Mittag–Leffler synchronization of fractional-order uncertain chaotic systems*

Wang Qiao(王 乔)†, Ding Dong-Sheng(丁冬生), and Qi Dong-Lian(齐冬莲)

College of Electrical Engineering, Zhejiang University, Hangzhou 310027, *China*

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This paper deals with the synchronization of fractional-order chaotic systems with unknown parameters and unknown disturbances. An adaptive control scheme combined with fractional-order update laws is proposed. The asymptotic stability of the error system is proved in the sense of generalized Mittag–Leffler stability. The two fractional-order chaotic systems can be synchronized in the presence of model uncertainties and additive disturbances. Finally these new developments are illustrated in examples and numerical simulations are provided to demonstrate the effectiveness of the proposed control scheme.

Keywords: chaos, synchronization, adaptive control, Mittag–Leffler stability

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1. Introduction

Fractional-order systems have become an active research field in recent years. Since the fractional calculus enables us to describe real world physical phenomena more accurately than the classical integer calculus, the research on the dynamics, control, and applications of fractional-order systems deserves considerable attention.[1,2] The controls of linear and nonlinear systems have been reported in a lot of literature related to the system stability analysis,[3–6] controller design,[7,8] system simulation, etc. Among many physical and biological fractional-order systems, it has been confirmed theoretically and practically that the chaotic behaviors exist in the fractional-order systems, such as fractional-order Chua's system,[9] fractional-order financial system,[10] and fractional-order unified chaotic system,[11] etc.

Synchronization of fractional-order chaotic systems is of significance for real world physical applications. Fractional-order chaotic systems have been found in physics, engineering, finance, and sociology. The control and synchronization of fractional-order chaotic systems have been used successfully in many real applications, such as permanent magnet synchronous motor,[12] circuit design,[13] and secret communication.[14] For classical chaotic system synchronization, much research work has been done in some areas, such as linear control,[15] nonlinear state feedback control, robust control,[16,17] and sliding mode control.[18–20] In recent years, control research on the synchronization of fractional-order chaotic systems were extensively investigated.[21–25] However, the integer-order stability analysis tools are inapplicable for fractional-order chaotic systems in the sense of Lyapunov stability.

For fractional-order nonlinear stability analysis, the generalized Mittag–Leffler stability[26] is a powerful tool, in terms of fractional Lyapunov function, to prove controlled system stability. Although an appropriate candidate Lyapunov function is often hard to obtain, some recent work[27,28] provides suitable fractional-order inequalities and analysis techniques to facilitate the process of controller design.

To our best knowledge, the work on the synchronization of fractional-order chaotic systems with uncertain parameters, model uncertainty or additive disturbance, is not so much in the sense of the generalized Mittag–Leffler stability. In practical applications, it is inevitable to encounter the situation where systems are often with various uncertainties,[29,30] which will be considered in this paper.

In our contributions, we use the generalized Mittag–Leffler stability to deal with the synchronization of fractional-order chaotic systems with uncertain parameters and uncertain disturbances. To our best knowledge, most of the studies are based on traditional Lyapunov stability, however we employ the generalized Mittag–Leffler stability, which implies Lyapunov asymptotic stability, to design the controller, by constructing a fractional Lyapunov function. The uncertainties are often caused by model uncertainty and additive noise, and these uncertainties usually can be dealt with under certain conditions that the uncertainties are assumed to be bounded either by a known or an unknown constant. For the case that uncertainties are bounded by a known constant, a robust or adaptive control proposed in the cited references in this paper can be employed to deal with this situation. Meanwhile, for the case of an unknown constant, limited work has addressed this issue before and we propose fractional-order update laws

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†Corresponding author. E-mail: qiao@zju.edu.cn

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for the unknown parameters and unknown upper bounds of the uncertainties respectively. An adaptive control combined with fractional-order state feedback and uncertainty estimates is proposed to synchronize two fractional-order chaotic systems. Although the proposed control law consists of multi-controllers, it presents a new insight into this issue theoretically and it holds true for most of the chaotic systems, even the synchronization between two different chaotic systems as will be discussed later in Corollary 2. Finally, illustrative examples and simulations are provided to demonstrate our control design.

