Quantum mechanical operator realization of the Stirling numbers theory studied by virtue of the operator Hermite polynomials method*

Fan Hong-Yi(范洪义)† and Lou Sen-Yue(楼森岳)

Department of Physics, Ningbo University, Ningbo 315211, *China*

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Based on the operator Hermite polynomials method (OHPM), we study Stirling numbers in the context of quantum mechanics, i.e., we present operator realization of generating function formulas of Stirling numbers with some applications. As a by-product, we derive a summation formula involving both Stirling number and Hermite polynomials.

Keywords: operator Hermite polynomials method (OHPM), Stirling numbers

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1. Introduction

In mathematical physics and quantum statistics, the Stirling numbers are frequently used.[1] The Stirling numbers of the first kind, which count the number of permutations of m elements with k disjoint cycles, are the coefficients in the expansion

$$(x)_m = \sum_{k=0}^{m} s(m,k)\, x^k, \tag{1}$$

where $(x)_m$ (a Pochhammer symbol) denotes the falling factorial,

$$(x)_m \equiv x(x-1)\cdots(x-m). \tag{2}$$

The Stirling numbers of the second kind[2,3] count the number of ways to partition a set of m elements into k nonempty subsets, defined by

$$x^m = \sum_{k=0}^{m} S(m,k)\,(x)_k. \tag{3}$$

The purpose of this paper is to study the Stirling numbers in the context of quantum mechanics, i.e., we shall present the operator realization of generating functions of Stirling numbers by virtue of the operator Hermite polynomials method (OHPM).[4–6] In other words, we search for operator realization for deriving generating functions $S(m,k)$ and $s(m,k)$, which bear some physical meaning in the realization itself. We also study some combination of Stirling numbers with Hermite polynomials.

2. Brief review of OHPM

Hermite polynomials $H_n(x)$, as a special function, is very useful in quantum mechanics and mathematical physics. Physically, $H_n(x)$ is the eigenfunction of harmonic oscillator.[7] The essence of OHPM is to replace the usual Hermite polynomials $H_n(x)$ by $H_n(X)$, where $X = (a+a^\dagger)/\sqrt{2}$ is the coordinate operator, a and $a^\dagger$ are annihilation operator and creation operator, respectively, obeying $\left[a, a^\dagger\right] = 1$, and we have proposed two operator Hermite polynomials identities[8,9]

$$H_n(X) = 2^n : X^n :, \tag{4}$$

and

$$X^k = (2\mathrm{i})^{-k} : H_k(\mathrm{i}X) :, \tag{5}$$

where : : denotes normal ordering. In some preceding papers,[10,11] we have employed them to derive the binomial theorem involving Hermite polynomials

$$\sum_{k=0}^{m} \binom{m}{k} y^k H_{m-k}(x) = H_m\left(x+\frac{y}{2}\right). \tag{6}$$

As its application, we can deduce the following new formula:

$$\begin{aligned}(X+\lambda)^n &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k X^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k (2\mathrm{i})^{k-n} : H_{n-k}(\mathrm{i}X) : \\ &= (2\mathrm{i})^{-n} : H_n(\mathrm{i}X + \mathrm{i}\lambda) :. \end{aligned} \tag{7}$$

Furthermore, performing derivative for the right-hand side of Eq. (6)

$$\frac{\partial^n}{\partial (y/2)^n} H_m\left(x+\frac{y}{2}\right) = \frac{2^n m!}{(m-n)!} H_{m-n}\left(x+\frac{y}{2}\right). \tag{8}$$

On the other hand, performing derivative for the left-hand side of Eq. (6)

$$\frac{\partial^n}{\partial (y/2)^n} \sum_{k=0}^{m} \binom{m}{k} y^k H_{m-k}(x)$$

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†Corresponding author. E-mail: fhym@ustc.edu.cn

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$$= 2^n \sum_{k=0}^{m} \binom{m}{k} \binom{k}{n} y^{k-n} H_{m-k}(x). \tag{9}$$

By comparing Eq. (9) with Eq. (8), we obtain another generalized binomial theorem involving Hermite polynomials

$$\sum_{k=0}^{m} \binom{m}{k} \binom{k}{n} y^{k-n} H_{m-k}(x) = \frac{m!}{(m-n)!} H_{m-n}\left(x+\frac{y}{2}\right). \tag{10}$$

In Sections 3 and 4 we shall respectively employ the OHPM to examine generating functions of Stirling numbers with some applications.

