Conditions for formation and trapping of the two-ion Coulomb cluster in the dissipative optical superlattice

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Conditions have been studied under which a polychromatic optical superlattice can form and trap the Coulomb cluster of two strongly interacting ions. In our previous work (Krasnov I V and Kamenshchikov L P 2014 *Opt. Comm.* **312** 192) this new all-optical method of obtaining and confining the Coulomb clusters was demonstrated by numerical simulations for special values of the optical superlattice parameters and in the case of Yb ions. In the present paper the conditions are explicitly formulated, under which the long-lived two-ion cluster in the superlattice cell is formed. The peculiarity of these conditions is the renormalization of the ion–ion Coulomb interaction. Notably, the renormalized Coulomb force is determined by the effective charge which depends on the light field parameters and can strongly differ from the "bare" ion charge. This result can be accounted for by the combined manifestation of the quantum fluctuations of optical forces, nonlinear dependence of these forces on the velocity, and non-Maxwellian (Tsallis type) velocity distribution of the ions in the optical superlattice. Explicit analytical formulas are also obtained for the parameters of the optical two-ion cluster.

Keywords: optical trap, ion trap, rectified gradient force, Coulomb clusters

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1. Introduction

The capability of laser radiation to control the translational degrees of freedom has intensively been studied for several decades. These studies offer a wide spectrum of important applications and useful theoretical models.[1,2] The mechanical action of light is used for the deep laser cooling of both neutral atoms and charged atomic ions as well as for the trapping of neutral atoms. Electromagnetic traps (similar to the Paul or Penning traps) are conventionally used for the ion confinement.

Recently the interest has arisen from the problem of all-optical trapping of ions, i.e., from ion trapping without applying radiofrequency or electrostatic and magnetostatic fields.[3,4] Unfortunately, the shallow depth $\Delta W_{\rm d}$ of monochromatic dipole trap (ODT) or optical lattice (OL) can be a serious obstacle for their use in the trapping of a few ions due to the Coulomb repulsion of ions (although ODT and OL are quite successfully used for all-optical trapping of neutral atoms[2,5–7]).

An alternative method for long-time trapping of the interacting ions, the confinement of the ions in a three-dimensional (3D) dissipative polychromatic optical superlattice (OSL),[8,9] was considered in Ref. [4]. The action of such a 3D OSL (with the spatial period $\sim L$ greatly exceeding the light wavelength λ, $L \gg \lambda$) on the ions is based on the effect of the gradient (dipole) force rectification in the non-monochromatic (bichromatic, polychromatic) optical fields.[10–12] Due to unique features of the rectified gradient force RcGF,[10] OSL is capable of inducing 3D superdeep (as compared with ODT or OL) potential wells (for the ions) with the value of depth

$$\Delta W \sim \hbar\Delta_{\rm s} L/\lambda \gg \hbar\Delta_{\rm s} \sim \Delta W_{\rm d},$$

where $\hbar\Delta_{\rm s}$ is the characteristic value of the light (Stark) shifts of ionic levels. Moreover, the light-induced friction force is exerted on the ions in OSL, resulting in the cooling of ions. The computer simulation results of the ion motion in OSL (for the case of two ytterbium ions) presented in Ref. [4] demonstrate that the OSL action on the ions can lead to their deep cooling and to the formation and long-time confinement of the two-ion (Coulomb) cluster. Such a cluster has an approximately fixed (with the accuracy up to small fluctuations) interionic distance (size) and the center-of-mass (CM) position. Note that in the considered two-ion system all the main factors hindering the all-optical ion trapping are present: the Coulomb repulsion, quantum fluctuations of the optical forces,[1,13] and finite depth of the light-induced potential wells.

The analysis of the computer simulation results allows one to formulate the following conditions for the long-lived two-ion cluster formation within a single unit cell Ω_0 of OSL.[4]

There must exist equilibrium points $\{\eta^*\}$ in the six-dimensional configuration space (of the ion spatial positions $\{\eta\} = (\boldsymbol{r}_1, \boldsymbol{r}_2)$), where the Coulomb repulsion of ions is balanced by the trapping force (RcGF $\boldsymbol{F}_{0\rm R}$ in our model) and the total potential energy $U_t(\{\eta\})$ achieves a local minimum, i.e., a point where

$$\boldsymbol{F}_{0\rm R} - \frac{\partial U_{\rm c}(|\boldsymbol{r}_1, \boldsymbol{r}_2|)}{\partial \boldsymbol{r}_\alpha} = 0,$$

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$$\{\eta\} = (\boldsymbol{r}_1, \boldsymbol{r}_2) = \{\eta^*\} \in \Omega_0, \quad (1)$$
$$\mathrm{d}^2 U_t(\{\eta^*\}) > 0,$$
$$U_t(\{\eta\}) = \sum_{\alpha=1}^{2} U_{\mathrm{R}}(\boldsymbol{r}_\alpha) + U_{\mathrm{c}}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|), \quad (2)$$

where $\alpha = 1,2$, $\boldsymbol{r}_\alpha$ is the position of the α-th ion, U_{R} is the potential of RcGF, $U_{\mathrm{c}}(r) = e^2/4\pi\varepsilon_0 r$ is the Coulomb energy of two ions separated by the distance r, e is the ion charge, $\mathrm{d}^2 U_t(\{\eta^*\})$ is the second differential of $U_t(\{\eta\})$ at the point $\{\eta^*\}$.

There exist equilibrium cluster configurations $\{\eta^*\}$ if the value of the light-induced potential well depth ΔW is sufficiently large as compared with the Coulomb energy $U_{\mathrm{c}}(L)$:[4]

$$\frac{U_{\mathrm{c}}(L)}{\Delta W} < \frac{\pi}{4}, \quad (3)$$

where L is the spatial period of OSL.

The cluster configuration $\{\eta^*\}$ is metastable[4] due to the quantum fluctuation of the optical trapping force. Notably, the metastable-state lifetime τ_{M} is a strongly increasing function of the ratio of the potential well depth ΔW to the effective cluster temperature T, which is established by the balance between the optical cooling of the ions and their heating due to the optical force fluctuations. Therefore, another important condition for the formation of the long-lived cluster in OSL can be presented by the inequality

$$\Upsilon = \frac{\Delta W}{T} \gg 1, \quad (4)$$

where T is expressed in energy units. Inequality (4) also provides the smallness of the random fluctuations of the interionic distances (at the time points $t < \tau_{\mathrm{M}}$).[4]

The presented conditions seem perfectly natural and simple for the interpretation. However they are valid only in the approximation of slow ions (SI), i.e., in the case of

$$\varepsilon = \frac{s^2}{v_{\mathrm{c}}^2} \ll 1, \quad (5)$$

where $s = \sqrt{T/m}$ is the thermal velocity of the ions, m is the ionic mass, and v_{c} is the so-called capture velocity[2] determining the region (in the velocity space) of the most efficient action of the optical forces, $|\boldsymbol{v}|^2 < v_{\mathrm{c}}^2$: outside this region the forces decrease rapidly with increasing ion velocity. The SI approximation and the quasi-classical theory of optical forces[1,13] allow treating the ion motion in OSL as the Brownian motion under the action of conservative trapping force (RcGF).[4] In this model, neither the friction coefficient nor the diffusion coefficient depends on ion velocity $\boldsymbol{v}$.

The present paper considers a new model where the parameter ε is not small enough and an explicit velocity dependence of RcGF, friction and diffusion coefficients are taken into account.

The necessity for the special analysis of this model is due to the following important circumstances. The low (sub-Doppler) limit of the ionic temperature in OSL is achieved at the low intensity of the optical field and is accompanied by the violation of condition (5).[9] The velocity profile of RcGF is determined (see e.g. Refs. [10], [12], and [14]) by the multipliers of the $\mathscr{L}(v_j/v_{\mathrm{c}})$ type, where v_j are the velocity projections onto the axis $j \in \{x,y,z\}$, and $\mathscr{L}(u)$ is the Lorentzian function:

$$\mathscr{L}(u) = \frac{1}{1+u^2}. \quad (6)$$

Therefore, RcGF is a non-conservative force, and equation (1) and condition (2) (as well as the outcoming relation (3)) loose their senses at $\varepsilon \sim 1$ and do not allow one to unambiguously predict possible equilibrium configurations of the two-ion cluster. The metastability condition (4) is also difficult to explain since it is not understandable in advance how to determine accurately the height of the energy barrier preventing the cluster from decaying in the case of the non-conservative RcGF.