The rest of this paper is organized as follows. In Section 2, fractional calculus properties and fractional-order system stability criteria are introduced. In Section 3, the adaptive control design and estimated update laws are presented. In Section 4, illustrative examples and numerical simulation results are provided. Finally, some conclusions are drawn from the present studies in Section 5.

2. Problem formulation and preliminaries

Definition 1 Let $f:[a,b]\to R$ and $f\in L^1[a,b]$. The Caputo fractional derivative of order α is defined as

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\tau, \tag{1}$$

where $\alpha\in R^+$ and $\Gamma(\cdot)$ is the Gamma function. The order of the chaotic system is $0<\alpha<1$. In this paper we employ D for representing the classical integer differential $D^1f(t)=\mathrm{d}f(t)/\mathrm{d}t$.

Furthermore, there are some properties for the fractional calculus.[10]

(i) For $\alpha=n$, where n is an integer, the fractional-order derivative coincides with the integer order derivative. Particularly, when $\alpha=0$, it appears as the identity operator, i.e.,

$$D^{\alpha}D^{-\alpha}f(t)=D^0f(t)=f(t). \tag{2}$$

(ii) The fractional operator is a linear operator

$$D^{\alpha}[af(t)+bg(t)]=aD^{\alpha}f(t)+bD^{\alpha}g(t), \tag{3}$$

where a and b are real constants.

Theorem 1[26] Let $x(t)=0$ be the equilibrium point of the fractional-order system $D^{\alpha}x=f(x,t)$, $x\in D\subset R^n$, where D contains the origin. Assume that a fractional Lyapunov function $V(x,t):R^n\times[0,\infty)\to R$ is a continuous differential function and locally Lipschitz with respect to x, and there exists class-K function γ_i such that

$$\begin{aligned}&\gamma_1(\|x\|)\le V(x,t)\le\gamma_2(\|x\|),\\&D^{\alpha}V(x,t)\le-\gamma_3(\|x\|),\end{aligned} \tag{4}$$

then $x(t)=0$ is asymptotically stable. Moreover, if the conditions hold globally on $D=R^n$, then $x(t)=0$ is globally asymptotically stable.

The above theorem deals with system stability in the sense of fractional order, and is also called the generalized Mittag–Leffler stability theorem. It should be noted that Mittag–Leffler stability implies Lyapunov asymptotic stability.[26]

Theorem 2[5] Let fractional-order linear time-invariant (LTI) system $D^{\alpha}x(t)=Ax(t)$, where $A\in R^{n\times n}$ and $x\in R^n$, if the following condition is satisfied:

$$|\arg(\mathrm{eig}(A))|>\alpha\frac{\pi}{2}, \tag{5}$$

then the LTI system is asymptotically stable.

Since the fractional Lyapunov function described in Theorem 1 must work within the framework of fractional-order inequalities, here we present some powerful inequalities concerning the fractional-order systems.

Lemma 1[27] Let $x(t)\in R$ be a continuous and derivable function, then, for any time $t\ge0$, the following inequality will always hold:

$$\frac{1}{2}D^{\alpha}x^2(t)\le x(t)D^{\alpha}x(t), \tag{6}$$

where $\alpha\in(0,1)$.

Lemma 2[28] Let $x(t)=[x_1(t),\ldots,x_n(t)]\in R^n$ be a real-valued continuous and derivable vector function, then, for any time $t\ge0$, the following inequality will always hold:

$$\frac{1}{2}D^{\alpha}x(t)^{\mathrm{T}}Px(t)\le x(t)^{\mathrm{T}}PD^{\alpha}x(t), \tag{7}$$

where $\alpha\in(0,1)$ and $P=\mathrm{diag}[p_1,\ldots,p_n]>0$.

Remark 1 The above two lemmas are conducive to constructing fractional Lyapunov candidate function through transforming the well-used quadratic function in Lyapunov direct method into system fractional differential equations by inequality.

