Stochastic stability of the derivative unscented Kalman filter*

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(Received 28 October 2014; revised manuscript received 15 January 2015; published online 29 May 2015)

This is the second of two consecutive papers focusing on the filtering algorithm for a nonlinear stochastic discrete-time system with linear system state equation. The first paper established a derivative unscented Kalman filter (DUKF) to eliminate the redundant computational load of the unscented Kalman filter (UKF) due to the use of unscented transformation (UT) in the prediction process. The present paper studies the error behavior of the DUKF using the boundedness property of stochastic processes. It is proved that the estimation error of the DUKF remains bounded if the system satisfies certain conditions. Furthermore, it is shown that the design of the measurement noise covariance matrix plays an important role in improvement of the algorithm stability. The DUKF can be significantly stabilized by adding small quantities to the measurement noise covariance matrix in the presence of large initial error. Simulation results demonstrate the effectiveness of the proposed technique.

Keywords: nonlinear stochastic system, stochastic process, unscented Kalman filter, stochastic stability

PACS: 02.30.Yy, 02.50.Ey, 02.50.Fz **DOI:** 10.1088/1674-1056/24/7/070202

1. Introduction

As a relatively new nonlinear filtering algorithm, the unscented Kalman filter (UKF) is an improvement to the extended Kalman filter (EKF).[1,2] It uses a finite number of sigma points to propagate the probability of state distribution through the nonlinear dynamics of a system, thus reducing linearization errors involved in the EKF. Consequently, the UKF has received great attention in many areas such as control theory, aerospace navigation, eye tracking, and information fusion.[3–9] However, in the case of nonlinear stochastic systems with linear system state equation, such as the target tracking system[5] and the tightly coupled INS/GPS integrated system,[10] the UKF causes a great amount of redundant computation in the prediction process. This is because the system state equation is linear while it is still used for each of the chosen sigma points in turn to yield a transformed sample to calculate predicted mean and covariance.

To overcome this limitation of the UKF, a derivative unscented Kalman filter (DUKF) was reported in our previous work[1] to reduce the computational complexity by incorporating the concise form of the original Kalman filter (KF) in the prediction process. The DUKF avoids the extra computational load involved in the implementation of the UKF and also inherits the excellent properties of the UKF in dealing with nonlinear filtering problems. However, similar to the UKF, the DUKF is sensitive to the initial error, that is, the DUKF may diverge if the initial state is not sufficiently close to the actual state.

Studies have been dedicated to the stability analysis of the UKF. Based on the new formulation of the first-order linearization technique developed by Boutayeb *et al.*,[11,12] Xiong *et al.* proved that, if the process noise covariance matrix is properly chosen, the UKF for a nonlinear system with linear measurements is exponentially bounded in mean-squared estimation error.[13] However, this result is not generic since the nonlinear system considered is a quite simple case in the filed of nonlinear filtering. By taking into account the approximation error of unscented transformation (UT) in the calculation of the cross-correlation covariance matrix between the predicted state and measurement, Xiong *et al.* extended the result to a general nonlinear stochastic system with nonlinear measurements.[6,14] However, this study requires a strong coupling relationship between the designed covariance matrices of the process and measurement noises and the sufficient conditions for the designed noise covariance matrices to guarantee the boundedness of estimation error are totally given by a sophisticated construction, thus leading to the difficulty in using the achieved result. Xu *et al.* also studied the stability analysis of the continuous-time UKF.[15] This study proves that the estimation error will be bounded if the system satisfies a detectability condition and both the initial estimation error and the disturbing noise terms are small enough.[16] However, in the implementation of the UKF, it is difficult to keep the initial estimation error small. Just recently, Li and Xia analyzed the error behavior of the UKF for a nonlinear stochastic system with intermittent measurement.[17] However, this study only provides some indications for the impact of intermittent obser-

*Project supported by the National Natural Science Foundation of China (Grant No. 61174193) and the Doctorate Foundation of Northwestern Polytechnical University, China (Grant No. CX201409).

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vations on the stochastic stability of the UKF, without concrete results. To the best of our knowledge, there has been limited research on the analysis of the error dynamics of the DUKF in the presence of large initial error.

This paper focuses on the stochastic stability of the DUKF with poor initial conditions. Based on some standard results about the boundedness of stochastic processes and using the second method of Lyapunov, it is proved that the estimation error of the DUKF remains bounded under certain conditions. Motivated by the assumption that the approximation error of UT in calculating the cross-correlation covariance matrix between the predicted state and measurement is small enough, the coupling relationship between the process noise covariance matrix and designed measurement noise covariance matrix is removed from the sufficient conditions to guarantee the stochastic stability of the DUKF. Furthermore, the impact of the measurement noise covariance matrix on the error behavior is analyzed. It is demonstrated that by adding small quantities to the measurement noise covariance matrix, the stability of the DUKF can be achieved while maintaining certain accuracy. The above results are verified through numerical simulations on three example systems in the presence of large initial error.

Throughout this paper, $\|\cdot\|$ denotes the Euclidian norm of real vectors or the spectral norm of real matrices, $E(\boldsymbol{x})$ is the expectation value of $\boldsymbol{x}$, and $E(\boldsymbol{x}|\boldsymbol{y})$ is the expectation value of $\boldsymbol{x}$ conditional on $\boldsymbol{y}$.

2. DUKF

In order to conduct the stability analysis, let us briefly review the concept of the DUKF at first.

Consider the nonlinear discrete-time system represented by

$$\boldsymbol{x}_k = \boldsymbol{\Phi}_{k/k-1}\boldsymbol{x}_{k-1} + \boldsymbol{w}_k, \tag{1}$$

$$\boldsymbol{z}_k = h(\boldsymbol{x}_k) + \boldsymbol{v}_k, \tag{2}$$

where $\boldsymbol{x}_k \in \boldsymbol{R}^n$ and $\boldsymbol{z}_k \in \boldsymbol{R}^m$ denote the state and measurement vector at time step k, $\boldsymbol{\Phi}_{k/k-1}$ is the discrete state transition matrix, $h(\cdot)$ is the nonlinear function describing the measurement model and is assumed to be continuously differentiable, and $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are the uncorrelated zero-mean Gaussian white noise processes with covariances

$$E\left[\boldsymbol{w}_k\boldsymbol{w}_k^{\mathrm{T}}\right] = \boldsymbol{Q}_k, \quad E\left[\boldsymbol{v}_k\boldsymbol{v}_k^{\mathrm{T}}\right] = \boldsymbol{R}_k. \tag{3}$$

The DUKF deterministically chooses a set of weighted sigma points to capture the specific information of the predicted mean and covariance rather than the priori ones. In terms of the prediction process, the DUKF has the same form as the KF, thus avoiding the extra computational load of the UKF caused by the UT. In the update process, the predicted measurement, measurement covariance, and cross-correlation covariance are calculated by the statistics of chosen sigma points and transformed sigma points yielded through the nonlinear measurement equation. Thus, the DUKF provides a filtering solution to retain the nonlinearity of the nonlinear measurement equation with at least second-order accuracy.

The procedure for implementing the DUKF can be summarized as follows.[1]

Step 1 Give the state estimate $\hat{\boldsymbol{x}}_{k-1}$ and the error covariance matrix $\hat{\boldsymbol{P}}_{k-1}$.

Step 2 Prediction As the system state equation is linear, the predicted mean and covariance are performed with the same equations as the KF,

$$\hat{\boldsymbol{x}}_{k/k-1} = \boldsymbol{\Phi}_{k/k-1}\hat{x}_{k-1}, \tag{4}$$

$$\hat{\boldsymbol{P}}_{k/k-1} = \boldsymbol{\Phi}_{k/k-1}\hat{P}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}} + Q_k. \tag{5}$$

Step 3 Sigma points selection A set of weighted sigma points are selected from the predicted mean and covariance of the state. These sigma points are obtained by

$$\begin{cases} \boldsymbol{\xi}_{i,k/k-1} = \hat{\boldsymbol{x}}_{k/k-1}, \quad i = 0, \\ \boldsymbol{\xi}_{i,k/k-1} = \hat{\boldsymbol{x}}_{k/k-1} + a\left(\sqrt{n\hat{\boldsymbol{P}}_{k/k-1}}\right)_i, \\ \quad i = 1,2,\ldots,n, \\ \boldsymbol{\xi}_{i,k/k-1} = \hat{\boldsymbol{x}}_{k/k-1} - a\left(\sqrt{n\hat{\boldsymbol{P}}_{k/k-1}}\right)_i, \\ \quad i = n+1, n+2, \ldots, 2n, \end{cases} \tag{6}$$

where $a \in \boldsymbol{R}$ is a tuning parameter denoting the spread of the sigma points around $\hat{\boldsymbol{x}}_{k/k-1}$ and is generally set as a small positive value, and $\left(\sqrt{n\hat{P}_{k/k-1}}\right)_i$ is the i-th column of the matrix square root of $n\hat{\boldsymbol{P}}_{k/k-1}$.