Finally, a reasonable question arises. Is it at all possible in this case (i.e., at $\varepsilon \sim 1$) to form in OSL a long-lived Coulomb ion cluster which is similar to that predicted by the computer simulation in Ref. [4].

Our answer to this question of principle is "yes", and a correct formulation of the basic conditions for the formation of such a cluster in OSL is given. These conditions (with an arbitrary value of ε) can be written in the form analogous to relations (1)–(4), where it is, however, necessary to perform suitable renormalization (re-definition) of the parameters and forces. In particular, for renormalization of the Coulomb interaction it is necessary to replace the charge e (in the expression for $U_{\mathrm{c}}(r)$) by the effective (renormalized) charge e^*:

$$e \to e^*(\chi, G), \quad (7)$$

where e^* is the function of the governing parameters of OSL, χ, and G.[9]

In other words, one shows that with the arbitrary value of the parameter ε and the time scales larger than the inverse friction coefficient $\varkappa^{-1}$, the light-induced Brownian motion of the ions in the configuration space $\{\eta\}$ can be described using a renormalized SI model.

Our analysis is based on the asymptotic expansion of the Wigner distribution function (for the ions in OSL) in the Knudsen number of problem[9]

$$Kn = \lambda_r / L \ll 1, \quad (8)$$

where $\lambda_r \sim s\varkappa^{-1}$ has the meaning of the effective free-path length of the ions in a viscous "fluid" of photons.

Another interesting aspect (not only for optical physics but also for statistical mechanics (e.g., see Ref. [15]) of the problem under consideration is connected with the non-Maxwellian characteristic of the velocity distribution of ions in OSL. It is shown that the main part of the velocity distribution can be well approximated by the 3D Tsallis distribution

$$\rho(\boldsymbol{v}) = \prod_{j\in\{x,y,z\}} \rho_{\rm s}(v_j), \quad \rho_{\rm s}(v) = \frac{1}{N}\exp_q\left(\frac{v^2}{2T_0}\right), \tag{9}$$

where $\exp_q(v^2) = \left[1+(q-1)v^2\right]^{1/1-q}$ is the so-called q-Gaussian, normalizing factor N, and the index q and T_0 each are a function of the governing parameters of OSL.

2. Model

2.1. Governing parameters of 3D OSL

We consider two identical ions with the tripod configuration of the energy levels (Fig. 1) located in the 3D polychromatic OSL[8,9] and assume that the light field drives the closed dipole transition $|F_b = 1, M_b \in \{-1,0,1\}\rangle \to |F_a = 0, M_a = 0\rangle$ where F_α and M_α are the full angular momenta and their projections for the ground ($\alpha = b$) and excited ($\alpha = a$) internal ionic states, respectively. The light field (with a complex amplitude $\boldsymbol{E}$) forming OSL is a superposition of three color coherent fields and a partially coherent (fluctuating) resonant field $\boldsymbol{E}'$ with the bandwidth Γ:

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_j \boldsymbol{e}_j\left(E_{j1}(\boldsymbol{r})\exp[-{\rm i}\Delta_j t] + E'_j(\boldsymbol{r},t)\right), \tag{10}$$

where $\boldsymbol{e}_j$ denotes the unit basis vectors of the Cartesian coordinate system, $j \in \{x,y,z\}$, Δ_j are the detunings from the resonant frequency ω_0, which are subject to the conditions

$$\Delta_j,\ |\Delta_j - \Delta_{j'}| \gg \Gamma \gg \frac{|\hat{V}_{j1}|^2}{\Delta_j}, \frac{|\hat{U}_j|^2}{\Gamma},\ \gamma,\ ks. \tag{11}$$

Here, $\hat{V}_{j1} = {\rm d}E^*_{j1}/\hbar$, $\hat{U}_j = {\rm d}E'^*_j/\hbar$ are the Rabi frequencies, $d = |d|/\sqrt{3}$, $|d|$ is the reduced dipole transition matrix element, $k = \omega_0/c$ is the wave number, s is the thermal velocity of the ions, $\gamma = \gamma'/3$, γ' is the rate of the spontaneous decay of the excited state. Inequalities (11) imply that the coherent components of the field are "far-off-resonant", and the fluctuating component E' is resonant with ionic transitions. Therefore, the coherent components of the field, $\propto E_{j1}(\boldsymbol{r})\boldsymbol{e}_j$, form the light (Stark) shifts of the ionic energy levels, and the fluctuating field component $\boldsymbol{E}'(\boldsymbol{r},t)$ provides incoherent excitation of the ions (their redistribution over the quantum states).

In the Cartezian representation, i.e., in the representation of the basis wave functions (of the intra-ionic motions) for the excited $|a\rangle$ and ground $|bj\rangle$ states (where $j \in \{x,y,z\}$), in which the matrix elements of the dipole moment $\hat{\boldsymbol{d}}$ are directed along the unit vectors $\boldsymbol{e}_j$ ($j \in \{x,y,z\}$),

$$\langle bj|\hat{\boldsymbol{d}}|a\rangle = d\boldsymbol{e}_j, \tag{12}$$

the effective Hamiltoninan $\hat{H}_{\rm eff}$ (in the interaction picture) describing the interaction of a single ion with the light field has the form[9]

$$\hat{H}_{\rm eff} = \hbar \sum_{j\in\{x,y,z\}} \Big[-\left(\hat{U}_j(\boldsymbol{r},t)|bj\rangle\langle a| + {\rm H.c.}\right) + \frac{|\hat{V}_{j1}(\boldsymbol{r})|^2}{\Delta_j}\left(|bj\rangle\langle bj| - |a\rangle\langle a|\right)\Big], \tag{13}$$

where the terms $\propto |\hat{V}_{j1}(\boldsymbol{r})|^2/\Delta_j = \Delta_{sj}(\boldsymbol{r})$ and $-\sum|\hat{V}_{j1}(\boldsymbol{r})|^2/\Delta_j = \Delta_{\rm s}(\boldsymbol{r})$ describe the Stark shifts of the atomic levels induced by the "far-off-resonant" field E_1.

The particular configurations of the fields $E_{j1}(\boldsymbol{r})$ and $E'_\alpha(\boldsymbol{r})$ are described in detail in Refs. [8] and [9]. Note the key features of these configurations.

The interference of different wave components of each "far-off-resonant" field E_{j1} ($j \in \{x,y,z\}$) leads to the spatial modulation of the light (Stark) shifts along the i axis

$$\Delta_{sj}(\boldsymbol{r}) = \frac{V_j^2}{2\Delta_j}\left[1-\cos(2\pi\Omega_i r_i)\right], \tag{14}$$

where (ji) is the ordered pair of indices (e.g., $i = y$ for $j = x$; $i = x$ for $j = z$; $i = z$ for $j = y$), $r_i = (\boldsymbol{r}\boldsymbol{e}_i)$, $\Omega_i \sim 2/\lambda$ is the frequency of the spatial modulation along the i axis, V_j ($j \in \{x,y,z\}$) are the amplitudes of the local Rabi frequencies $\hat{V}_{j1}(\boldsymbol{r})$. Then, taking Eq. (13) into account, one can see that $\hbar\Delta_{sj}(r_i)$ and $-\sum_j\hbar\Delta_{sj}(r_i)$ are the potentials experienced by the ion in the states $|bj\rangle$, $j \in \{x,y,z\}$ and $|a\rangle$, respectively.