3. Synchronization of fractional-order uncertain chaotic systems

Consider a pair of fractional-order uncertain chaotic systems. The drive system can be described by

$$\begin{cases}D^{\alpha}x_1(t)=f_1(x)-\varphi_1^{\mathrm{T}}(x)\theta_1,\\D^{\alpha}x_2(t)=f_2(x)-\varphi_2^{\mathrm{T}}(x)\theta_2,\\\vdots\\D^{\alpha}x_n(t)=f_n(x)-\varphi_n^{\mathrm{T}}(x)\theta_n.\end{cases} \tag{8}$$

The response system with unknown uncertainties can be described by

$$\begin{cases}D^{\alpha}y_1(t)=f_1(y)+\varphi_1^{\mathrm{T}}(y)\hat{\theta}_1+\Delta f_1(y,t)+d_1(t)+u_1,\\D^{\alpha}y_2(t)=f_2(y)+\varphi_2^{\mathrm{T}}(y)\hat{\theta}_2+\Delta f_2(y,t)+d_2(t)+u_2,\\\vdots\\D^{\alpha}y_n(t)=f_n(y)+\varphi_n^{\mathrm{T}}(y)\hat{\theta}_n+\Delta f_n(y,t)+d_n(t)+u_n,\end{cases} \tag{9}$$

where the system state $x \in R^n$, $y \in R^n$, and $\alpha \in (0,1)$ is the order of the fractional derivative; f_i and φ_i^{T} $(i = 1,\ldots,n)$ are continuous nonlinear system vector functions; $\theta_i \in R^{m_i}$ $(i = 1,\ldots,n)$ are the unknown parameters in the drive system; $\hat{\theta}_i \in R^{m_i}$ $(i = 1,\ldots,n)$ are the estimates of θ_i in the drive system; $\Delta f_i(y,t)$ $(i = 1,\ldots,n)$ are unknown model uncertainty terms; $d_i(t)$ $(i = 1,\ldots,n)$ are external disturbances of the system; u_i $(i = 1,\ldots,n)$ are system controllers.

The system error is defined as $e = y - x$, and systems (8) and (9) are said to be synchronized under arbitrary initial conditions $x(0)$ and $y(0)$ if

$$\lim_{t\to\infty} ||e(t)|| = ||y(t) - x(t)|| = 0. \tag{10}$$

The error system can be represented by the following equations:

$$\begin{cases} D^\alpha e_1(t) = f_{e1}(e) + \varphi_1^{\mathrm{T}}(x)\theta_1 + \Delta f_{e1} + u_1, \\ D^\alpha e_2(t) = f_{e2}(e) + \varphi_2^{\mathrm{T}}(x)\theta_2 + \Delta f_{e2} + u_2, \\ \vdots \\ D^\alpha e_n(t) = f_{en}(e) + \varphi_n^{\mathrm{T}}(x)\theta_n + \Delta f_{en} + u_n, \end{cases} \tag{11}$$

where $f_{ei}(e) = f_i(y) - f_i(x) + \varphi_i^{\mathrm{T}}(y)\hat{\theta}_i$ $(i = 1,\ldots,n)$ and $\Delta f_{ei} = \Delta f_i(y,t) + d_i(t)$.

Assumption 1 The unknown model uncertain terms $\Delta f_i(y,t)$ and external disturbances $d_i(t)$ are all bounded by unknown positive constants, which implies

$$|\Delta f_{ei}| = |\Delta f_i(y,t) + d_i(t)| \le \delta_i, \tag{12}$$

where δ_i $(i = 1,\ldots,n)$ are unknown positive constants.

Remark 2 The term Δf_{ei} is introduced for simplifying the system expression, and one can also consider $\Delta f_i(y,t)$ and $d_i(t)$ directly instead of Δf_{ei}, which makes no difference in the error system asymptotic stability. Furthermore, it is also available to consider uncertain terms and external disturbances in the drive system (8), and this consideration only leads to the change of Δf_{ei}. It can be seen that our control method is valid under the Assumption 1, thus it will also work if the uncertainties are considered in both drive system and response system.

Theorem 3 The error system (11) with uncertainties can be globally stabilized asymptotically by the adaptive feedback control

$$u_i = -[f_{ei} + \varphi_i^{\mathrm{T}}(x)\hat{\theta}_i + \lambda_i e_i + s_i + m_i + k_i \mathrm{sgn}(s_i)], \tag{13}$$

where $s_i = e_i + \lambda_i D^{-\alpha} e_i$,

$$m_i = \frac{s_i \hat{\delta}_i^2}{|s_i|\,\hat{\delta}_i + s_i^2};$$

λ_i and k_i are constants, update laws

$$D^\alpha \hat{\theta}_i = \Gamma_i \varphi_i(x) s_i, \tag{14}$$

and

$$D^\alpha \hat{\delta}_i = r_i |s_i|, \tag{15}$$

with $\hat{\theta}_i$ being the estimate of the unknown parameter θ_i, and $\hat{\delta}_i$ being the estimate of the unknown upper bound δ_i in Assumption 1; $\Gamma_i = \mathrm{diag}[p_{i1},\ldots,p_{im}] > 0$ and $r_i > 0$ are the gains of the update laws respectively.