Step 4 Update The transformed sigma points yielded through the nonlinear measurement model are

$$\gamma_{i,k/k-1} = h(\boldsymbol{\xi}_{i,k/k-1}), \ i = 0, 1, \ldots, 2n. \tag{7}$$

The predicted measurement and the measurement covariance are given by

$$\hat{\boldsymbol{z}}_{k/k-1} = \sum_{i=0}^{2n} \omega_i \gamma_{i,k/k-1}, \tag{8}$$

$$\begin{aligned} \hat{\boldsymbol{P}}_{\hat{\boldsymbol{z}}_{k/k-1}} = & \sum_{i=0}^{2n} \omega_i \left(\gamma_{i,k/k-1} - \hat{\boldsymbol{z}}_{k/k-1}\right) \\ & \times \left(\gamma_{i,k/k-1} - \hat{\boldsymbol{z}}_{k/k-1}\right)^{\mathrm{T}} + \boldsymbol{R}_k, \end{aligned} \tag{9}$$

and the cross-correlation covariance matrix between the predicted state and measurement is expressed as

$$\begin{aligned} \hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}} = & \sum_{i=0}^{2n} \omega_i \left(\boldsymbol{\xi}_{i,k/k-1} - \hat{\boldsymbol{x}}_{k/k-1}\right) \\ & \times \left(\gamma_{i,k/k-1} - \hat{\boldsymbol{z}}_{k/k-1}\right)^{\mathrm{T}}, \end{aligned} \tag{10}$$

where

$$\begin{cases} \omega_i = 1 - \dfrac{1}{a^2}, & i = 0, \\ \omega_i = \dfrac{1}{2na^2}, & i = 1,2,\ldots,2n. \end{cases}$$

The Kalman gain is determined as

$$\boldsymbol{K}_k = \hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}} \hat{\boldsymbol{P}}^{-1}_{\hat{\boldsymbol{z}}_{k/k-1}}. \tag{11}$$

Then, the state $\hat{\boldsymbol{x}}_k$ and the corresponding error covariance matrix $\boldsymbol{P}_k$ can be updated by

$$\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k/k-1}), \tag{12}$$

$$\hat{\boldsymbol{P}}_k = \hat{\boldsymbol{P}}_{k/k-1} - \boldsymbol{K}_k \hat{\boldsymbol{P}}_{\hat{\boldsymbol{z}}_{k/k-1}} \boldsymbol{K}_k^{\mathrm{T}}. \tag{13}$$

Step 5 Repeat Steps 1 to 4 for the next sample.

3. Stochastic stability analysis of DUKF

3.1. Instrumental matrix and extra positive definite matrix

Define the estimation error and prediction error by

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k, \tag{14}$$

$$\tilde{\boldsymbol{x}}_{k/k-1} = \boldsymbol{x}_k - \hat{x}_{k/k-1}. \tag{15}$$

Substituting Eqs. (1) and (4) into Eq. (15) gives

$$\begin{aligned} \tilde{\boldsymbol{x}}_{k/k-1} &= \boldsymbol{\Phi}_{k/k-1}(\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k-1}) + \boldsymbol{w}_k \\ &= \boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1} + \boldsymbol{w}_k. \end{aligned} \tag{16}$$

Define the innovation vector as

$$\tilde{\boldsymbol{z}}_k = z_k - \hat{\boldsymbol{z}}_{k/k-1}. \tag{17}$$

Expanding $\boldsymbol{z}_k$ as a Taylor series about $\hat{x}_{k/k-1}$ yields

$$\begin{aligned} \boldsymbol{z}_k &= h(\hat{\boldsymbol{x}}_{k/k-1}) + \nabla h(\hat{x}_{k/k-1})\tilde{\boldsymbol{x}}_{k/k-1} \\ &\quad + \nabla^2 h(\hat{x}_{k/k-1})\tilde{\boldsymbol{x}}^2_{k/k-1} + \cdots + \boldsymbol{v}_k, \end{aligned} \tag{18}$$

where the i-th term in the Taylor series for $h(\cdot)$ is given by

$$\nabla^i h(\hat{\boldsymbol{x}}_{k/k-1})\tilde{\boldsymbol{x}}^i_{k/k-1} = \frac{1}{i!}\left(\sum_{j=1}^{n} \tilde{x}_j \frac{\partial}{\partial x_j}\right)^i h(\boldsymbol{x})\bigg|_{\boldsymbol{x}=\hat{x}_{k/k-1}},$$

where x_j denotes the j-th component of $\boldsymbol{x}$.

Expanding $\hat{\boldsymbol{z}}_{k/k-1}$ given by Eq. (8) by a Taylor series yields

$$\begin{aligned} \hat{\boldsymbol{z}}_{k/k-1} &= \left(1 - \frac{1}{a^2}\right) h(\hat{\boldsymbol{x}}_{k/k-1}) \\ &\quad + \frac{1}{2na^2}\sum_{i=1}^{n} h\left[\hat{\boldsymbol{x}}_{k/k-1} + a\left(\sqrt{n\hat{\boldsymbol{P}}_{k/k-1}}\right)_i\right] \\ &\quad + \frac{1}{2na^2}\sum_{i=n+1}^{2n} h\left[\hat{x}_{k/k-1} - a\left(\sqrt{n\hat{\boldsymbol{P}}_{k/k-1}}\right)_{i-n}\right] \\ &= h(\hat{\boldsymbol{x}}_{k/k-1}) + \frac{1}{2}\nabla^2 h(\hat{\boldsymbol{x}}_{k/k-1})\hat{\boldsymbol{P}}_{k/k-1} + \cdots. \end{aligned} \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (17), the innovation vector $\tilde{\boldsymbol{z}}_k$ becomes

$$\tilde{\boldsymbol{z}}_k = \boldsymbol{H}_k\tilde{\boldsymbol{x}}_{k/k-1} + \Delta(\tilde{\boldsymbol{x}}_{k/k-1}) + \boldsymbol{v}_k, \tag{20}$$

where $\boldsymbol{H}_k = \partial h(\boldsymbol{x})/\partial \boldsymbol{x}|_{\boldsymbol{x}=\hat{x}_{k/k-1}}$, and $\Delta(\tilde{x}_{k/k-1})$ denotes the second- and higher-order moments in the Taylor series.

In order to simplify the error expression, the unknown instrumental diagonal matrices $\boldsymbol{\alpha}_k = \mathrm{diag}(\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{m,k})$ are introduced to model errors due to the first-order linearization,[11,12] leading to the following exact equality:

$$\tilde{\boldsymbol{z}}_k = \boldsymbol{\alpha}_k\boldsymbol{H}_k\tilde{\boldsymbol{x}}_{k/k-1} + \boldsymbol{v}_k. \tag{21}$$

The real cross-correlation covariance matrix between the predicted state and measurement is

$$\begin{aligned} \boldsymbol{P}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}} &= \boldsymbol{E}\left[\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k/k-1}\right)\left(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k/k-1}\right)^{\mathrm{T}}\right] \\ &= \boldsymbol{E}\left[\tilde{x}_{k/k-1}\left(\boldsymbol{\alpha}_k\boldsymbol{H}_k\tilde{\boldsymbol{x}}_{k/k-1} + \boldsymbol{v}_k\right)^{\mathrm{T}}\right] \\ &= \hat{\boldsymbol{P}}_{k/k-1}\left(\boldsymbol{\alpha}_k\boldsymbol{H}_k\right)^{\mathrm{T}}. \end{aligned} \tag{22}$$