3. Using OHPM to study generating function of the Stirling numbers of the second kind

Lemma 1 Let the power series expansion (in terms of x) of $\frac{1}{k!}(\mathrm{e}^x-1)^k$ be in the form

$$\frac{1}{k!}(\mathrm{e}^x-1)^k = \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} x^n, \tag{11}$$

then $A(n,k)$ is equal to

$$A(n,k) = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n. \tag{12}$$

Proof We use OHPM. Instead of directly considering $\frac{1}{k!}(\mathrm{e}^x-1)^k$, we examine $\frac{1}{k!}:\left(\mathrm{e}^{2X}-1\right)^k:$, where $X = (a+a^\dagger)/\sqrt{2}$ and $\left[a,a^\dagger\right]=1$. Then according to Eqs. (11) and (4) we have

$$\frac{1}{k!}:\left(\mathrm{e}^{2X}-1\right)^k: = :\sum_{n=0}^{\infty} \frac{A(n,k)}{n!} (2X)^n: = \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} H_n(X). \tag{13}$$

On the other hand

$$\frac{1}{k!}:\left(\mathrm{e}^{2X}-1\right)^k: = \frac{1}{k!}: \sum_{l=0}^{k} \binom{k}{l} \mathrm{e}^{2Xl} (-1)^{k-l}: . \tag{14}$$

Using the Baker–Hausdorff formula we have

$$:\mathrm{e}^{2Xl}: = \mathrm{e}^{2Xl-l^2} = \sum_{m=0}^{\infty} \frac{l^m}{m!} H_m(X). \tag{15}$$

Substituting Eq. (15) into Eq. (14) yields

$$\frac{1}{k!}:\left(\mathrm{e}^{2X}-1\right)^k: = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{\infty} \frac{l^m}{m!} H_m(X). \tag{16}$$

By comparing Eq. (16) with Eq. (13), we can identify

$$\frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{m=0}^{\infty} \frac{l^m}{m!} H_m(X) = \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} H_n(X), \tag{17}$$

which determines

$$A(n,k) = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n, \tag{18}$$

and converts

$$\frac{1}{k!}:\left(\mathrm{e}^{2X}-1\right)^k: =: \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} (2X)^n:$$

back to

$$\frac{1}{k!}(\mathrm{e}^x-1)^k = \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} x^n$$

has no obstacles, thus the Lemma is proved.

Next we should examine if $A(n,k)$ in Eq. (12) is just equal to the Stirling number of the second kind $S(n,k)$. This constitutes Theorem 1.

We still adopt quantum mechanical operator approach. By considering operator $N = a^\dagger a$, $a^\dagger|m\rangle = \sqrt{m+1}\,|m+1\rangle$, and $|m\rangle$ span Fock space $\sum_{m=0}^{\infty} |m\rangle\langle m| = 1$, we have

$$\begin{aligned} a^{\dagger k} a^k &= \sum_{m=0}^{\infty} a^{\dagger k} |m\rangle\langle m| a^k \\ &= \sum_{m=0}^{\infty} (m+1)\cdots(m+k)\,|m+k\rangle\langle m+k| \\ &= N(N-1)\cdots(N-k+1), \end{aligned} \tag{19}$$

then

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} N^k &= \mathrm{e}^{\lambda a^\dagger a} =: \mathrm{e}^{\left(\mathrm{e}^\lambda-1\right)a^\dagger a}: \\ &= \sum_{k=0}^{\infty} \frac{a^{\dagger k} a^k}{k!} \left(\mathrm{e}^\lambda-1\right)^k. \end{aligned} \tag{20}$$

Substituting

$$\left(\mathrm{e}^\lambda-1\right)^k = k! \sum_{n=0}^{\infty} \frac{A(n,k)}{n!} \lambda^n \tag{21}$$

and Eq. (19) into Eq. (20) yields

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} N^k = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} A(n,k) N(N-1)\cdots(N-k+1), \tag{22}$$

which leads to

$$N^k = \sum_{l=0}^{\infty} A(n,k) N(N-1)\cdots(N-k+1). \tag{23}$$

This is in the same form as the definition of the Stirling number of the second kind in Eq. (3). Hence,

$$A(n,k) = S(n,k). \tag{24}$$

The proof is completed. Thus, equation (23) reveals the meaning of Stirling numbers in the Fock space.

As a by-product, now we know from Eqs. (13) and (4) that

$$\sum_{n=0}^{\infty} \frac{S(n,k)}{n!} H_n(X)$$

$$= \sum_{n=0}^{\infty} \frac{S(n,k)}{n!} : 2^n X^n :$$

$$= : \frac{1}{k!} \left(\mathrm{e}^{2X} - 1\right)^k : = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{e}^{2Xl-l^2}. \quad (25)$$

Replacing $X \to x$, we have a new summation formula involving both Stirling number and Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{S(n,k)}{n!} H_n(x) = \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \mathrm{e}^{2xl-l^2}. \quad (26)$$

This new formula is useful, because it provides opportunity of employing Hermite polynomials' property to deduce characters of the Stirling numbers, for example,

$$S(n+1,k) = kS(n,k) + S(n,k-1). \quad (27)$$

We also know

$$(1+t)^N = \mathrm{e}^{a^\dagger a \ln(1+t)} =: \mathrm{e}^{ta^\dagger a} : = \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{\dagger n} a^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} N(N-1)(N-2)\cdots(N-n+1). \quad (28)$$

Letting $N \to x$, we have

$$(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} x(x-1)(x-2)\cdots(x-n+1), \quad (29)$$

which is another kind of Newton binomial theorem.