Proof The procedure of the proof consists of two steps.

Step 1 By introducing $s_i = e_i + \lambda_i D^{-\alpha} e_i$ for the i-th equation in error system (11), and using the Caputo derivative, we obtain

$$D^\alpha s_i = D^\alpha e_i + \lambda_i e_i. \tag{16}$$

Choosing Lyapunov function for the i-th equation

$$V_i(e, \tilde{\theta}_i, \tilde{\delta}_i) = \frac{1}{2} s_i^2 + \frac{1}{2} \tilde{\theta}_i^{\mathrm{T}} \Gamma_i^{-1} \tilde{\theta}_i + \frac{1}{2r_i} \tilde{\delta}_i^2, \tag{17}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ is the parameter estimate error and $\tilde{\delta}_i = \delta_i - \hat{\delta}_i$ is the upper bound estimate error, using the Caputo derivative, and according to Lemma 1 and Lemma 2 we have

$$D^\alpha V_i \le s_i D^\alpha s_i + \tilde{\theta}_i^{\mathrm{T}} \Gamma_i^{-1} D^\alpha \tilde{\theta}_i + \frac{1}{r_i} \tilde{\delta}_i D^\alpha \tilde{\delta}_i. \tag{18}$$

By Eq. (16), we have

$$\begin{aligned} D^\alpha V_i \le\ & s_i (f_{ei} + \varphi_i^{\mathrm{T}}(x)\theta_i + \Delta f_{ei} + u_i + \lambda_i e_i) \\ & - \tilde{\theta}_i^{\mathrm{T}} \Gamma_i^{-1} D^\alpha \hat{\theta}_i - \frac{1}{r_i} \tilde{\delta}_i D^\alpha \hat{\delta}_i \\ =\ & s_i (f_{ei} + \varphi_i^{\mathrm{T}}(x)\hat{\theta}_i + u_i + \lambda_i e_i) + s_i \Delta f_{ei} \\ & + \tilde{\theta}_i^{\mathrm{T}} (\varphi_i^{\mathrm{T}}(x) s_i - \Gamma_i^{-1} D^\alpha \hat{\theta}_i) - \frac{1}{r_i} \tilde{\delta}_i D^\alpha \hat{\delta}_i. \end{aligned} \tag{19}$$

Substitute control u_i and note that

$$s_i \Delta f_{ei} \le |s_i \Delta f_{ei}| \le |s_i|\,|\Delta f_{ei}| \le |s_i|\,\delta_i = |s_i|\,(\tilde{\delta}_i + \hat{\delta}_i),$$

we have

$$\begin{aligned} D^\alpha V_i \le\ & -s_i k_i \mathrm{sgn}(s_i) + \tilde{\theta}_i^T (\phi_i^T(x) s_i - \Gamma_i^{-1} D^\alpha \hat{\theta}_i) \\ & - \tilde{\delta}_i (|s_i| - \frac{1}{r_i} D^\alpha \hat{\delta}_i) + |s_i|\,\hat{\delta}_i - s_i^2 - s_i m_i. \end{aligned} \tag{20}$$

By use of update laws (14) and (15) and noting that $|s_i|\,\hat{\delta}_i - s_i^2 - s_i m_i \le 0$, we have

$$D^\alpha V_i \le -s_i k_i \mathrm{sgn}(s_i) = -k_i |s_i|. \tag{21}$$

Step 2 Choose Lyapunov function $V = \sum_{i=1}^{n} V_i$ for the whole error system (11), then we have

$$D^\alpha V(e, \tilde{\theta}_1,\ldots,\tilde{\theta}_n, \tilde{\delta}_1,\ldots,\tilde{\delta}_n) \le -\sum_{i=1}^{n} k_i |s_i|. \tag{22}$$

If $D^\alpha V = 0$, which implies $s_i = 0$, thus the error system becomes

$$D^\alpha e_i(t) = -\lambda_i e_i. \tag{23}$$

By choosing λ_i appropriately to satisfy Theorem 2, the error system achieves asymptotic stability.