Similar to Eq. (21), the unknown instrumental matrices $\boldsymbol{\gamma}_k \in \boldsymbol{R}^{m\times m}$ are introduced to describe the approximation error between $\boldsymbol{P}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}$ and $\hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}$ such that

$$\hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}} = \boldsymbol{P}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}\boldsymbol{\gamma}_k = \hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k. \tag{23}$$

From the presentation of $\tilde{\boldsymbol{z}}_k$ given by Eq. (21), the real measurement covariance matrix can be expressed as

$$\begin{aligned} \boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} &= \boldsymbol{E}\left[\left(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k/k-1}\right)\left(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k/k-1}\right)^{\mathrm{T}}\right] \\ &= \boldsymbol{E}\left[\left(\boldsymbol{\alpha}_k\boldsymbol{H}_k\tilde{\boldsymbol{x}}_{k/k-1} + \boldsymbol{v}_k\right)\left(\boldsymbol{\alpha}_k\boldsymbol{H}_k\tilde{\boldsymbol{x}}_{k/k-1} + \boldsymbol{v}_k\right)^{\mathrm{T}}\right] \\ &= \boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k + \boldsymbol{R}_k. \end{aligned} \tag{24}$$

Define

$$\Delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} = \hat{\boldsymbol{P}}_{\hat{\boldsymbol{z}}_{k/k-1}} - \boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}},$$

$$\delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} = \boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k - \boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k.$$

A slight modification is made by adding an extra positive definite matrix $\Delta\boldsymbol{R}_k$ to the calculated measurement covariance matrix $\hat{P}_{\hat{\boldsymbol{z}}_{k/k-1}}$, where the role of $\Delta\boldsymbol{R}_k$ to guarantee the stability of the DUKF is shown in Section 3.2. Then the calculated measurement covariance matrix may be written as

$$\begin{aligned} \hat{\boldsymbol{P}}_{\hat{\boldsymbol{z}}_{k/k-1}} &= \sum_{i=0}^{2n}\omega_i\left(\boldsymbol{\gamma}_{i,k/k-1} - \hat{\boldsymbol{z}}_{k/k-1}\right)\left(\boldsymbol{\gamma}_{i,k/k-1} - \hat{\boldsymbol{z}}_{k/k-1}\right)^{\mathrm{T}} \\ &\quad + \boldsymbol{R}_k + \Delta\boldsymbol{R}_k \\ &= \boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k \\ &\quad + \Delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} + \delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} + \boldsymbol{R}_k + \Delta\boldsymbol{R}_k \\ &= \boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k + \boldsymbol{R}_k^*, \end{aligned} \tag{25}$$

where $\boldsymbol{R}_k^* = \Delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} + \delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}} + \boldsymbol{R}_k + \Delta\boldsymbol{R}_k$.

Thus, from Eqs. (23) and (25), it is verified that

$$\begin{aligned} \boldsymbol{K}_k &= \hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}\hat{\boldsymbol{P}}^{-1}_{\hat{\boldsymbol{z}}_{k/k-1}} \\ &= \hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k \end{aligned}$$

$$\times \left(\gamma_k^{\mathrm{T}} \alpha_k \boldsymbol{H}_k \hat{\boldsymbol{P}}_{k/k-1} \boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k + \boldsymbol{R}_k^*\right)^{-1}. \quad (26)$$

Substituting Eq. (26) into Eq. (13) and applying the matrix inversion lemma yield

$$\hat{\boldsymbol{P}}_k^{-1} = \hat{\boldsymbol{P}}_{k/k-1}^{-1} + \boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k \left(\boldsymbol{R}_k^*\right)^{-1} \gamma_k^{\mathrm{T}} \alpha_k \boldsymbol{H}_k. \quad (27)$$

Based on the matrix inversion lemma and Eq. (27), equation (26) can also be rewritten as

$$\begin{aligned} \boldsymbol{K}_k &= \hat{\boldsymbol{P}}_{k/k-1} \left(\hat{\boldsymbol{P}}_k^{-1} - \boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k \left(\boldsymbol{R}_k^*\right)^{-1} \gamma_k^{\mathrm{T}} \alpha_k \boldsymbol{H}_k\right) \\ &\quad \times \hat{\boldsymbol{P}}_k \boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k \left(\boldsymbol{R}_k^*\right)^{-1} \\ &= \hat{\boldsymbol{P}}_k \boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k \left(\boldsymbol{R}_k^*\right)^{-1}. \end{aligned} \quad (28)$$

3.2. Boundedness of the estimation error for DUKF

In order to analyze the dynamics of the estimation error (14), we introduce the following standard result for the boundedness of stochastic processes.

Lemma 1 Assume that ς_k is a stochastic process, and there is a stochastic process $V_k(\varsigma_k)$ together with real numbers $v_{\min}$, $v_{\max} > 0$, $\mu > 0$, and $0 < \lambda \leq 1$ such that

$$v_{\min} \|\varsigma_k\|^2 \leq \boldsymbol{V}_k(\varsigma_k) \leq v_{\max} \|\varsigma_k\|^2, \quad (29)$$

and

$$E\left[V_k(\varsigma_k) | \varsigma_{k-1}\right] - V_{k-1}(\varsigma_{k-1}) \leq \mu - \lambda V_{k-1}(\varsigma_{k-1}). \quad (30)$$

Then, the stochastic process ς_k is exponentially bounded in mean square, i.e.,

$$\begin{aligned} E\left\{\|\varsigma_k\|^2\right\} \leq \frac{v_{\max}}{v_{\min}} E\left\{\|\varsigma_0\|^2\right\} (1-\lambda)^k \\ + \frac{\mu}{v_{\min}} \sum_{i=1}^{k-1} (1-\lambda)^i. \end{aligned} \quad (31)$$

This lemma can be found in Ref. [18].

Theorem 1 Consider the nonlinear stochastic systems given by Eqs. (1) and (2) and the modified DUKF described by Eqs. (4)–(8), (10)–(13), and (25). Let the following assumptions hold for every $k \geq 0$.

(i) There exist real constants $\phi_{\min}$, $\phi_{\max}$, $c_{\min}$, $c_{\max}$, $h_{\min}$, $h_{\max}$, $\eta_{\max}$, $\kappa_{\max} > 0$ such that the following bounds are fulfilled:

$$\phi_{\min} \leq \left\|\boldsymbol{\Phi}_{k/k-1}\right\| \leq \phi_{\max}, \quad (32)$$

$$c_{\min} \leq \left\|\gamma_k^{\mathrm{T}} \boldsymbol{\alpha}_k H_k\right\| \leq c_{\max}, \quad (33)$$

$$h_{\min} \leq \left\|\boldsymbol{\alpha}_k \boldsymbol{H}_k\right\| \leq h_{\max}, \quad (34)$$

$$\max_{i,j=1,2,\ldots,n} \left[\left|\left(\boldsymbol{H}_k^{\mathrm{T}} \alpha_k \gamma_k\right)_{i,j}\right|\right] \leq \eta_{\max}, \quad (35)$$

$$\max_{i,j=1,2,\ldots,n} \left[\left|\left(\boldsymbol{\alpha}_k \boldsymbol{H}_k\right)_{i,j}\right|\right] \leq \kappa_{\max}, \quad (36)$$

where $h_{\min} \leq c_{\max}$.

(ii) There exist real constants $r_{\min}^*$, $r_{\max}^*$, $r_{\max}$, $q_{\min}$, $q_{\max}$, $p_{\min}$, $p_{\max} > 0$ such that the following bounds on various matrices are fulfilled:

$$r_{\min}^* \boldsymbol{I} \leq \boldsymbol{R}_k^* \leq r_{\max}^* \boldsymbol{I}, \quad (37)$$

$$\boldsymbol{R}_k \leq r_{\max} \boldsymbol{I}, \quad (38)$$

$$p_{\min} \boldsymbol{I} \leq \hat{\boldsymbol{P}}_k \leq p_{\max} \boldsymbol{I}, \quad (39)$$

$$q_{\min} \boldsymbol{I} \leq \boldsymbol{Q}_k \leq q_{\max} \boldsymbol{I}. \quad (40)$$

(iii) There exists a real constant $\varepsilon_{\max} > 0$ such that the following matrix norm is bounded via:

$$\left\|\gamma_k^{\mathrm{T}} \boldsymbol{\alpha}_k H_k - \boldsymbol{\alpha}_k H_k\right\| \leq \varepsilon_{\max}, \quad (41)$$

where $\varepsilon_{\max}^2 \leq r_{\min}^* / \phi_{\max}^2 \left(p_{\max} + p_{\max}^2 \phi_{\max}^2 / q_{\min}\right)$. Then the estimation error $\tilde{\boldsymbol{x}}_k$ is exponentially bounded in mean square.