The partially coherent fields $E'_j(\boldsymbol{r},t)$, ($j \in \{x,y,z\}$) fluctuating independently of each other provide incoherent (due to conditions (11)) excitation of the ion with the transition $|bj\rangle \to |a\rangle$ rates $R_j(\boldsymbol{r})$. It is highly important that these fields should be non-uniform, i.e., include the plane-wave components capable of interfering with each other (we assume that the coherence length $l_{\rm c} = c/\Gamma$ is much greater than the characteristic size $\bar{D}$ of the region where the ions are located, $l_{\rm c} \gg \bar{D}$[12,16]). Therefore, the transition rates $R_j(\boldsymbol{r})$ also experience a suitable spatial modulation,

$$\begin{aligned} R_j(\boldsymbol{r}) &= R\left((1+a_1P_i(\boldsymbol{r})\right), \quad R \simeq \frac{U^2}{\Gamma}, \\ P_j(\boldsymbol{r}) &= \frac{1}{1+b}\left[\cos(2\pi\Omega'_i r_i) - b\cos(2\pi\Omega_i r_i)\right], \end{aligned} \tag{15}$$

where U is the amplitude of the local Rabi frequencies $\hat{U}_j(\boldsymbol{r},t)$, a_1 and b are the independent parameters determined by the amplitudes of interfering wave components of the field $E'_j(\boldsymbol{r})$, $a_1 < 1$ is the modulation depth, $b/(1+b)$ and $1/(1+b)$ are the relative modulation depths of the components oscillating with the frequencies Ω_i and Ω'_i, respectively, spatial frequencies Ω'_i are close to the spatial frequencies Ω_i: $\delta\Omega_i = \Omega'_i - \Omega_i \leqslant \Omega'_i, \Omega_i$.

So the resonant field $\boldsymbol{E}'$ induces the spatial modulation of the ionic state populations (due to the modulation of the transition rates $R_j(\boldsymbol{r})$) and the "far-off-resonant" fields $E_{j1}\boldsymbol{e}_j$ induce the effective potentials ($\propto \Delta_{sj}(\boldsymbol{r})$, $\Delta_{\rm s}(\boldsymbol{r})$), which determine the ion motion depending on its internal state. As a result, OSL is formed.[8,9] Figure 1 illustrates the general physical picture of the OSL action on the particles with tripod configuration of levels. The OSL periods L_i ($i \in \{x,y,z\}$) are determined by the spatial beat frequencies, $L_i = (1/\delta\Omega_i) \gg \lambda$. These three

spatial periods L_x, L_y, and L_z can be controlled independently by changing the OSL angular detunings.[8,9]

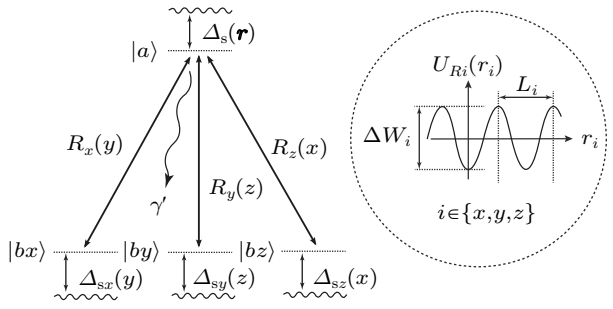


Fig. 1. Schematic picture of the OSL action on tripod-type particles. $|a\rangle$ and $|bj\rangle$ ($j\in\{x,y,z\}$) are the internal ionic states in the representation, in which the matrix elements of the dipole moments $\langle bj|\hat{\boldsymbol{d}}|a\rangle$ are directed along the Cartesian axes (See Eq. (12)); $\Delta_{sj}(r_i)$ and $\Delta_s(\boldsymbol{r})$ are the light shifts of the states $|bj\rangle$ and $|a\rangle$, $(ji)\in\{(xy),(yz),(zx)\}$; $R_j(r_i)$ are the rates of incoherent transitions $|bj\rangle\to|a\rangle$. The spatial frequencies Ω_i, $\Omega_i'\sim 2/\lambda$ of the oscillations of $\Delta_{sj}(r_i)$ and $R_j(r_i)$ are close to each other. As a result, deep potential wells along the x, y, z axes appear (see the inset), which have the periods $L_i=1/|\Omega_i-\Omega_i'|\gg\lambda$ and the depths $\Delta W_i\sim\hbar\Delta_{sj}L_i/\lambda\gg\hbar\Delta_{sj}$.

Apart from the periods L_i, the most important governing parameters of OSL are the parameters determined by the intensity values of the fields ($j\in\{x,y,z\}$)

$$G_j=\frac{V_j^2}{\Delta_j\gamma}=\frac{I_j\gamma}{I_s\Delta_j},\quad \chi=\frac{R}{\gamma}=\frac{I'\gamma}{I_s\Gamma}, \tag{16}$$

where I_j is the intensity of the waves forming the "far-off-resonant" field $\propto E_{j1}\boldsymbol{e}_j$, I' is the intensity of the partially-coherent field E', $I_s=\hbar\omega_0k^2\gamma/6\pi$ is the intensity of the optical radiation saturating the quantum transition. These parameters determine the values of the optical forces.

Further, we restrict ourselves to the most interesting case

$$\chi\ll 1, \tag{17}$$

i.e., to the case of the weak field $\boldsymbol{E}'$, when sub-Doppler cooling of ions becomes possible.[9]

In order to simplify the notations and calculations, it is also assumed that the parameter $b\gg 1$ (i.e., the term $\propto\cos(2\pi\Omega r_i)$ dominates in Eq. (15)) and all the parameters G_j (in Eq. (16)) are equal: $G_i=G>0$ for $i\in\{x,y,z\}$ (i.e., $I_j/\Delta_j=I_{j'}/\Delta_{j'}$, for $\forall j\neq j'$). Besides, it is assumed that the "far-off-resonant field" $\boldsymbol{E}_1$ is not too strong and $G\lessapprox 1/2$.

2.2. Fokker–Planck equation (FPE)

Our study is based on the quantum kinetic equation for the two-particle Wigner phase-space distibution function $\breve{f}(\boldsymbol{r}_1,\boldsymbol{v}_1,\boldsymbol{r}_2,\boldsymbol{v}_2,t)$ which describes the ion motion in OSL.

We introduce the dimensionless quantities by measuring positions in units of $L_x=L$, time in units of ω_R^{-1} (where $\omega_R=\hbar k^2/m$ is the photon recoil frequency) and velocities in units of $s_0=\sqrt{T_D/m}$ (where $T_D=\hbar\gamma'/2$ is the characteristic temperature determining a so-called Doppler cooling temperature limit[1,2]).

Then, FPE for the distribution function (DF) averaged over the spatial micro-oscillations, $f=\langle\breve{f}\rangle_s$, can be written as follows (in the case of a moderate spatial modulation of $R_j(\boldsymbol{r})$, $a_1^2\ll 1$):

$$\hat{L}_{FP}f=\frac{\partial f}{\partial t}+\zeta\sum_{\alpha,i}\left(v_{\alpha i}\frac{\partial}{\partial r_{\alpha i}}+\hat{L}_F^{(\alpha i)}\right)f,$$
$$\hat{L}_{FP}=\sum_{\alpha,i}\hat{L}_{FP}^{(ai)},$$
$$\hat{L}_{FP}^{(\alpha i)}=\frac{\partial}{\partial v_{\alpha i}}\left(\breve{\varkappa}(v_{\alpha i})v_{\alpha i}+[D_R(v_{\alpha i})+D_1]\frac{\partial}{\partial v_{\alpha i}}\right), \tag{18}$$

where $\alpha\in\{1,2\}$, $i\in\{x,y,z\}$, $v_{\alpha i}=(\boldsymbol{v}_\alpha\boldsymbol{e}_i)$, $r_{\alpha i}=(\boldsymbol{r}_\alpha\boldsymbol{e}_i)$, $\zeta=s_0/\omega_R L$ is the parameter determining the scale of the Knudsen number value (see Eq. (8)), $\zeta\sim Kn$, and the friction and diffusion coefficients $\breve{\varkappa}(v)$, $D_R(v)$, and D_1 are given by the following expressions