Theorem 1 Let the power series expansion (in terms of t) of $\frac{1}{k!}\log^k(1+t)$ be in the form

$$\frac{1}{k!} \ln^k(1+t) = \sum_{n=0}^{\infty} \frac{s(n,k)}{n!} t^n, \quad (30)$$

then $s(n,k)$ is the Stirling number of the first kind.

Proof Instead of directly considering $\frac{1}{k!}\log^k(1+t)$, we examine

$$\sum_{k=0}^{\infty} \frac{\left(a^\dagger a\right)^k}{k!} \ln^k(1+t)$$

$$= \exp\left[a^\dagger a \ln(1+t)\right] = (1+t)^N$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} N(N-1)(N-2)\cdots(N-n+1). \quad (31)$$

On the other hand, according to Eq. (30)

$$\sum_{k=0}^{\infty} \frac{\left(a^\dagger a\right)^k}{k!} \ln^k(1+t) = \sum_{k=0}^{\infty} \left(a^\dagger a\right)^k \sum_{n=0}^{\infty} \frac{s(n,k)}{n!} t^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \left(a^\dagger a\right)^k s(n,k). \quad (32)$$

Comparing Eq. (30) with Eq. (31) leads to

$$N(N-1)(N-2)\cdots(N-n+1) = \sum_{k=0}^{\infty} s(n,k) N^k. \quad (33)$$

This is just the definition of Stirling number of the first kind in Eq. (1), so equation (28) reflects the generating function of $s(n,k)$.

4. Application of Eq. (11)

Using Eqs. (24) and (4), we consider

$$\frac{1}{k!} : \left(\mathrm{e}^{4X^2} - 1\right)^k : = : \sum_{n=0}^{\infty} \frac{S(n,k)}{n!} \left(2^2 X^2\right)^n :$$

$$= \sum_{n=0}^{\infty} \frac{S(n,k)}{n!} H_{2n}(X). \quad (34)$$

On the other hand,

$$\frac{1}{k!} : \left(\mathrm{e}^{4X^2} - 1\right)^k : = \frac{1}{k!} : \sum_{l=0}^{k} \binom{k}{l} \mathrm{e}^{4X^2 l} (-1)^{k-l} : . \quad (35)$$

Using the technique of integration within ordered product (IWOP) of operators[11,12] we can obtain

$$: \mathrm{e}^{4X^2 l} : = \sqrt{\frac{1}{1+4l}}\, \mathrm{e}^{4X^2 l/(1+4l)}. \quad (36)$$

By substituting Eq. (36) into Eq. (35) and comparing with Eq. (34), we have

$$\frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sqrt{\frac{1}{1+4l}}\, \mathrm{e}^{\frac{4l}{1+4l}X^2}$$

$$= \sum_{n=0}^{\infty} \frac{S(n,k)}{n!} H_{2n}(X). \quad (37)$$

Comparing Eq. (37) with Eq. (12) we obtain the generating function formula of even-order Hermite polynomials

$$\sqrt{\frac{1}{1+4l}}\, \mathrm{e}^{\frac{4l}{1+4l}X^2} = \sum_{m=0}^{\infty} \frac{l^m}{m!} H_{2m}(X), \quad (38)$$

we can identify $\sum_{m=0}^{\infty} \frac{l^m}{m!} H_{2m}(X)|0\rangle$ as a single-mode squeezed state, because using Eq. (4) we have

$$H_{2m}(X)|0\rangle = 2^{2m} : X^{2m} : |0\rangle = 2^m a^{\dagger 2m} |0\rangle, \quad (39)$$

so

$$\sum_{m=0}^{\infty} \frac{l^m}{m!} H_{2m}(X)|0\rangle = \sum_{m=0}^{\infty} \frac{l^m}{m!} 2^m a^{\dagger 2m} |0\rangle = \mathrm{e}^{2la^{\dagger 2}} |0\rangle \quad (40)$$

is really a squeezed state.

5. Conclusion

In summary, for the first time we use OHPM to study the Stirling numbers theory, we have found quantum mechanical operator realization for generating functions $S(m,k)$ and $s(m,k)$, which bear some physical meaning in the realization itself. As a by-product, we derived a summation formula involving both Stirling number and Hermite polynomials.

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