Proof Define a Lyapunov function

$$V_k(\tilde{\boldsymbol{x}}_k) = \tilde{x}_k^{\mathrm{T}} \hat{\boldsymbol{P}}_k^{-1} \tilde{\boldsymbol{x}}_k. \quad (42)$$

From Eq. (39) it follows that

$$\frac{1}{p_{\max}} \|\tilde{\boldsymbol{x}}_k\|^2 \leq \boldsymbol{V}_k(\tilde{\boldsymbol{x}}_k) \leq \frac{1}{p_{\min}} \|\tilde{\boldsymbol{x}}_k\|^2. \quad (43)$$

To satisfy the requirement of Lemma 1, it requires an upper bound on $E\left[V_k(\tilde{\boldsymbol{x}}_k) | \tilde{x}_{k-1}\right] - V_{k-1}(\tilde{\boldsymbol{x}}_{k-1})$ as given by Eq. (30).

Substituting Eq. (12) into Eq. (14) and using Eqs. (15)–(17) and (21), it can be verified that

$$\begin{aligned} \tilde{\boldsymbol{x}}_k &= \boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k/k-1} - \boldsymbol{K}_k(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k/k-1}) \\ &= \tilde{\boldsymbol{x}}_{k/k-1} - \boldsymbol{K}_k \tilde{z}_k \\ &= \boldsymbol{\Phi}_{k/k-1} \tilde{\boldsymbol{x}}_{k-1} + \boldsymbol{w}_k - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k \tilde{\boldsymbol{x}}_{k/k-1} - \boldsymbol{K}_k \boldsymbol{v}_k \\ &= \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k\right) \boldsymbol{\Phi}_{k/k-1} \tilde{\boldsymbol{x}}_{k-1} \\ &\quad + \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k\right) \boldsymbol{w}_k - \boldsymbol{K}_k \boldsymbol{v}_k. \end{aligned} \quad (44)$$

Due to the fact that $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are the uncorrelated Gaussian white noise processes and $\tilde{x}_{k-1}$ is uncorrelated to $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$, substituting Eq. (44) into Eq. (30) and taking the conditional expectation yield

$$\begin{aligned} &E[V_k(\tilde{\boldsymbol{x}}_k) | \tilde{\boldsymbol{x}}_{k-1}] \\ &= E\{[(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{\Phi}_{k/k-1} \tilde{\boldsymbol{x}}_{k-1} \\ &\quad + (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{w}_k - \boldsymbol{K}_k \boldsymbol{v}_k]^{\mathrm{T}} \hat{\boldsymbol{P}}_k^{-1} \\ &\quad \times [(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{\Phi}_{k/k-1} \tilde{x}_{k-1} \\ &\quad + (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{w}_k - \boldsymbol{K}_k \boldsymbol{v}_k] | \tilde{\boldsymbol{x}}_{k-1}\} \\ &= E\{[(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{\Phi}_{k/k-1} \tilde{\boldsymbol{x}}_{k-1}]^{\mathrm{T}} \\ &\quad \times \hat{\boldsymbol{P}}_k^{-1} [(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{\Phi}_{k/k-1} \tilde{\boldsymbol{x}}_{k-1}] \\ &\quad + [(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{w}_k]^{\mathrm{T}} \hat{\boldsymbol{P}}_k^{-1} [(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{\alpha}_k \boldsymbol{H}_k) \boldsymbol{w}_k] \\ &\quad + (\boldsymbol{K}_k \boldsymbol{v}_k)^{\mathrm{T}} \hat{\boldsymbol{P}}_k^{-1} (\boldsymbol{K}_k \boldsymbol{v}_k) | \tilde{\boldsymbol{x}}_{k-1}\}. \end{aligned} \quad (45)$$

Considering the property of the condition expectation $E(\tilde{\boldsymbol{x}}_{k-1} | \tilde{\boldsymbol{x}}_{k-1}) = \tilde{\boldsymbol{x}}_{k-1}$, equation (45) can be further expressed as

$$E[V_k(\tilde{\boldsymbol{x}}_k) | \tilde{\boldsymbol{x}}_{k-1}]$$

$$
\begin{aligned}
&= [(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\quad\times[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]\\
&\quad+E\{[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]\}\\
&\quad+E[(\boldsymbol{K}_k\boldsymbol{v}_k)^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}(\boldsymbol{K}_k\boldsymbol{v}_k)]. \qquad (46)
\end{aligned}
$$

Substituting Eq. (28) into the first item of Eq. (46) gives

$$
\begin{aligned}
&[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\times[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]\\
=\ &(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{x}_{k-1})\\
&-(\boldsymbol{\Phi}_{k/k-1}\tilde{x}_{k-1})^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k)(\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\alpha}_k\boldsymbol{H}_k)(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})\\
&+(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\hat{P}_k\\
&\times(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k)(\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\alpha}_k\boldsymbol{H}_k)(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})\\
&-(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{R}_k^*)^{-1}\\
&\times(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}). \qquad (47)
\end{aligned}
$$

Substituting Eq. (27) into the first item of Eq. (47) yields

$$
\begin{aligned}
&(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})\\
=\ &(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_{k/k-1}^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})+(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\\
&\times(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k)(\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}). \qquad (48)
\end{aligned}
$$

From Eqs. (26) and (28), it can be verified that

$$
\begin{aligned}
&(\boldsymbol{R}_k^*)^{-1}-(\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\hat{\boldsymbol{P}}_k(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k)(\boldsymbol{R}_k^*)^{-1}\\
&=(\boldsymbol{R}_k^*)^{-1}[\boldsymbol{I}-(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{K}_k]\\
&=(\boldsymbol{R}_k^*)^{-1}[\boldsymbol{I}-(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k\\
&\quad\times(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}]\\
&=(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}>\boldsymbol{0}. \qquad (49)
\end{aligned}
$$

Then, by substituting Eqs. (48) and (49) into Eq. (47), we obtain

$$
\begin{aligned}
&[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\times[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]\\
=\ &(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_{k/k-1}^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})+(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\\
&\times\{[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]^{\mathrm{T}}(\boldsymbol{R}_k^*)^{-1}[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]\\
&-(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}\\
&\times(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}). \qquad (50)
\end{aligned}
$$

Substituting Eq. (5) into $(\boldsymbol{\Phi}_{k/k-1}\tilde{x}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_{k/k-1}^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})$ and using the matrix inversion lemma, we obtain

$$
\begin{aligned}
&(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_{k/k-1}^{-1}(\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1})\\
&=\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\hat{\boldsymbol{P}}_{k-1}^{-1}\tilde{x}_{k-1}-\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}[\hat{\boldsymbol{P}}_{k-1}+\hat{\boldsymbol{P}}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}\\
&\quad\times\boldsymbol{Q}_k^{-1}\boldsymbol{\Phi}_{k/k-1}\hat{\boldsymbol{P}}_{k-1}]^{-1}\tilde{\boldsymbol{x}}_{k-1}. \qquad (51)
\end{aligned}
$$

Considering Eq. (51), equation (50) can be expressed as

$$
\begin{aligned}
&[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\times[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\tilde{\boldsymbol{x}}_{k-1}]\\
=\ &\boldsymbol{V}_{k-1}(\tilde{\boldsymbol{x}}_{k-1})+\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\{\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]^{\mathrm{T}}(\boldsymbol{R}_k^*)^{-1}\\
&\times[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]\boldsymbol{\Phi}_{k/k-1}\\
&-\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\\
&\times\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\\
&-[\hat{\boldsymbol{P}}_{k-1}+\hat{\boldsymbol{P}}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{\Phi}_{k/k-1}\hat{\boldsymbol{P}}_{k-1}]^{-1}\}\tilde{\boldsymbol{x}}_{k-1}. \qquad (52)
\end{aligned}
$$