$$\breve{\varkappa}(v)=\breve{\varkappa}_0\mathscr{L}\left(\frac{v}{v_c}\right)+\breve{\varkappa}_1,\ D_R(v)=D_0\mathscr{L}\left(\frac{v}{v_c}\right),$$
$$D_1=\left(\frac{2\chi}{9}\right)\left(1+\frac{2^3G^2}{3^3}\right), \tag{19}$$

with $\breve{\varkappa}_0=2a_1G/9\chi$, $\breve{\varkappa}_1=(2\chi^2/9)\breve{\varkappa}_0$, and $D_0=2G^2/3^3\chi$, where v_c is the half-width of the Lorentzian ($\mathscr{L}$) velocity profile (or capture velocity)

$$v_c=\left(\frac{\gamma}{6\omega_R}\right)^{1/2}\chi. \tag{20}$$

The operators $L_F^{(\alpha i)}$ describe the action of RcGF and Coulomb forces on the ions and can be written as

$$\hat{L}_F^{(\alpha i)}=-\frac{\partial}{\partial v_{\alpha i}}\cdot\left(\frac{\partial\breve{U}_{Ri}(r_{\alpha i},v_{\alpha i})}{\partial r_{\alpha i}}+\frac{\partial\breve{U}_c(|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{\partial r_{\alpha i}}\right), \tag{21}$$

where

$$\breve{U}_{Ri}(r,v)=-\frac{\Delta\breve{W}}{2p_i}\cos(2\pi p_i r)\left(\mathscr{L}\left(\frac{v}{v_c}\right)+\frac{2\chi}{3}\right),$$
$$\breve{U}_c(r)=\frac{e^2}{4\pi\varepsilon_0 rLT_D},\quad \Delta\breve{W}\simeq\frac{4a_1GL}{3^3\lambda b}, \tag{22}$$

$\Delta\breve{W}(1+2\chi/3)$ is the dimensionless depth of the light-induced potential wells for slow ions (along the x axis), $p_i=L/L_i$, $i\in\{x,y,z\}$, $p_x=1$, p_y, $p_z>1$, $\breve{U}_c(r)$ is the dimensionless Coulomb energy of the ions separated by the distance r.

The first term on the right-hand side of Eq. (21) is associated with RcGF and the second term is related to the repulsion Coulomb force.

Note, that we retain (in the expressions for coefficients) small (at $\chi\ll 1$) but velocity-independent terms since they (to be seen later) determine the very far tails of DF (i.e., DF at $v_\alpha\gg v_c$).

Equation (18) is the generalization of FPE (for the particles in OSL) presented in Ref. [9]. This equation takes into account the rectified radiation force dependence on the ion velocity and the ion–ion Coulomb interaction. Here, the situation is considered, where the characteristic frequency of the

ionic translational motion induced by the Coulomb interaction is considerably lower than the decay rate γ' of the excited state $|a\rangle$ and the rate R of the transitions $|ja\rangle - |b\rangle$ (in practice, this is provided by the smallness of the parameters ω_R/γ, ω_R/R). Therefore, the ion–ion Coulomb interaction does not influence the process of the optical force formation. Then, the expressions for the RcGF, friction, and diffusion coefficients can be derived in the framework of the one-particle model by using the standard techniques of the quasi-classical kinetic theory of the light pressure force[1,17,18] and the kinetic theory of rectified radiation forces.[9,14]

2.3. Langevin equations

FPE (18) is equivalent to the following system of stochastic differential equations in the Stratonovich sense[19]

$$\begin{aligned}\frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} &= \zeta v_{\alpha i},\\ \frac{\mathrm{d}v_{\alpha i}}{\mathrm{d}t} &= -\breve{\varkappa}'(v_{\alpha i})v_{\alpha i} - \zeta\frac{\partial}{\partial r_{\alpha i}}\big(\breve{U}_{\mathrm{R}i}(r_{\alpha i},v_{\alpha i}) + \\ &\quad + \breve{U}_{\mathrm{c}}(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\big) + \sqrt{2D(v_{\alpha i})}\circ\Gamma_{\alpha i}(t),\end{aligned} \tag{23}$$

where $\alpha\in\{1,2\}$, $i\in\{x,y,z\}$, $D(v)=D_{\mathrm{R}}(v)+D_1$, $\breve{\varkappa}'(v)=\breve{\varkappa}(v)-\partial D(v)/2v\partial v$, and $\Gamma_{\alpha i}(t)$ are the Langevin stochastic forces (Gaussian white noise) with the averages $\langle\!\langle\Gamma_{\alpha i}(t)\rangle\!\rangle=0$ and correlations given by

$$\langle\!\langle\Gamma_{\alpha i}(t)\Gamma_{\alpha' i'}(t')\rangle\!\rangle = \delta_{\alpha\alpha'}\delta_{ii'}\delta(t-t').$$

Thus, equation (23) describes the light-induced (Brownian) motion of ions in the 12-dimensional phase space $(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{v}_2)$. A significant difference from the model in Ref. [4] is due to the multiplicative characteristic of noises since the coefficients $D(v_{\alpha j})$ in Eq. (23) depend on the particle velocity. Within the range $v_\alpha^2 \ll v_{\mathrm{c}}^2$ the ion motion model with additive noises[4] is obtained from Eqs. (23).

3. Ion motion in the configuration space

In this section the equations of the ion motion in the configuration space $\{\eta\}=(\boldsymbol{r}_1,\boldsymbol{r}_2)$ are derived. These equations are to be used further to obtain correct conditions for the formation of the metastable two-ion (Coulomb) cluster in OSL when inequality (5) is violated. The overdamped characteristic of the ion motion in OSL, which is expressed by inequality (8), $Kn\propto\zeta\ll 1$, allows using the method of Chapman–Enskog type (e.g., see Refs. [20] and [21]) for finding the solution of Eq. (18) on a diffusion time scale $t\gg(\zeta\breve{\varkappa})^{-1}$.

3.1. Chapman–Enskog expansion

Introduce the "slow" time $\tau=\zeta^2 t$ and the operators

$$\hat{L}_1=\sum_{\alpha,i}\hat{L}_1^{(\alpha i)},\quad \hat{L}_1^{(\alpha i)}=v_{\alpha i}\frac{\partial}{\partial r_{\alpha i}}+\hat{L}_{\mathrm{F}}^{(\alpha i)}, \tag{24}$$

and rewrite Eq. (18) as

$$\frac{1}{\zeta}\hat{L}_{\mathrm{FP}}f=\zeta\frac{\partial f}{\partial\tau}+\hat{L}_1 f. \tag{25}$$

In the manner of the Chapman–Enskog method we implement the following asymptotic expansion of f and of the time derivative in the parameter ζ:

$$\begin{aligned}f &= n(\{\eta\},\tau)f_0(\{v\}) + \zeta f_1(\{\eta\},\{v\}|n)\\ &\quad + \zeta^2 f_2(\{\eta\},\{v\}|n)+\cdots,\end{aligned} \tag{26}$$

$$\frac{\partial}{\partial\tau}=\partial_\tau^{(0)}+\zeta\partial_\tau^{(1)}+\cdots, \tag{27}$$

where $\{v\}=(\boldsymbol{v}_1,\boldsymbol{v}_2)$, n is the two-particle space distribution function,

$$n(\{\eta\},\tau)=\langle f\rangle_v=\int\mathrm{d}\{v\}f,$$

the functions f_m depend on time τ as a functional of $n(\{\eta\},\tau)$ and should satisfy the condition $\langle f_m\rangle_v=0$. Substituting Eq. (26) into Eq. (25), one obtains a chain of equations:

$$\hat{L}_{\mathrm{FP}}f_0=0,\quad \hat{L}_{\mathrm{FP}}f_1=\hat{L}_1(nf_0), \tag{28}$$

$$\hat{L}_{\mathrm{FP}}f_2=f_0\partial_\tau^{(0)}n+\hat{L}_1 f_1,\cdots. \tag{29}$$

Then, the expansion of the probability density currents $\boldsymbol{J}_\alpha=\langle f\boldsymbol{v}_\alpha\rangle_v$ ($\alpha\in\{x,y,z\}$) has the form

$$\boldsymbol{J}_\alpha=\zeta\boldsymbol{J}_\alpha^{(1)}+\zeta^3\boldsymbol{J}_\alpha^{(3)}\cdots. \tag{30}$$

Here, we take into account that $\langle f_0\boldsymbol{v}_\alpha\rangle_v=\langle f_2\boldsymbol{v}_\alpha\rangle_v=0$.