If $D^\alpha V < 0$, and since $V(e,\tilde{\theta}_1,\ldots,\tilde{\theta}_n,\tilde{\delta}_1,\ldots,,\tilde{\delta}_n)$ is positive-definite, based on the relationship between class-K functions and positive-definite functions, there exists a class-K function γ such that $D^\alpha V \le -\gamma(\|\eta\|)$, where $\eta = [e^{\mathrm{T}},\tilde{\theta}_1^{\mathrm{T}},\ldots,\tilde{\theta}_n^{\mathrm{T}},\tilde{\delta}_1,\ldots,,\tilde{\delta}_n]^{\mathrm{T}}$. According to Theorem 1, the error system is asymptotically stable. This completes the proof.

Corollary 1 Theorem 3 makes it also possible to design a simpler feedback control in the way of replacing s_i by e_i. In this case, the items $k_i\mathrm{sgn}(s_i)$ and m_i in u_i become $k_i e_i$ and

$$m_i = \frac{e_i\hat{\delta}_i^2}{|e_i|\,\hat{\delta}_i + k_i e_i^2}$$

respectively. Follow the two steps in the proof, we have

$$\begin{cases} V_i = \dfrac{1}{2}e_i^2 + \dfrac{1}{2}\tilde{\theta}_i^{\mathrm{T}}\Gamma_i^{-1}\tilde{\theta}_i + \dfrac{1}{2r_i}\tilde{\delta}_i^2, \\ D^\alpha V_i \le -k_i e_i^2, \end{cases} \tag{24}$$

and

$$\begin{cases} V(e,\tilde{\theta}_1,\ldots,\tilde{\theta}_n,\tilde{\delta}_1,\ldots,,\tilde{\delta}_n) = \sum\limits_{i=1}^{n} V_i, \\ D^\alpha V(e,\tilde{\theta}_1,\ldots,\tilde{\theta}_n,\tilde{\delta}_1,\ldots,\tilde{\delta}_n) \le -\sum\limits_{i=1}^{n} k_i e_i^2. \end{cases} \tag{25}$$

If $D^\alpha V = 0$, this implies $e_i = 0$; otherwise $D^\alpha V < 0$, thus the error system is asymptotically stable according to Theorem 1.

Corollary 2 It is also available to achieve the synchronization of two different fractional-order chaotic systems, based on Theorem 3. In this case, the response system (9) becomes $D^\alpha y_i(t) = g_i(y) + \Delta f_i(y,t) + d_i(t) + u_i$ and $f_{ei}(e) = g_i(y) - f_i(x)$ in the error system (11) correspondingly. Following the procedure of the proof of Theorem 3, we can design an adaptive control for the synchronization between two different chaotic systems, which case will be illustrated in Example 3 in the simulations.

4. Illustrative examples and simulations

In this section, we will give numerical simulations to illustrate the effectiveness of the proposed adaptive control.

Example 1 Consider the fractional-order Van der Pol oscillator

$$\begin{aligned} D^\alpha x_1(t) &= x_2(t), \\ D^\alpha x_2(t) &= -x_1(t) - \varepsilon(x_1^2(t)-1)x_2(t), \end{aligned} \tag{26}$$

where $\alpha \in (0,1)$ and ε is the system parameter. The response system within unknown ε is expressed as

$$\begin{aligned} D^\alpha y_1(t) &= y_2(t) + \Delta f_1(y,t) + d_1(t) + u_1, \\ D^\alpha y_2(t) &= -y_1(t) - \hat{\varepsilon}(y_1^2(t)-1)y_2(t) \\ &\quad + \Delta f_2(y,t) + d_2(t) + u_2, \end{aligned} \tag{27}$$

the adaptive control is

$$\begin{aligned} u_1 &= -[y_2 - x_2 + \lambda_1(y_1 - x_1) + s_1 + m_1 + k_1\mathrm{sgn}(s_1)], \\ u_2 &= -[-y_1 - \hat{\varepsilon}(y_1^2-1)y_2 + \hat{\varepsilon}(x_1^2-1)x_2 + x_1 \\ &\quad + \lambda_2(y_2 - x_2) + s_2 + m_2 + k_2\mathrm{sgn}(s_2)], \end{aligned} \tag{28}$$

and the update laws are

$$\begin{aligned} D^\alpha \hat{\varepsilon} &= p(x_1^2-1)x_2 s_2, \\ D^\alpha \hat{\delta}_i &= r_i\,|s_i|\,. \end{aligned} \tag{29}$$