Hence, the following equality is obtained by substituting Eq. (52) into Eq. (46):

$$
\begin{aligned}
&E[V_k(\tilde{\boldsymbol{x}}_k)|\tilde{\boldsymbol{x}}_{k-1}]-V_{k-1}(\tilde{\boldsymbol{x}}_{k-1})\\
=\ &\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\{\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]^{\mathrm{T}}(\boldsymbol{R}_k^*)^{-1}\\
&\times[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]\boldsymbol{\Phi}_{k/k-1}-\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)\\
&\times(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}\\
&-[\hat{\boldsymbol{P}}_{k-1}+\hat{\boldsymbol{P}}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{\Phi}_{k/k-1}\hat{\boldsymbol{P}}_{k-1}]^{-1}\}\tilde{\boldsymbol{x}}_{k-1}\\
&+E\{[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]\}\\
&+E[(\boldsymbol{K}_k\boldsymbol{v}_k)^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}(\boldsymbol{K}_k\boldsymbol{v}_k)]. \qquad (53)
\end{aligned}
$$

Denote

$$
\begin{aligned}
\mu_k\ =\ &E\{[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}[(\boldsymbol{I}-\boldsymbol{K}_k\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]\}\\
&+E[(\boldsymbol{K}_k\boldsymbol{v}_k)^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}(\boldsymbol{K}_k\boldsymbol{v}_k)] \qquad (54)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_k\ =\ &-\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\{\Phi_{k/k-1}^{\mathrm{T}}[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]^{\mathrm{T}}(\boldsymbol{R}_k^*)^{-1}\\
&\times[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]\Phi_{k/k-1}\\
&-\Phi_{k/k-1}^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}\\
&\times(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\Phi_{k/k-1}\\
&-[\hat{\boldsymbol{P}}_{k-1}+\hat{\boldsymbol{P}}_{k-1}\Phi_{k/k-1}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\Phi_{k/k-1}\hat{\boldsymbol{P}}_{k-1}]^{-1}\}\\
&\times\tilde{\boldsymbol{x}}_{k-1}/V_{k-1}(\tilde{\boldsymbol{x}}_{k-1}). \qquad (55)
\end{aligned}
$$

Since both sides of Eq. (54) are scalars, taking the trace does not change their values. Substituting Eq. (28) into Eq. (54) and using $\mathrm{Tr}(\boldsymbol{AB})=\mathrm{Tr}(\boldsymbol{BA})$, we have

$$
\begin{aligned}
\mu_k\ =\ &\mathrm{Tr}\{E\{[(\boldsymbol{I}-\hat{\boldsymbol{P}}_k\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\times[(\boldsymbol{I}-\hat{\boldsymbol{P}}_k\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{w}_k]\}\}\\
&+\mathrm{Tr}\{E[(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{v}_k)^{\mathrm{T}}\hat{\boldsymbol{P}}_k\\
&\times(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{v}_k)]\}\\
=\ &\mathrm{Tr}[(\boldsymbol{I}-\hat{\boldsymbol{P}}_k\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{\alpha}_k\boldsymbol{H}_k)^{\mathrm{T}}\hat{\boldsymbol{P}}_k^{-1}\\
&\times(\boldsymbol{I}-\hat{\boldsymbol{P}}_k\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{Q}_k]\\
&+\mathrm{Tr}[(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1})^{\mathrm{T}}\hat{\boldsymbol{P}}_k(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k(\boldsymbol{R}_k^*)^{-1}\boldsymbol{R}_k)]. \qquad (56)
\end{aligned}
$$

Apparently, $\mu_k>0$. Using Eqs. (35)–(40), we have

$$
\begin{aligned}
\mu_k\ \le\ &\frac{n^2q_{\max}}{p_{\min}}\left(\frac{m\eta_{\max}\kappa_{\max}}{r_{\min}^*}+1\right)^2\\
&+\frac{nmr_{\max}p_{\max}\eta_{\max}^2}{(r_{\min}^*)^2}\triangleq\mu_{\max}. \qquad (57)
\end{aligned}
$$

Under assumptions (32) and (37)–(41), it can be verified that

$$\begin{aligned}&\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\{\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]^{\mathrm{T}}(\boldsymbol{R}_k^*)^{-1}\\&\times[(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k)-\boldsymbol{\alpha}_k\boldsymbol{H}_k]\boldsymbol{\Phi}_{k/k-1}\\&-[\hat{\boldsymbol{P}}_{k-1}+\hat{\boldsymbol{P}}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{\Phi}_{k/k-1}\hat{P}_{k-1}]^{-1}\}\tilde{\boldsymbol{x}}_{k-1}\\\le&\left(\frac{\phi_{\max}^2\varepsilon_{\max}^2}{r_{\min}^*}-\frac{1}{p_{\max}+p_{\max}^2\phi_{\max}^2/q_{\min}}\right)\tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}\tilde{\boldsymbol{x}}_{k-1}\le 0.\end{aligned}\tag{58}$$

Thus, from Eqs. (32)–(34) and (37)–(40), we can obtain

$$\begin{aligned}\lambda_k\ge&\ \tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}[\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\hat{\boldsymbol{P}}_{k/k-1}\\&+\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}\\&\times(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}]\tilde{\boldsymbol{x}}_{k-1}/V_{k-1}(\tilde{\boldsymbol{x}}_{k-1})\\=&\ \tilde{\boldsymbol{x}}_{k-1}^{\mathrm{T}}[\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k)(\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\boldsymbol{\Phi}_{k/k-1}\hat{\boldsymbol{P}}_{k-1}\boldsymbol{\Phi}_{k/k-1}^{\mathrm{T}}\\&+\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{H}_k\boldsymbol{Q}_k\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{\alpha}_k\boldsymbol{\gamma}_k+\boldsymbol{R}_k^*)^{-1}\\&\times(\boldsymbol{\alpha}_k\boldsymbol{H}_k)\boldsymbol{\Phi}_{k/k-1}]\tilde{\boldsymbol{x}}_{k-1}/V_{k-1}(\tilde{\boldsymbol{x}}_{k-1})\\\ge&\ \frac{p_{\min}h_{\min}^2\phi_{\min}^2}{p_{\max}c_{\max}^2\phi_{\max}^2+q_{\max}h_{\max}^2r_{\max}^*}\triangleq\lambda_{\min}.\end{aligned}\tag{59}$$

Obviously, it follows from Eq. (59) that

$$0<\lambda_{\min}\le 1.\tag{60}$$

Consequently, applying Lemma 1 with Eqs. (53), (57), and (59), the inequality

$$\begin{aligned}&E[V_k(\tilde{\boldsymbol{x}}_k)|\tilde{\boldsymbol{x}}_{k-1}]-V_{k-1}(\tilde{x}_{k-1})\\\le&\ \mu_{\max}-\lambda_{\min}V_{k-1}(\tilde{\boldsymbol{x}}_{k-1})\end{aligned}\tag{61}$$

is satisfied. This complements the proof of the boundedness of $\tilde{x}_k$.

Theorem 1 provides the sufficient conditions to guarantee the stochastic stability of the DUKF. It also clearly defines the connection between the design of the measurement noise covariance matrix and the filter stability. To ensure the stability of the DUKF, it requires the matrix $\boldsymbol{R}_k^*$ to be positive definite. However, since $\Delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}}$ and $\delta\boldsymbol{P}_{\hat{\boldsymbol{z}}_{k/k-1}}$ may not be positive definite matrices (see Eq. (25)) due to the large initial error, extra additive matrix $\Delta\boldsymbol{R}_k$ should be introduced as a slight modification to the DUKF so that $\boldsymbol{R}_k^*\ge r_{\min}^*\boldsymbol{I}$ will be satisfied. Apparently, if $\Delta\boldsymbol{R}_k$ is sufficiently large, assumption (37) is always fulfilled. This means that the stability of the DUKF can be enhanced by adding small quantities to the measurement noise covariance matrix. On the other hand, as can be seen in Eq. (27), the error covariance matrix $\boldsymbol{P}_k$ is dependent on $\boldsymbol{R}_k^*$. The higher the value of $\Delta\boldsymbol{R}_k$ is, the larger the estimation error will be. Therefore, the value of the extra positive definite matrix $\Delta\boldsymbol{R}_k$ can be viewed as a tradeoff between the stability and accuracy. It can also be seen from Theorem 1 that the design of the modified measurement noise covariance matrix $\boldsymbol{R}_k^*$ is more intuitive and much easier than that presented in Ref. [6], since it does not involve any coupling relationship with the process noise covariance matrix.