Intergrating Eq. (25) over the velocity space $\{v\}$ and taking Eq. (30) into account, we obtain a two-particle continuity equation for $n(\{\eta\},t)$

$$\frac{\partial n}{\partial\tau}=-\sum_{\alpha=1}^{2}\frac{\partial\boldsymbol{J}_\alpha^{(1)}}{\partial\boldsymbol{r}_\alpha}, \tag{31}$$

where the correction term of the order $O(\zeta^2)$ is ignored. It is important that equation (31) is the solvability condition for Eq. (29). So, the explicit form of the continuity (Eq. (31)) can be obtained with the solution of Eqs. (28).

3.2. Ion velocity distribution function

The solution of Eq. (28) for $f_0(\{v\})$ can be presented as the product of the q-Gaussian (Tsallis) distribution and Gaussian distribution functions

$$\begin{aligned}f_0(\{v\}) &= \prod_{\alpha,i}\breve{\rho}(v_{\alpha i}),\quad \alpha\in\{1,2\},\quad i\in\{x,y,z\},\\ \breve{\rho}(v) &= \rho_{\mathrm{s}}(v)\exp\left(-\frac{v^2}{2T_1}\right),\end{aligned} \tag{32}$$

where $\rho_{\mathrm{s}}(v)$ is given by Eq. (9), $T_0=(D_{\mathrm{R}}(0)+D_1)/\breve{\varkappa}(0)\approx(G/3a_1)(1+g^{-1})$, $g=G^2/3\chi^2$, $T_1=D_1/\breve{\varkappa}_1\approx 9/2Ga_1$, index $q=(1/\mathscr{A})+1$, $\mathscr{A}=(v_{\mathrm{s}}^2/2T_0)(1-(T_0/T_1))$, $v_{\mathrm{s}}\approx v_{\mathrm{c}}(1+g)^{1/2}$. Note that T_0/T_1 and $\mathscr{A}T_0/T_1=\chi_1\ll 1$ in the case of $G\lessapprox 1/2$ under consideration. Further, we set the lower bounds of G and $\mathscr{A}$ using the inequality $\mathscr{A}\approx Ga_1\gamma/12\omega_{\mathrm{R}}>5/2$. Then, in the interval $v^2\lesssim T_1$, the distribution $\breve{\rho}(v)$ is well approximated by

the Tsallis distribution ($\check{\rho}(v) \to \rho_s(v)$) with power-low tails: $\rho_s(v) \propto (v_s^2/v^2)^{\mathscr{A}}$ at $v_s^2 \ll v^2 \lesssim T_1$, only in the very far tails of the distribution, i.e., at $v^2 > T_1 \gg v_s^2$, this power law is broken: $\check{\rho}(v) \propto (v_s^2/v^2)^{\mathscr{A}} \exp(-v^2/2T_1)$.

The indicated peculiarities of the distribution function allow calculating the zeroth, second, and fourth moments of the $\check{\rho}(v)$ using the relations $\langle v^{2n}\check{\rho}(v)\rangle_v \approx \langle v^{2n}\rho_s(v)\rangle_v$ for $n = 0,1,2$, which are true when $\chi_1 \ll 1$. The calculations of the zeroth and second moments result in determining the normalizing constant N (in Eq. (32)) and the effective kinetic temperature T of the ions

$$N \approx \sqrt{\pi} v_s \frac{\Gamma(\mathscr{A}-0.5)}{\Gamma(\mathscr{A})}, \quad T = T_0 \frac{\mathscr{A}}{\mathscr{A}-1.5}, \tag{33}$$

where $\Gamma(\mathscr{A})$ is the gamma function. The estimate for the lowest ion cooling limit: $T > T_m = 24\omega_R/\gamma a_1^2$, is obtained from Eq. (33), and this limit is achieved at $\mathscr{A} = 3$ and $g \gg 1$ (i.e., at $3\chi^2 \ll G^2$). To prove this relation it is necessary to take into account that $T_0 \propto \mathscr{A}(1+g^{-1})$.

Also note that $T \approx T_0$ at $\mathscr{A} \gg 3/2$, i.e., at large $\mathscr{A}$ the kinetic temperature T tends to the value T_0 corresponding to the SI model (where the velocity profile of the radiation forces is not taken into account).[4]

Further, the solution of Eq. (28) for f_1 is (at $\chi_1 \ll 1$):

$$f_1 = \sum_{\alpha,i} f_{\alpha i}, \quad f_{\alpha i} = \Psi_{\alpha i}(v_{\alpha i}|n) f_0(\{v\}), \tag{34}$$

$$\Psi_{\alpha i} = \Psi_{\alpha i}^{D} + \Psi_{\alpha i}^{C} + \Psi_{\alpha i}^{R}, \tag{35}$$

$$\Psi_{\alpha i}^{D} = -\frac{\partial n}{\partial r_{\alpha i}} \left(\frac{v_s \mathscr{A}(\mathscr{A}-1)^{-1}}{\check{\varkappa}(0)} \right) \left(u_{\alpha i} + \frac{g_1}{3} u_{\alpha i}^3 \right), \tag{36}$$

$$\Psi_{\alpha i}^{C} = -2\mathscr{A} n \frac{\partial \check{U}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{\partial r_{\alpha i} \check{\varkappa}(0) v_s} \times (g_1 u_{\alpha i} - g \arctan(u_{\alpha i})), \tag{37}$$

$$\Psi_{\alpha i}^{R} = -2\mathscr{A} n \frac{\partial \check{U}_{Ri}(r_{\alpha i})}{\partial r_{\alpha i} \check{\varkappa}(0) v_s} \times \left[\frac{2}{3} \chi g_1 u_{\alpha i} + \left(1 - \frac{2\chi g}{3} \arctan(u_{\alpha i}) \right) \right], \tag{38}$$

where $u_{\alpha i} = v_{\alpha i}/v_s$, $g_1 = (1+g) = v_s^2/v_c^2$, $\check{U}_{Ri}(r) = -(\Delta\check{W}/2p_i)\cos(2\pi p_i r)$.

From expressions (35)–(38) it follows that ζf_1 is the small correction to f_0 at $\zeta_1 = \zeta\sqrt{T_0}/\check{\varkappa}(0) \ll 1$ and $\zeta_2 = 1/b(1+g)^{1/2} \ll 1$.

One can also see that the perturbations of DF $\propto \Psi_{\alpha i}^{D}$, $\Psi_{\alpha i}^{C}$, and $\Psi_{\alpha i}^{R}$ are caused by the gradient of n, Coulomb force, and RcGF, respectively.