Figures 1 and 2 show the convergence of the Van der Pol error system and the system synchronization, with initial conditions $x(0) = [2,-3]^\top$, $y(0) = [5,4]^\top$, $\hat{\varepsilon}(0) = 2$; gains of update law $p = 1$, $r_i = 2$, model uncertainties $\Delta f_i = 0.5\cos(\pi t)$, external disturbances $d_i(t) = 0.1\sin(5t)$; control parameters $k_i = 2$ and $\lambda_i = 1$; system order $\alpha = 0.96$.

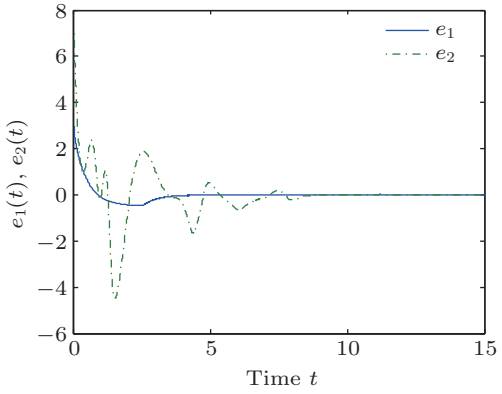


Fig. 1. (color online) Convergence of Van der Pol error system.

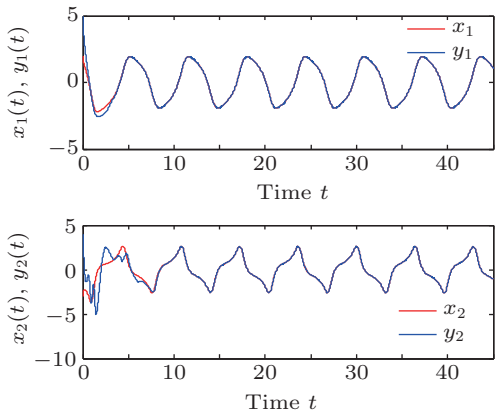


Fig. 2. (color online) Synchronization of Van der Pol systems.

Example 2 Consider the fractional-order Genesio–Tesi system

$$\begin{aligned} D^\alpha x_1(t) &= x_2(t), \\ D^\alpha x_2(t) &= x_3(t), \\ D^\alpha x_3(t) &= -\beta_1 x_1(t) - \beta_2 x_2(t) - \beta_3 x_3(t) + \beta_4 x_1^2(t), \end{aligned} \tag{30}$$

where $\alpha \in (0,1)$ and β_i are the system parameters. The response system with unknown β_1 and β_2 is expressed as

$$D^\alpha y_1(t) = y_2(t) + \Delta f_1(y,t) + d_1(t) + u_1,$$

$$
\begin{aligned}
D^{\alpha}y_2(t) &= y_3(t)+\Delta f_2(y,t)+d_2(t)+u_2,\\
D^{\alpha}y_2(t) &= -\hat{\beta}_1 y_1(t)-\hat{\beta}_2 y_2(t)-\beta_3 y_3(t)\\
&\quad +\beta_4 y_1^2(t)+\Delta f_3(y,t)+d_3(t)+u_3, \qquad (31)
\end{aligned}
$$

the adaptive control is

$$
\begin{aligned}
u_1 &= -[y_2-x_2+\lambda_1(y_1-x_1)+s_1+m_1+k_1\mathrm{sgn}(s_1)],\\
u_2 &= -[y_3-x_3+\lambda_2(y_2-x_2)+s_2+m_2+k_2\mathrm{sgn}(s_2)],\\
u_3 &= -[-\hat{\beta}_1 y_1(t)-\hat{\beta}_2 y_2(t)-\beta_3 y_3(t)+\beta_4 y_1^2(t)\\
&\quad +\hat{\beta}_1 x_1(t)+\hat{\beta}_2 x_2(t)+\beta_3 x_3(t)-\beta_4 x_1^2(t)\\
&\quad +\lambda_3(y_3-x_3)+s_3+m_3+k_3\mathrm{sgn}(s_3)], \qquad (32)
\end{aligned}
$$

and the update laws are

$$
\begin{aligned}
D^{\alpha}\hat{\beta}_i &= p_i x_i s_i,\\
D^{\alpha}\hat{\delta}_i &= r_i|s_i|. \qquad (33)
\end{aligned}
$$