Remark 1 The condition $h_{\min}\le c_{\max}$ can be relaxed as $p_{\min}h_{\min}^2\phi_{\min}^2\le p_{\max}c_{\max}^2\phi_{\max}^2+q_{\max}h_{\max}^2+r_{\max}^*$, which is easier to satisfy. The assumption (41) means the approximation error between $\boldsymbol{P}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}$ and $\hat{\boldsymbol{P}}_{\hat{x}_{k/k-1}\hat{\boldsymbol{z}}_{k/k-1}}$ is small enough. It is precise because of the assumption (41) that the coupling relationship between the modified measurement noise covariance matrix $\boldsymbol{R}_k^*$ and the process noise covariance matrix $\boldsymbol{Q}_k$ can be released. Actually, if the error introduced by $\hat{\boldsymbol{P}}_{\hat{\boldsymbol{x}}_{k/k-1}\hat{z}_{k/k-1}}$ is negligible, i.e., $\boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{\alpha}_kH_k=\boldsymbol{\alpha}_kH_k$ or equivalently $\boldsymbol{\gamma}_k=\boldsymbol{I}$, the condition $h_{\min}\le c_{\max}$ and assumption (41) are obviously fulfilled.

Remark 2 The rationality of assumption (41) lies in the fact that the error introduced by the UT for estimating the posterior mean and covariance of the state of any Gaussian and nonlinear system is in at least second-order accuracy.[9,19]

Remark 3 The assumption (39) will be fulfilled if the matrices $\boldsymbol{\Phi}_{k/k-1}$ and $\boldsymbol{H}_k$ satisfy the nonlinear uniform observability condition.[20,21]

Remark 4 Even though the condition (37) in Theorem 1 provides a theoretical criteria to obtain $\Delta\boldsymbol{R}_k$, it is difficult to acquire the analytical solution for the bounds on $\Delta\boldsymbol{R}_k$. This is because the determination of the parameters $r_{\min}^*$ and $r_{\max}^*$ in condition (37) requires a large amount of computational load. Therefore, $\Delta\boldsymbol{R}_k$ is generally selected by empiricism or test computations.[6,17]

4. Simulations and analysis

The result described in Section 3 proves that the estimation error of the DUKF will remain bounded if the considered nonlinear system satisfies certain conditions. It shows that the stability of the DUKF can be significantly improved by adding small quantities to the measurement noise covariance matrix. It also excludes the requirement of small initial estimation error.

In this section, simulations have been conducted based on three real-world systems to illustrate the effectiveness of the proposed technique. The performance of the modified DUKF with an extra positive definite matrix $\Delta\boldsymbol{R}_k$ is comprehensively evaluated in comparison with the original DUKF.

4.1. Radar target tracking system

Consider a radar target tracking system based on two-dimensional Cartesian coordinates.[5] The target moves according to a constant velocity (CV) model, and the radar measures the azimuth angle and slope distance of the target. Denote the state vector as $\boldsymbol{x}_k=[x_{1,k},x_{2,k},x_{3,k},x_{4,k}]^{\mathrm{T}}$, where $x_{1,k}$ and $x_{2,k}$ are the displacement and velocity in the x axis, $x_{3,k}$ and $x_{4,k}$ are the corresponding components in the y axis. The

system state equation and the measurement equation are described as

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \\ x_{3,k-1} \\ x_{4,k-1} \end{bmatrix} + \boldsymbol{w}_k, \quad (62)$$

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} (x_{1,k}^2 + x_{3,k}^2)^{1/2} \\ \tan^{-1}(x_{3,k}/x_{1,k}) \end{bmatrix} + \boldsymbol{v}_k. \quad (63)$$

The covariance matrices of $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are

$$\begin{aligned} \boldsymbol{Q}_k &= \mathrm{diag}(0, 0.02^2, 0, 0.02^2), \\ \boldsymbol{R}_k &= \mathrm{diag}(5^2, 0.005^2). \end{aligned} \quad (64)$$

The initial conditions for the system and filters are taken as

$$\begin{aligned} &x_{1,0} = 104, \quad x_{2,0} = 65, \quad x_{3,0} = 100, \quad x_{4,0} = 20, \\ &\hat{x}_{1,0} = 50^2, \quad \hat{x}_{2,0} = 65, \quad \hat{x}_{3,0} = 100, \quad \hat{x}_{4,0} = 20. \end{aligned} \quad (65)$$

The initial covariance matrix is chosen as

$$\hat{\boldsymbol{P}}_0 = 2\boldsymbol{I}_4. \quad (66)$$

Figure 1 depicts the state $x_{2,k}$ and its estimates $\hat{x}_{2,k}$ obtained by the DUKF and the modified DUKF with an extra additive matrix $\Delta\boldsymbol{R}_k = \mathrm{diag}(7^2, 0.001^2)$ (named the modified DUKF1), respectively. As shown in Fig. 1, in the case of large initial error for $x_{1,k}$, the DUKF fails to converge with the noise covariance matrices (64). However, as the designed extra positive definite matrix $\Delta\boldsymbol{R}_k$ is added to $\boldsymbol{R}_k$ so that the assumption (37) in Theorem 1 is fulfilled, the estimation error of the modified DUKF1 remains bounded for the above initial conditions. This verifies that the stability of the DUKF can be improved by adding small quantities to the measurement noise covariance matrix.

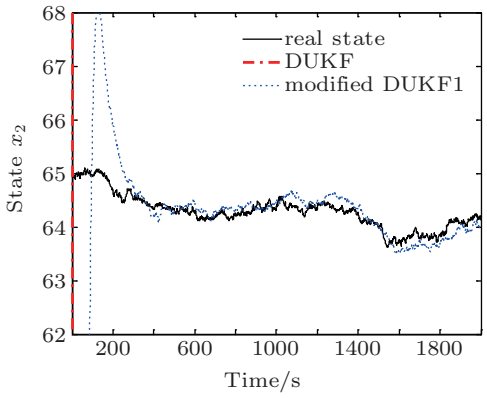


Fig. 1. (color online) State $x_{2,k}$ and its estimates $\hat{x}_{2,k}$ for the radar target tracking system.

Monte Carlo simulations were conducted to analyze the impact of the extra positive definite matrix $\Delta\boldsymbol{R}_k$ on the error behavior. In addition to the DUKF and the modified DUKF1 with $\Delta\boldsymbol{R}_k = \mathrm{diag}(7^2, 0.001^2)$, the modified DUKF with another extra additive matrix $\Delta\boldsymbol{R}_k = \mathrm{diag}(30^2, 0.015^2)$ (named the modified DUKF2) was also used to estimate the state $x_{2,k}$. The Monte Carlo simulations were carried out for 10 times, yielding 10 realizations of the state and measurement trajectories. The mean square errors (MSEs) of the three filters were calculated by averaging the square error between $x_{2,k}$ and $\hat{x}_{2,k}$ over 10 trials. The results are shown in Fig. 2.

It can be seen from Fig. 2 that the MSE of the DUKF diverges. However, due to the enlargement of the measurement noise covariance matrix, the MSEs of the modified DUKF1 and the modified DUKF2 remain bounded. Figure 2 also shows that the MSE of the modified DUKF2 is much worse than that of the modified DUKF1. It confirms that the larger the extra positive definition matrix $\Delta\boldsymbol{R}_k$ is, the larger the estimation error will be. Therefore, it requires the extra positive definition matrix $\Delta\boldsymbol{R}_k$ to be chosen properly to balance the stability and accuracy of the modified DUKF.

