_{33}q_x\\
&+\frac{1}{2}b_{12}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)-b_{34}(rp-sq)=0, \qquad (84)\\
&b_{33t}+b_{23}\overline{q}_x+b_{43}\overline{r}_x+b_{32}q_x+b_{34}r_x\\
&+\frac{1}{2}b_{13}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)\\
&+\frac{1}{2}b_{31}(p^2+q^2+r^2+s^2)=0, \qquad (85)\\
&b_{34t}+b_{24}\overline{q}_x+b_{44}\overline{r}_x+b_{31}s_x-b_{33}r_x\\
&+\frac{1}{2}b_{14}(\overline{p}^2+\overline{q}^2+\overline{r}^2+\overline{s}^2)+b_{32}(rp-sq)=0, \qquad (86)\\
&b_{41t}+b_{11}\overline{s}_x-b_{31}\overline{r}_x-b_{42}p_x-b_{44}s_x\\
&+b_{21}(\overline{rp}-\overline{sq})-\frac{1}{2}b_{43}(p^2+q^2+r^2+s^2)=0, \qquad (87)\\
&b_{42t}+b_{12}\overline{s}_x-b_{32}\overline{r}_x+b_{41}p_x-b_{43}q_x\\
&+b_{22}(\overline{rp}-\overline{sq})-b_{44}(rp-sq)=0, \qquad (88)\\
&b_{43t}+b_{13}\overline{s}_x-b_{33}\overline{r}_x+b_{42}q_x+b_{44}r_x\\
&+b_{23}(\overline{rp}-\overline{sq})+\frac{1}{2}b_{41}(p^2+q^2+r^2+s^2)=0, \qquad (89)\\
&b_{44t}+b_{14}\overline{s}_x-b_{34}\overline{r}_x+b_{41}s_x-b_{43}r_x\\
&+b_{24}(\overline{rp}-\overline{sq})+b_{42}(rp-sq)=0. \qquad (90)
\end{aligned}
$$

Here, we only give the proof of Eq. (88) for brevity.

From Eq. (53) and $\phi_x=U\phi$, we read

$$
\begin{aligned}
\left(\frac{\phi_4}{\phi_2}\right)_x &= \frac{b_{12}-b_{32}}{2}r+\frac{b_{12}+b_{32}}{2}s\\
&\quad +b_{42}\left(\frac{b_{12}-b_{32}}{2}q+\frac{b_{12}+b_{32}}{2}p\right). \qquad (91)
\end{aligned}
$$

Also, from Eqs. (56) and (57), we have

$$
\begin{aligned}
\overline{r}_x &= \frac{1}{2}(r_x-s_x)+\frac{\phi_4}{2\phi_2}(p_x+q_x)+\left(\frac{b_{12}-b_{32}}{4}r+\frac{b_{12}+b_{32}}{4}s\right.\\
&\quad \left.+b_{42}\frac{b_{12}-b_{32}}{4}q+b_{42}\frac{b_{12}+b_{32}}{4}p\right)(p+q), \qquad (92)\\
\overline{s}_x &= \frac{1}{2}(r_x+s_x)+\frac{\phi_4}{2\phi_2}(q_x-p_x)+\left(\frac{b_{12}-b_{32}}{4}r+\frac{b_{12}+b_{32}}{4}s\right.\\
&\quad \left.+b_{42}\frac{b_{12}-b_{32}}{4}q+b_{42}\frac{b_{12}+b_{32}}{4}p\right)(q-p). \qquad (93)
\end{aligned}
$$

Again, from Eqs. (53) and $\phi_t=N\phi$, we propose that

$$
\begin{aligned}
-b_{42t} &= \left(\frac{\phi_4}{\phi_2}\right)_t\\
&= (\lambda_1 r-s_x)\frac{b_{32}-b_{12}}{2}+(-rp+sq)\\
&\quad -(\lambda_1 s+r_x)\frac{b_{32}+b_{12}}{2}+b_{42}(\lambda_1 q-p_x)\frac{b_{32}-b_{12}}{2}\\
&\quad -b_{42}(\lambda_1 p+q_x)\frac{b_{32}+b_{12}}{2}-b_{42}^2(rp-sq). \qquad (94)
\end{aligned}
$$

We know that equation (88) is true by substituting Eqs. (53)–(57), (92)–(94) into Eq. (88).

4. Remark

As the subalgebras of $so(4,\mathbb{C})$, $G_1=\mathrm{span}\{e_1,e_2,e_3\}$ and $G_2=\mathrm{span}\{e_1,e_5,e_6\}$, both are isomorphic to $so(3,\mathbb{C})$, and both are isomorphic to $sl(2,\mathbb{C})$ over the complex field $\mathbb{C}$, we know that equations (16) and (17), as the reductions of Eq. (11), are equivalent to AKNS hierarchy over the complex field $\mathbb{C}$. Therefore, equation (9) was regarded as the integrable coupling of AKNS and itself, which was also called the bi-AKNS system.

In the past, general coupling form[18–23] by making use of Lie algebra A_1, A_2 and their subalgebras was presented in the following

$$
\begin{cases}
u_t=K(u),\\
v_t=S(u,v),
\end{cases} \qquad (95)
$$

and general super-integrable hierarchy form through use of Lie subalgebra $B(0,1)$ was proposed by Li,[24] Ma[25] as follows:

$$
\begin{cases}
u_t=K(u,v),\\
v_t=S(u,v).
\end{cases} \qquad (96)
$$

In this paper, we obtained the integrable coupling by making use of orthogonal Lie algebra $so(4,\mathbb{C})$, which possessed the form of Eq. (96), and is not super-integrable coupling. It is the first time that we have found the results.

Acknowledgments

Authors would like to thank Professor Ma Wen-Xiu for his useful discussion during his visit to Shandong University of Science and Technology of China.

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