Figures 3 and 4 show the convergence of the Genesio–Tesi error system and the system synchronization, with initial conditions $x(0)=[0.1,0.5,-0.2]^{\top}$, $y(0)=[-1,-2,1]^{\top}$, and $\hat{\beta}_i(0)=2$; gains of update law $p_i=2$, $r_i=3$; model uncertainties $\Delta f_i=0.4\sin(\pi t)$; external disturbances $d_i(t)=0.2\cos(5t)$; control parameters $k_i=1$ and $\lambda_i=3$; system order $\alpha=0.98$.

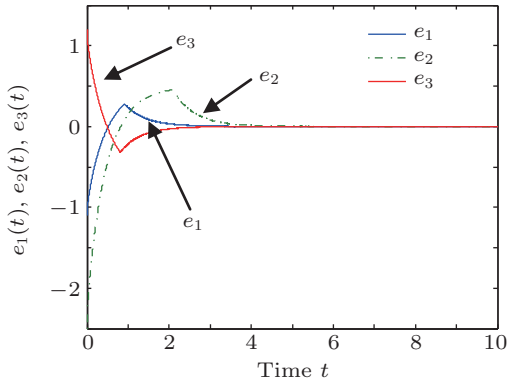


Fig. 3. (color online) Convergence of Genesio–Tesi error system.

Example 3 Consider the synchronization of two different chaotic systems as discussed in Corollary 2. The drive system is the Lorenz system

$$
\begin{aligned}
D^{\alpha}x_1(t) &= a(x_2(t)-x_1(t)),\\
D^{\alpha}x_2(t) &= x_1(t)(b-x_3(t))-x_2(t),\\
D^{\alpha}x_3(t) &= x_1(t)x_2(t)-cx_3(t), \qquad (34)
\end{aligned}
$$

where $\alpha\in(0,1)$; a and c are the unknown parameters. The response system is chosen to be the Lü system with uncertainties

$$
\begin{aligned}
D^{\alpha}y_1(t) &= \tau(y_2(t)-y_2(t))+\Delta f_1(y,t)+d_1(t)+u_1,\\
D^{\alpha}y_2(t) &= -y_1(t)y_3(t)+\rho y_2(t)+\Delta f_2(y,t)+d_2(t)+u_2,\\
D^{\alpha}y_2(t) &= y_1(t)y_2(t)-\beta y_3(t)+\Delta f_3(y,t)+d_3(t)+u_3, \qquad (35)
\end{aligned}
$$

where $\alpha\in(0,1)$; τ, ρ, and β are the system parameters. The adaptive control is

$$
\begin{aligned}
u_1 &= -[\tau(y_2-y_1)+(\hat{a}-a)(x_2(t)-x_1(t))\\
&\quad +\lambda_1(y_1-x_1)+s_1+m_1+k_1\mathrm{sgn}(s_1)],\\
u_2 &= -[-y_1(t)y_3(t)+\rho y_2(t)-x_1(t)(b-x_3(t))-x_2(t)\\
&\quad +\lambda_2(y_2-x_2)+s_2+m_2+k_2\mathrm{sgn}(s_2)],\\
u_3 &= -[y_1(t)y_2(t)-\beta y_3(t)-x_1(t)x_2(t)+(c-\hat{c})x_3(t)\\
&\quad +\lambda_3(y_3-x_3)+s_3+m_3+k_3\mathrm{sgn}(s_3)], \qquad (36)
\end{aligned}
$$

and the update laws are

$$
\begin{aligned}
&D^{\alpha}\hat{a}=p_1(x_1-x_2)s_1, \quad D^{\alpha}\hat{c}=p_3x_3s_3,\\
&D^{\alpha}\hat{\delta}_i=r_i|s_i|. \qquad (37)
\end{aligned}
$$