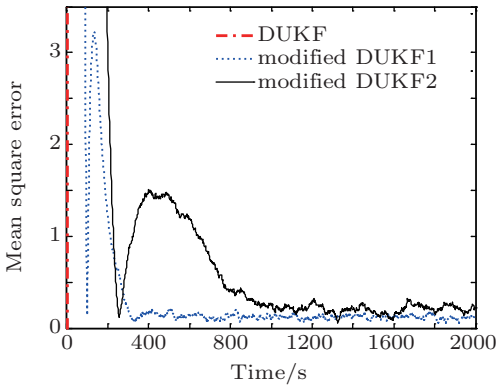


Fig. 2. (color online) Mean square errors of the three filters for state $x_{2,k}$ in the radar target tracking system.

4.2. Tightly coupled INS/GPS integrated system

Trials were conducted to evaluate the efficiency of the proposed technique in stabilizing the DUKF with large initial error for a tightly coupled INS/GPS integrated system. The system state equation and measurement equation of the tightly coupled INS/GPS integration for the DUKF are given in Refs. [1] and [22]–[24], respectively. The system state vector is defined as

$$\begin{aligned} \boldsymbol{x}_k = [&\delta v_{\mathrm{E}}, \delta v_{\mathrm{N}}, \delta v_{\mathrm{U}}, \phi_{\mathrm{E}}, \phi_{\mathrm{N}}, \phi_{\mathrm{U}}, \delta L, \delta\lambda, \delta h, \\ &\varepsilon_x, \varepsilon_y, \varepsilon_z, \nabla_x, \nabla_y, \nabla_z, b_{\mathrm{p}}, b_{\mathrm{f}}]^{\mathrm{T}}, \end{aligned} \quad (67)$$

where $(\delta v_{\mathrm{E}}, \delta v_{\mathrm{N}}, \delta v_{\mathrm{U}})$ is the velocity error, $(\phi_{\mathrm{E}}, \phi_{\mathrm{N}}, \phi_{\mathrm{U}})$ the attitude error, $(\delta L, \delta\lambda, \delta h)$ the position error, $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ the gyro constant drift, $(\nabla_x, \nabla_y, \nabla_z)$ the accelerometer zero-bias, and $(b_{\mathrm{p}}, b_{\mathrm{f}})$ the range bias and range drift related to the GPS receiver clock.

The measurement vector is taken as

$$\boldsymbol{z}_k = \delta\boldsymbol{\rho}, \quad (68)$$

where $\delta\rho$ is the range error which can be obtained by subtracting the estimated ranges of the INS from the pseudorange measurements of the GPS receiver.[1]

The aircraft trajectory, which was designed for a complex high-dynamic aircraft flight in the real world, is shown in Fig. 3. The flight involves various maneuvers such as climbing, pitching, rolling and turning. The simulation parameters are given in Table 1. It can be seen that the initial position error is very poor. The simulation time is 3600 s and the filtering period is 1 s.

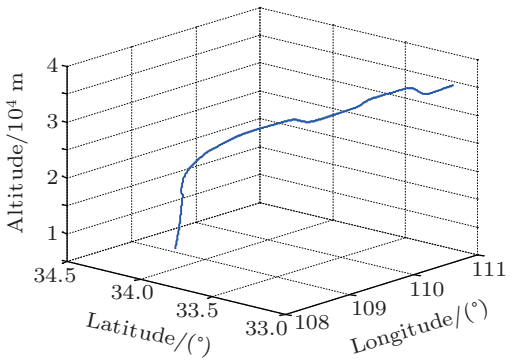


Fig. 3. (color online) Aircraft trajectory.

The initial error covariance matrix is taken as

$$\hat{\boldsymbol{P}}_0 = \mathrm{diag}\{(0.5\ \mathrm{m/s})^2\boldsymbol{I}_{3\times3}, (1')^2\boldsymbol{I}_{3\times3}, (1200\ \mathrm{m})^2\boldsymbol{I}_{3\times3}, (0.1^\circ/\mathrm{h})^2\boldsymbol{I}_{3\times3}, (10^{-3}\ \mathrm{g})^2\boldsymbol{I}_{3\times3}, (5\ \mathrm{m})^2\boldsymbol{I}_{2\times2}\}. \quad (69)$$

The covariance matrices of the process noise and measurement noise are

$$\begin{aligned}\boldsymbol{Q}_k &= \mathrm{diag}\{(10^{-4}\ \mathrm{g})^2\boldsymbol{I}_{3\times3}, (0.01^\circ/\mathrm{h})^2\boldsymbol{I}_{3\times3}, \\ &\quad \times\boldsymbol{0}_{9\times9}, 0.0114, 0.0039\}, \\ \boldsymbol{R}_k &= (25\ \mathrm{m})^2\boldsymbol{I}_{4\times4}. \end{aligned} \quad (70)$$

Figures 4 and 5 depict the position errors obtained by the DUKF and the modified DUKF with an extra additive matrix $\Delta\boldsymbol{R}_k = (20\ \mathrm{m})^2\boldsymbol{I}_{4\times4}$ (named the modified DUKF1). It can be seen that the position errors of the DUKF diverge significantly. This indicates that the DUKF is sensitive to the large initial error and performs in a subpar fashion. However, with the particular design of the modified measurement noise covariance matrix $\boldsymbol{R}_k^* = \boldsymbol{R}_k + \Delta\boldsymbol{R}_k$, the modified DUKF1 produces the stable estimation results. This illustrates that the instability of the DUKF in the presence of large initial error can be prevented by introducing a properly chosen extra positive definite matrix $\Delta\boldsymbol{R}_k$ to the measurement noise covariance matrix $\boldsymbol{R}_k$.

Furthermore, the MSEs of the position errors by the DUKF, the modified DUKF1 and the modified DUKF with another extra additive matrix $\Delta\boldsymbol{R}_k = (50\ \mathrm{m})^2\boldsymbol{I}_{4\times4}$ (named the modified DUKF2) were computed from 10 times of Monte Carlo simulations. The results are shown in Fig. 6, demonstrating the similar trend to that in Fig. 2. It can be observed that the MSE curve of the DUKF fails to converge, while the modified DUKF1 and DUKF2 remain bounded. Compared with the modified DUKF1, the MSE of the modified DUKF2 is much larger. These plots confirm that with a properly chosen extra positive definition matrix $\Delta\boldsymbol{R}_k$, the modified DUKF is robust. However, setting a very large value for $\Delta\boldsymbol{R}_k$ will decrease the MSE accuracy of the modified DUKF.

Table 1. Simulation parameters for the tightly coupled INS/GPS integrated system.

Initial position	north latitude	34.17°
	east longitude	108.93°
	altitude	6000 m
Initial velocity	east	150 m/s
	north	50 m/s
	up	30 m/s
Initial orientation	Yaw	0°
	Pitch	0°
	Roll	0°
Initial position error	north latitude	1200 m
	east longitude	1200 m
	altitude	1200 m
Initial velocity error	east	0.5 m/s
	north	0.5 m/s
	up	0.5 m/s
Initial attitude error	Yaw	1′
	Pitch	1′
	Roll	1′
Gyro parameters	constant drift	0.1°/h
	correlation time	1 h
	white noise	0.01°/h
Accelerometer parameters	zero bias	10^{-3} g
	correlation time	1 h
	white noise	10^{-4} g
GPS receiver parameters	pseudorange measurement error	25 m
	data update rate	1 Hz

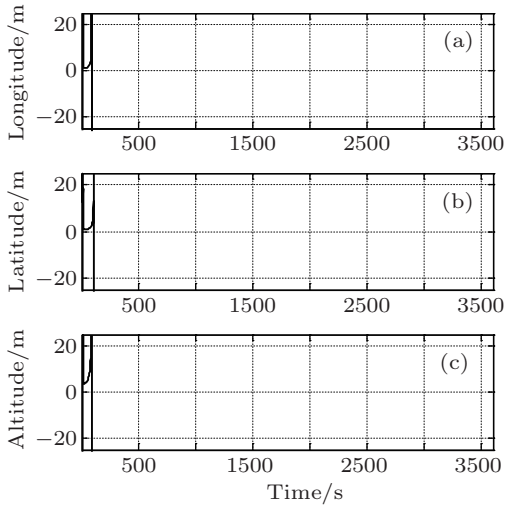


Fig. 4. Position errors ((a) longitude, (b) latitude, (c) altitude) obtained by the DUKF for the tightly coupled INS/GPS integrated system.