3.3. Two-particle Smoluchowski equation and reduced Langevin equations

Using expression (34) to calculate the probability density currents, $\boldsymbol{J}_\alpha^{(1)} = \langle f_1 \boldsymbol{v}_\alpha \rangle_v$, and inserting $\boldsymbol{J}_\alpha^{(1)}$ into continuity equation (31), the two-particle Smoluchowski equation is obtained, which describes the Brownian motion of ions in the configuration space:

$$\frac{\partial n}{\partial \tau} - \frac{1}{\check{\varkappa}(0)} \sum_{\alpha,i} \frac{\partial}{\partial r_{\alpha i}} \left(\frac{\partial \check{U}^{\text{eff}}}{\partial r_{\alpha i}} n \right) = \frac{T_{\text{eff}}}{\check{\varkappa}(0)} \sum_{\alpha i} \frac{\partial^2 n}{\partial r_{\alpha i}^2}, \tag{39}$$

$$\check{U}^{\text{eff}} = \sum_{\alpha i} \check{U}_{Ri}^{\text{eff}}(r_{\alpha i}) + \check{U}_c^{\text{eff}}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|), \tag{40}$$

where the effective potentials and the temperature T_{eff} are introduced. They are determined by the relations:

$$\check{U}_{Ri}^{\text{eff}}(r_i) = \check{U}_{Ri}(r_i)\Lambda_R, \quad T_{\text{eff}} = T\Lambda_{sp}, \tag{41}$$

$$\check{U}_c^{\text{eff}}(r_{12}) = \check{U}_c(r_{12})\Lambda_c = \frac{(e^*)^2}{4\pi\varepsilon_0 r_{12} L T_D},$$

$$e^* = e\sqrt{\Lambda_c}, \tag{42}$$

with $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and corrective multipliers depending on the governing parameters of OSL:

$$\Lambda_R = \left(\frac{\mathscr{A}}{\mathscr{A}-1} + \frac{2}{3}\Lambda_c\chi \right),$$

$$\Lambda_c = \frac{\mathscr{A}(\mathscr{A}-1+g/2)}{(\mathscr{A}-3/2)(\mathscr{A}-1)},$$

$$\Lambda_{sp} = \frac{\mathscr{A}(\mathscr{A}-2+g/2)}{(\mathscr{A}-1)(\mathscr{A}-5/2)}. \tag{43}$$

The effective temperature T_{eff} is determined so that the effective temperature, friction, and spatial diffusion coefficients (i.e., T_{eff}, $\check{\varkappa}(0)$, and D_{sp}) should be related by the standard Einstein relation $T_{\text{eff}} = D_{sp}\check{\varkappa}(0)$. The introduced potentials determine the effective forces

$$\mathscr{F}_{\alpha i}^{R} = -\frac{\partial \check{U}_{Ri}^{\text{eff}}(r_{\alpha i})}{\partial r_{\alpha i}}, \quad \mathscr{F}_{\alpha i}^{c} = -\frac{\partial \check{U}_c^{\text{eff}}(r_{12})}{\partial r_{\alpha i}}. \tag{44}$$

Using the effective forces one can express the density currents $\delta J_{\alpha i}^{R} = n\mathscr{F}_{\alpha i}^{R}/\check{\varkappa}(0)$ and $\delta J_{\alpha i}^{c} = n\mathscr{F}_{\alpha i}^{c}/\check{\varkappa}(0)$ induced by the RcGF and Coulomb interaction respectively. Equation (42) can be explained as the renormalization (re-definition) of the ion charge $e \to e^*$, necessary to accurately consider the factor of the ion velocity dependence of the friction coefficient, velocity diffusion, and RcGF. It is significant that the effective (renormalized) charge is always higher than the "bare" charge of ions $e^* > e$ and is the function of the light field parameters, i.e., $e^* = e^*(\chi, G)$.

Equation (39) is equivalent to the following system of stochastic differential equations (reduced Langevin equations):

$$\frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}\tau} = \frac{\mathscr{F}_{\alpha i}^{R}(r_{\alpha i})}{\check{\varkappa}(0)} + \frac{\mathscr{F}_{\alpha i}^{c}}{\check{\varkappa}(0)} + \sqrt{\frac{2T_{\text{eff}}}{\check{\varkappa}(0)}} \Gamma_{\alpha i}(\tau), \tag{45}$$

where $\alpha \in \{1,2\}$, $i \in \{x,y,z\}$. Equation (45) is considerably simpler for the analysis than Eq. (23) since it is a system of six stochastic differential equations with additive noises, while equation (23) is a system of twelve stochastic differential equations with multiplicative noises.

On the other hand, it is possible to introduce a renormalized version of the model based on Eq. (23) by means of the

following substitutions:

$$\breve{\varkappa}'(v_{\alpha i}) \to \breve{\varkappa}(0),$$
$$D(v_{\alpha i}) \to D(0)\frac{\mathscr{A}}{\mathscr{A}-3/2}\Lambda_{\rm sp} = D^{\rm eff}, \quad (46)$$
$$\breve{U}_{{\rm R}i}(r_{\alpha i}, v_{\alpha i}) \to \breve{U}^{\rm eff}_{{\rm R}i}(r_{\alpha i}), \ \ \breve{U}_{\rm c}(r_{12}) \to \breve{U}^{\rm eff}_{\rm c}(r_{12}). \quad (47)$$

This renormalized model formally corresponds to the SI approximation (since $\breve{\varkappa}(0)$, $D^{\rm eff}$, $\breve{U}^{\rm eff}_{{\rm R}i}$, $\breve{U}^{\rm eff}_{\rm c}$ do not depend on the velocity) and at $t \gg (\zeta\breve{\varkappa}(0))^{-1}$ and $\zeta \ll 1$ results in Eq. (45). So, the SI model[4] allows one to correctly describe light-induced ion motion in the configuration space with an arbitrary relation between the capture velocity $v_{\rm c}$ and thermal velocity, if the suitable renormalization of the Coulomb interaction and radiation forces is implemented.

How this renormalization (and, consequently, the factor of nonlinear velocity dependence of optical forces) influences the formation conditions of the two-ion (Coulomb) cluster in OSL, is considered in the following section.

4. Cluster formation conditions

Taking into account the results of the previous section it is possible to reformulate the formation conditions of the two-ion (Coulomb) cluster in OSL (see Eqs. (1)–(4)) by substituting the "bare" Coulomb force and RcGF by the renormalized forces. The influence of this renormalization appears to be rather significant for the OSL parameters which correspond to the minimum ion temperatures (see Subsection 3.2): $\mathscr{A} \sim 3$, $g \gg 1$. Indeed, the effect of renormalization can be ignored

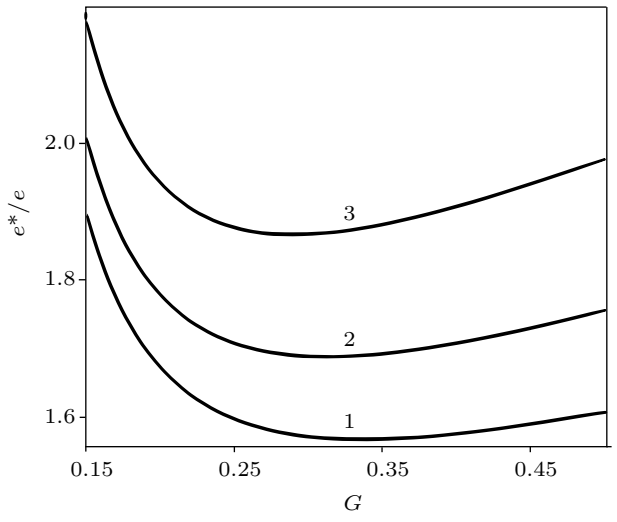


Fig. 2. Ratios of the effective (renormalized) ion charge to the "bare" charge versus G. The governing parameter of OSL $G = G_i$ (for $i \in \{x,y,z\}$) is proportional to the intensity of the "far-off-resonant" field, which is in accordance with Eq. (16). The governing parameter of OSL χ is proportional to the intensity of the resonant field $\boldsymbol{E}'$, $\chi = 0.07$ (line 1), $\chi = 0.06$ (line 2), $\chi = 0.05$ (line 3). The OSL and ion parameters are $a_1^2 = 0.2$, $b = 5$, $\gamma = 4.8\times10^7$ s^{-1}, $m = 171$ a.u. (The units a.u. is atomic unit), and $\lambda = 370$ nm. The parameter $\mathscr{A} = 3$ at $G \approx 0.17$ and $\mathscr{A} = 9$ at $G = 0.5$.

if the corrective multipliers $\Lambda_{\rm c}$ and $\Lambda_{\rm sp}$ are close to unity: $\Lambda_{\rm c} \approx 1$ and $\Lambda_{\rm sp} \approx 1$, i.e., when

$$\mathscr{A} \gg \frac{g}{2} + 2. \quad (48)$$

Note that this condition is equivalent to inequality (5).