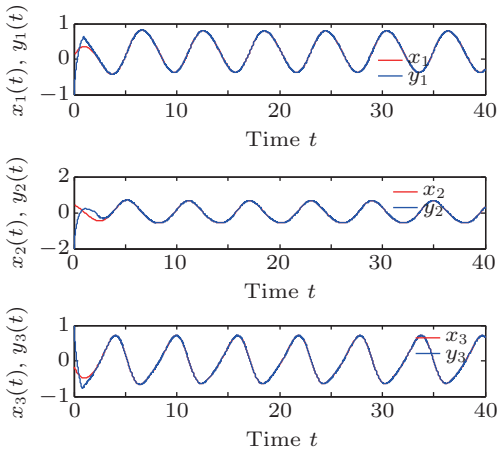


Fig. 4. (color online) Synchronization of Genesio–Tesi systems.

Figures 5 and 6 show the convergence of the errors and the synchronization between two different chaotic systems, with initial conditions $x(0)=[0.1,0.1,0.1]^{\top}$, $y(0)=[6,-6,5]^{\top}$, $\beta=20$, $\rho=3$, $\tau=36$, $b=1$, $\hat{c}(0)=3$, and $\hat{a}(0)=9$; gains of update law $p_i=6$, $r_i=4$; model uncertainties $\Delta f_i=0.2\sin(\pi t)$; external disturbances $d_i(t)=0.1\cos(8t)$; control parameters $k_i=3$ and $\lambda_i=3$; system order $\alpha=0.98$.

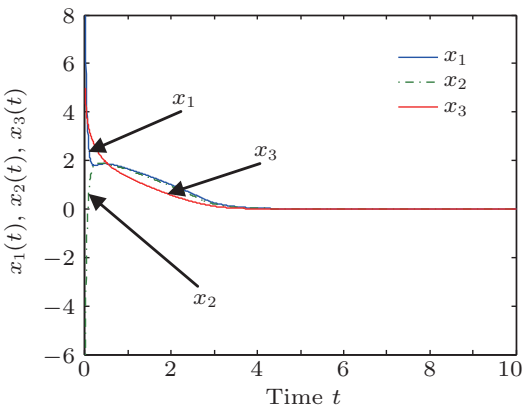


Fig. 5. (color online) Convergence of errors between Lorenz and Lü systems.

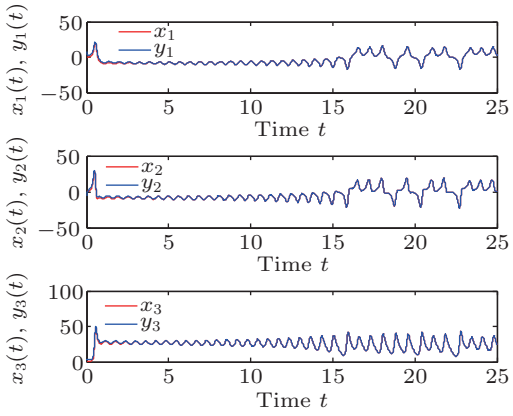


Fig. 6. (color online) Synchronization between Lorenz and Lü systems.

From Fig. 1 to Fig. 6, it is obvious that Van der Pol oscillator system, Genesio–Tesi system and Lorenz system are well synchronized by the adaptive control proposed in this paper. Based on the comparison of the three examples in the simulations, it is concluded that both the control parameters and the gains of update laws have influences on the convergence speed. When increasing the control parameters and update law gains, the error systems converge faster. Furthermore, when fractional orders of the systems increase, the error systems converge faster. The system unknown parameters, model uncertainties, and external disturbances can be well dealt with under arbitrary initial conditions, which confirms the effectiveness of our control method.

5. Conclusions

In this paper, we consider the synchronization of two fractional-order chaotic systems with unknown parameters and uncertainties. Based on fractional Lyapunov function, an adaptive control combined with the fractional-order nonlinear state feedback and estimated update laws is proposed to stabilize the fractional-order error system with asymptotic convergence. The unknown parameters and unknown uncertainties can be well dealt with under fractional order the update laws of the proposed controller. Illustrative examples and simulation results are provided to demonstrate our control scheme.

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