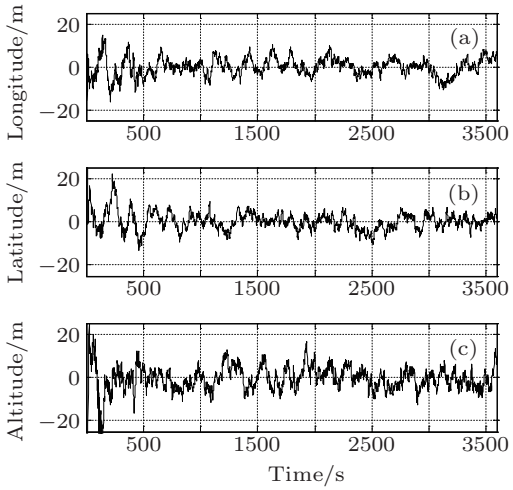


Fig. 5. (color online) Position errors ((a) longitude, (b) latitude, (c) altitude) obtained by the modified DUKF1 for the tightly coupled INS/GPS integrated system.

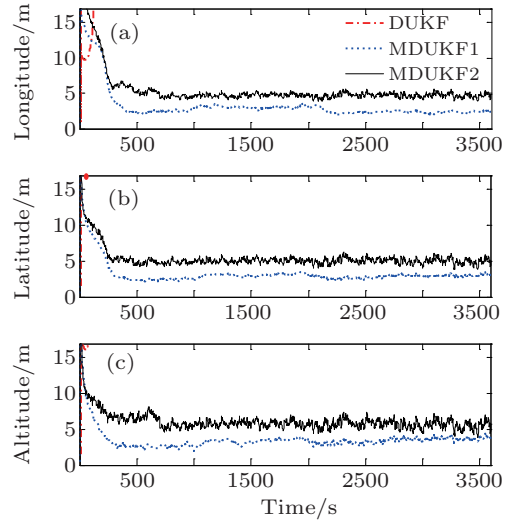


Fig. 6. (color online) MSEs of the position errors ((a) longitude, (b) latitude, (c) altitude) by the DUKF, modified DUKF1 and modified DUKF2 for the tightly coupled INS/GPS integrated system.

4.3. Bearings-only tracking system

The bearings-only tracking system[25] is also considered to verify the effectiveness of the proposed technique. The bearings-only tracking system is composed of two sensors to track a moving target, where each sensor obtains only an angle measurement related to the system state. The state vector is denoted as $\boldsymbol{x}_k = [x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k}]^{\mathrm{T}}$, where the meanings of the components are the same as those in Section 4.1. The target moves according to the CV model described by Eq. (62).

According to the principle of two-position cross-point, the measurement equation can be established as

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} \cos^{-1}\left(\dfrac{x_{1,k}}{\sqrt{x_{1,k}^2 + x_{3,k}^2}}\right) \\ \cos^{-1}\left(\dfrac{x_{1,k} - d}{\sqrt{(x_{1,k} - d)^2 + x_{3,k}^2}}\right) \end{bmatrix} + \boldsymbol{v}_k, \quad (71)$$

where $d = 5000$ m is the distance between the two sensors.

The covariance matrices of $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are selected as

$$\begin{aligned} \boldsymbol{Q}_k &= \mathrm{diag}(0, 0.02^2, 0, 0.02^2), \\ \boldsymbol{R}_k &= \mathrm{diag}(0.0001^2, 0.0001^2) \times \pi/180. \end{aligned} \quad (72)$$

The initial conditions for the system and filters are

$$\begin{aligned} &x_{1,0} = 100, \quad x_{2,0} = 4.62, \quad x_{3,0} = 100, \\ &x_{4,0} = 9.62, \quad \hat{x}_{1,0} = 100^2, \quad \hat{x}_{2,0} = 4.62, \\ &\hat{x}_{3,0} = 100, \quad \hat{x}_{4,0} = 9.62. \end{aligned} \quad (73)$$

The initial covariance matrix is taken as

$$\hat{\boldsymbol{P}}_0 = \mathrm{diag}(1, 0.1^2, 1, 0.1^2). \quad (74)$$

Figure 7 depicts the state $x_{2,k}$ and its estimates $\hat{x}_{2,k}$ by the DUKF and the modified DUKF with an extra additive matrix $\Delta\boldsymbol{R}_k = \mathrm{diag}(0.00005^2, 0.00005^2) \times \pi/180$ (named the modified DUKF1). Figure 8 illustrates the MSEs of state $x_{2,k}$ by the DUKF, the modified DUKF1, and the modified DUKF with another extra additive matrix $\Delta\boldsymbol{R}_k = \mathrm{diag}(0.0025^2, 0.0025^2) \times \pi/180$ (named the modified DUKF2) calculated from 10 times of Monte Carlo simulations. The similar phenomena to that in Sections 4.1 and 4.2 can also be observed from these figures. On one hand, in the presence of large initial error for $x_{1,k}$, the DUKF diverges significantly with the noise covariance matrices (72). However, as the designed extra positive definite matrix $\Delta\boldsymbol{R}_k = \mathrm{diag}(0.00005^2, 0.00005^2) \times \pi/180$ is added to $\boldsymbol{R}_k$, the estimation error of the modified DUKF1 remains bounded for the large initial error. On the other hand, with the enlargement of the extra additive matrix from $\Delta\boldsymbol{R}_k = \mathrm{diag}(0.00005^2, 0.00005^2) \times \pi/180$ to $\Delta\boldsymbol{R}_k = \mathrm{diag}(0.0025^2, 0.0025^2) \times \pi/180$, the MSEs of the modified DUKF2 become much larger than that of the modified DUKF1. These verify the result described in Section 3 that the stability of the DUKF can be improved by adding small quantities to the measurement noise covariance matrix, while a very large extra additive matrix will decrease the accuracy of the modified DUKF.

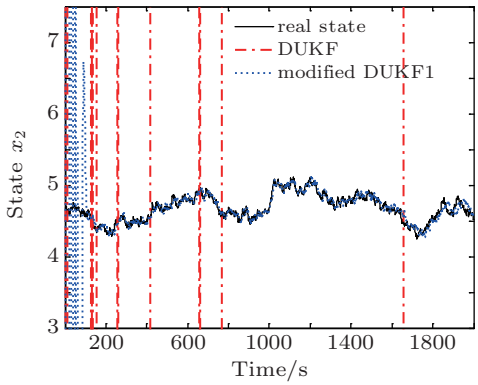


Fig. 7. (color online) State $x_{2,k}$ and its estimates $\hat{x}_{2,k}$ for the bearings-only tracking system.

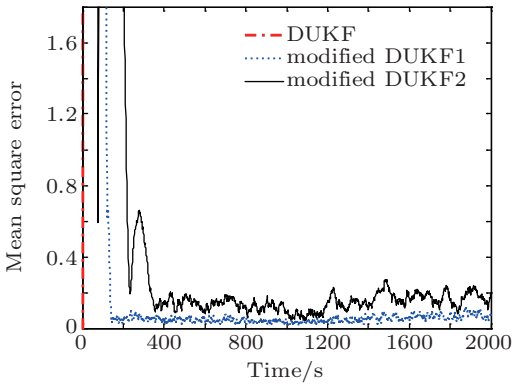


Fig. 8. (color online) Mean square errors of the three filters for state $x_{2,k}$ in the bearings-only tracking system.

5. Conclusions

The paper studies the error behavior of the DUKF for a nonlinear stochastic discrete-time system with linear system state equation. It proves that the estimation error of the DUKF will remain bounded in mean square if the system satisfies certain conditions. It also clearly defines the connection between the design of the measurement noise covariance matrix and the stochastic stability of the DUKF. The proposed technique shows that, by adding small quantities to the measurement noise covariance matrix, the DUKF can be stabilized without the assumption of small initial estimation error. Theoretical analysis and simulation results demonstrate that, with an enlargement of the measurement noise covariance matrix, the stability of the DUKF can be significantly improved despite large initial error. However, the value of the extra positive definite matrix $\Delta \boldsymbol{R}_k$ needs to be chosen properly to balance the stability and accuracy of the modified DUKF.

Future research work will focus on the combination of the proposed technique with artificial intelligence such as advanced expert systems, neural networks, and genetic algorithms to automatically determine the value of $\Delta \boldsymbol{R}_k$ for achieving the optimal balance between the stability and accuracy.

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