Inequality (48) imposes rather strong constraints on the governing parameters G and χ and is unlikely to be satisfied at $\mathscr{A} \sim 3$. Figure 2 illustrates the dependence of the renormalized charge e^* on the parameter G for the case of the $^{171}{\rm Yb}^+$ ions ($\gamma = 4.81\times10^7$ s^{-1}, $\lambda = 370$ nm). One can see from Fig. 2 that the effective charge e^* can exceed the "bare" charge by more than 1.5 times and, consequently, the effective Coulomb force $\propto (e^*)^2$ can increase several times in comparison with the "bare" Coulomb force.

The equations of the force balance (compared with Eq. (1)) should be rewritten as follows:

$$\mathscr{F}^{\rm R}_{\alpha i} + \mathscr{F}^{\rm c}_{\alpha i} = 0, \quad (49)$$

where $\alpha \in \{1,2\}$, $i \in \{x,y,z\}$. According to Ref. [4], we introduce relative and center-of-mass coordinates $\{\eta\} \to \{\eta_1\} = (\boldsymbol{r}, \boldsymbol{R})$:

$$\boldsymbol{r} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{2}, \quad \boldsymbol{R} = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}. \quad (50)$$

For certainty, the situation is considered when two ions are located in the same OSL unit cell Ω_0, with the center of the cell Ω_0 coinciding with the origin of coordinates. Then, the equilibrium (metastable) configurations $\{\eta_1^*\} = (\boldsymbol{r}^*, \boldsymbol{R}^*)$ of the ion cluster are determined by the following relations which result from the minimization of the effective total potential energy $\breve{U}^{\rm eff}(\eta_1))$ and from the second-order condition for the minimization of $\breve{U}^{\rm eff}(\eta_1)$:

$$\xi_i^2 \sin(2\pi|\xi_i|) = p_i^2 \frac{\breve{U}_{\rm c}(1)}{4\pi\Delta\breve{W}}\left(\frac{\Lambda_{\rm c}}{\Lambda_{\rm R}}\right), \quad (51)$$
$$\boldsymbol{R}^* = 0, \ \ \xi_e = \xi_j = 0, \ \ l \neq i \neq j, \quad (52)$$
$$\mathrm{d}^2\breve{U}^{\rm eff}(\{\eta^*_{1i}\}) = H_{ii}(dR_i)^2 + h_{ii}(\mathrm{d}r_i)^2 + \sum_{l\neq i}\left(H_{li}(\mathrm{d}R_l)^2 + h_{li}(\mathrm{d}r_l)^2\right) > 0, \quad (53)$$

where $\xi_i = p_i r_i^*$, H_{ii} and h_{ii} are the non-zero elements of the Hessian matrix for the equilibrium i-configuration $\{\eta^*_{1i}\}$ of the ions oriented along the i axis,

$$H_{ii} = 4\Delta W^{\rm eff}\pi^2\cos(2\pi\xi_i), \ H_{li} = 4\Delta W^{\rm eff}\pi^2 p_l,$$
$$h_{ii} = 4\Delta W^{\rm eff}\pi^2 p_i\left(\cos[2\pi\xi_i] + 2\mathrm{sinc}_\pi[2\xi_i]\right),$$
$$h_{li} = 4\Delta W^{\rm eff}\pi^2\left(p_l - p_i\mathrm{sinc}_\pi[2\xi_i]\right), \quad l \neq i,$$
$$\Delta W^{\rm eff} = \Delta\breve{W}\Lambda_{\rm R}, \quad \mathrm{sinc}_\pi(r) = \frac{\sin(\pi r)}{\pi r}. \quad (54)$$

Inequality (53) implies that the second differential of $\breve{U}^{\rm eff}(\{\eta_1\})$ at the point $\{\eta^*_{1i}\}$ has a positive-definite quadratic form. For each fixed index $i \in \{x,y,z\}$ equation (51) and condition (53) allow us to find the equilibrium i-configuration $\{\eta^*_{1i}\}$. Since $p_x = 1$, and p_y, $p_z > 1$, then, the deepest light-induced wells are the wells along the x axis. Thus, the stable

cluster configurations (at least, x-configurations) can appear only if the potential well depth $\Delta\breve{W}$ along the x axis is sufficiently large as compared with the Coulomb energy $\breve{U}_c(L)$:

$$\frac{\breve{U}_c(1)}{\Delta\breve{W}}\left(\frac{\Lambda_c}{\Lambda_R}\right) < \frac{\pi}{4}. \tag{55}$$

Indeed, in this case there is the positive root $\xi^* < 1/4$ of Eq. (51) for $i = x$ ($p_x = 1$) and the elements of the Hessian matrix are positive (H_{xx}, H_{lx}, h_{xx}, $h_{lx} > 0$, $l \in \{y,z\}$), i.e., condition (53) is satisfied. When inequality (48) is satisfied, condition (55) is transformed into earlier obtained condition (3).[4] It follows from Eq. (55) that the formation of the two-ion cluster is only possible in the case where the period of OSL exceeds the critical length ($L > L_{cr}$) at which $\breve{U}_c(1)\Lambda_c = \pi\Delta\breve{W}\Lambda_R/4$. Since $\xi^* < 1/4$, then the critical (minimal) size of the cluster (i.e., interionic distance) l_{cr} is equal to $L_{cr}/2$.

It is easily understood if it is taken into account that $\breve{U}_c(1) \propto 1/L$ and $\Delta\breve{W} \propto L$ and $l_{cr} = 2L_{cr}\xi^*$. In the present paper a physical situation is investigated when the effects of the nonlinear dependence of the optical forces on the ion velocity are manifested and, thus, inequalities (5) and (48) are not satisfied. In this case, $\Lambda_c/\Lambda_R > 1$ and the critical period L_{cr} exceeds its value corresponding to the SI model. It is a result of the renormalization of the bare ion charge and Coulomb interactions due to the nonlinear velocity dependence of the optical forces. Figure 3 demonstrates the appreciable increase of the critical period in the renormalized model in comparison with the model of the slow ions.

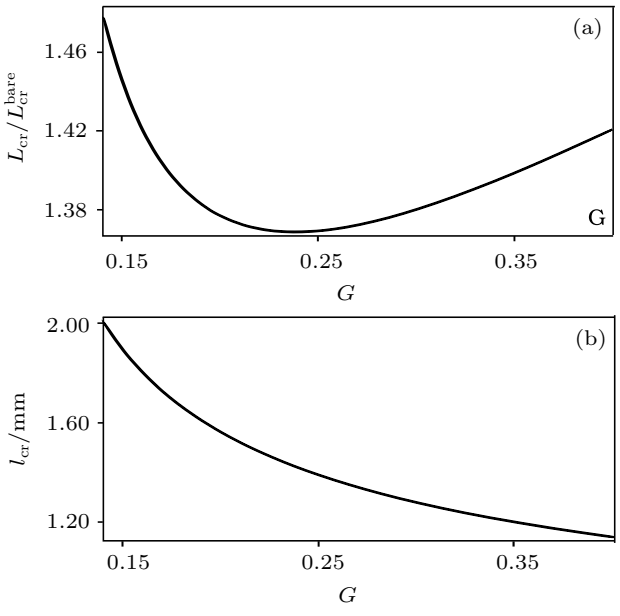


Fig. 3. (a) The ratio of the normalized critical OSL period L_{cr} to the critical OSL period L_{cr}^{bare} in the "bare" SI model versus the governing parameter of OSL G. (b) The critical size of the two-ion cluster versus the parameter G. The governing parameter of OSL $\chi = 0.07$, all the other parameters are the same as those in Fig. 2.

The condition analogous to inequality (4) for the renormalized model is modified to the following inequality:

$$\Upsilon^* = \frac{\Delta\breve{W}}{T}\left(\frac{\Lambda_R}{\Lambda_{sp}}\right) \gg 1. \tag{56}$$

Inequality (56) provides a possibility of forming a long-lived two-ion cluster with the lifetime $\tau \sim \tau_M \gg \tau_{cl}$, where τ_{cl} is the characteristic time of the cluster formation.[4] Besides, from inequality (56) it follows that at time $\tau \gg \tau_{cl}$ the fluctuations of the ions about their equilibrium positions are small as compared with the dimensionless cluster size $2\xi^*$.

Now we consider the last question in more detail. To be more exact, we are to investigate the dynamics of the fluctuations of ions about their equilibrium positions corresponding to x-configuration with $p_y = p_z > 1$. The linear approximation of Eq. (45) can be written in the compact matrix form

$$\frac{d\tilde{r}}{d\tau} = -\frac{1}{\breve{\varkappa}(0)}\mathscr{H}\tilde{r} + \sqrt{\frac{T_{eff}}{\breve{\varkappa}(0)}}\Gamma(\tau), \tag{57}$$

where $\mathscr{H}$ is the diagonal Hessian matrix ($\mathscr{H} = \mathrm{Diag}(h_{xx}, h_{xy}h_{xz}, H_{xx}, H_{xy}H_{xz})$) which is defined by Eq. (54) with $i = x$, $p_y = p_z = p > p_x = 1$, $\tilde{r}$ is the column vector, $\tilde{r} = \mathrm{col}(\tilde{r}_x, r_y, r_z, R_x, R_y, R_z)$, $\tilde{r}_x = r_x - \xi^*$, and ξ^* is the positive root of Eq. (51) for $i = x$,

$$\Gamma(\tau) = \mathrm{col}(\Gamma_x^{(-)}(\tau),\ \Gamma_y^{(-)}(\tau),\ \Gamma_z^{(-)}(\tau),\ \Gamma_x^{(+)}(\tau), \Gamma_y^{(+)}(\tau),\ \Gamma_z^{(+)}(\tau)),\ \Gamma_i^{\beta}(\tau))$$

is the Gaussian white noise ($i \in \{x,y,z\}$, $\beta \in \{+,-\}$) and correlator $\langle\langle\Gamma_i^{(\beta)}(\tau)\Gamma_{i'}^{(\beta')}(\tau')\rangle\rangle = \delta_{\beta\beta'}\delta_{ii'}\delta(\tau - \tau')$.

Equation (57) can be used for the time shorter than the characteristic lifetime of the ion cluster τ_M, whose finiteness is due to the manifestation of the large rare fluctuations,[22] i.e., the cluster metastability. This equation leads to the estimation of the characteristic formation time of the cluster τ_{cl}: $\tau_{cl} \sim \breve{\varkappa}(0)/h$ with $h = \min\{H_{xx}, h_{xx}, h_{yx} = h_{zx}\}$. The numerical simulations[4] indicate that under condition (56) the following is always true: $\tau_M \gg \tau_{cl}$. Using the standard method of the analysis of stochastic differential equations[19,22] one can easily obtain (established when $\tau_{cl} \ll \tau < \tau_M$) quasi-steady values of the average and standard deviation σ for the column vector $\tilde{r}$ components.

In particular, the following important relations are obtained:

$$\langle\langle\tilde{r}\rangle\rangle = 0, \quad \delta R_i < \left(\frac{1}{4\pi\xi^*}\right)(2\Upsilon^*\cos(2\pi\xi^*))^{-1/2},$$
$$\delta r = (1/2\pi\xi^*)(2\Upsilon^*[\cos(2\pi\xi^*) + 2\mathrm{sinc}_\pi(2\xi^*)])^{-1/2}, \tag{58}$$

where $\delta r = \sigma\{|\boldsymbol{r}|\}/2\xi^*$ and $\delta R_i = \sigma\{R_i\}/2\xi^*$ are the values characterizing the relative level of fluctuations (which is compared with the size of the cluster $2\xi^*$), the interionic distance, and CM position. These fluctuations are suppressed when the parameter Υ^* increases. Figure 4 shows the characteristic scale of the value of Υ^* for certain values of the

OSL parameters. For the parameters in Fig. 4 the δr and δR_i are about 0.07, temperature $T \approx 0.3$, $8.5 > \mathscr{A} > 2.65$ and the coupling parameters, Γ_c, $\Gamma_c^{\rm eff}$, considerably exceed unity: $\Gamma_c = \check{U}_c(2\xi^*)/T \approx 60$, $\Gamma_c^{\rm eff} = \check{U}_c^{\rm eff}(2\xi^*)/T_{\rm eff} \approx 50$ at $G \approx 0.28$. Moreover, in this case the parameters of the expansion of the DF, ζ_1, ζ_2, are significantly lower than unity: $\lesssim 0.05$, the intensity of the partially coherent optical field I' does not exceed the value 40 mW/cm^2, and intensity of the "far-off-resonant" field $I_i \lesssim 1$ W/cm^2 for $\Delta_y \sim 1.5\Delta_x$, $\Delta_z \sim 2\Delta_x$, $\Delta_x \sim 80\gamma$, $\Gamma = 10\gamma$.

This example demonstrates a real possibility to satisfy all the formation conditions of the two-ion (Coulomb) cluster in OSL in the case where the SI approximation (inequality (5)) is not applicable and one has to use the renormalized model constructed in the present work.

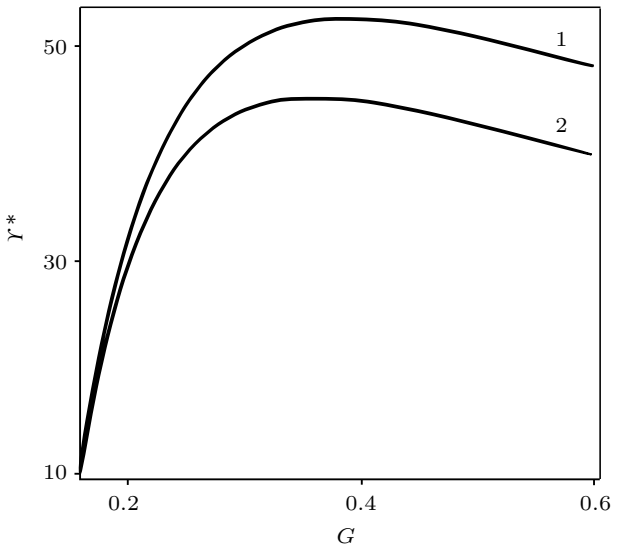


Fig. 4. The parameter Υ^* versus the governing parameter of OSL G. The value of ξ^* is fixed to be equal to 0.18, the governing parameter of OSL $\chi = 0.07$ (line 1), $\chi = 0.0.5$ (line 2). All the other parameters are the same those as in Fig. 2.

5. Conclusion

It is shown that in order to correctly formulate the conditions for the formation of the two-ion (Coulomb) cluster in the 3D polychromatic dissipative superlattice cell, suitable renormalization of the optical and Coulomb force terms in the force balance equation is necessary. Such a renormalization allows one to take into account the important factor of the nonlinear dependence of the optical forces on the velocity and, the strongly non-Maxwellian (Tsallis type) characteristic of the velocity distribution of ions in OSL.

The renormalization effects are most significant for the case of weak light fields, where the lowest (sub-Doppler) temperatures of ions are reached. In this case the effective charge e^* (which determines the renormalized Coulomb ion–ion interaction) is a function of OSL governing parameters, G and χ, and always exceeds the value of the "bare" ion charge. The consequence of this circumstance is the increase of the critical (minimal) period of the OSL at which the effect of the two-ion cluster formation is achieved.

Another important result of the present work which has stand-alone value, is the derivation of a closed system of stochastic differential equations describing the light-induced Brownian motion of ions in the configuration space $(\boldsymbol{r}_1, \boldsymbol{r}_2)$. This system of reduced Langevin equations with additive noises is considerably simpler than the original system of the Langevin equations with multiplicative noises. Using these equations to describe the ion dynamics in OSL is not limited to the narrow frames of the slow ion approximation[9] and allows implementing the numerical simulations of the long-time dynamics of the ions for much longer time than in Ref. [4]. This is extremely important for the investigation of the dependence of the cluster life-time on the OSL parameters. The cluster life-time is determined by the rare large fluctuations, thus, its accurate calculation requires carrying out simulations in long-time intervals.

In conclusion, it may be suggested that the considered effect of renormalization of the Coulomb interaction should be used in problems of the optical creation and confinement of ultracold electron–ion plasma.[8]

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