Solutions of the D-dimensional Schrödinger equation with Killingbeck potential: Lie algebraic approach

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Algebraic solutions of the D-dimensional Schrödinger equation with Killingbeck potential are investigated using the Lie algebraic approach within the framework of quasi-exact solvability. The spectrum and wavefunctions of the system are reported and the allowed values of the potential parameters are obtained through the $sl(2)$ algebraization.

Keywords: quasi-exactly solvable, Schrödinger equation, Killingbeck potential, $sl(2)$ Lie algebra, representation theory

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1. Introduction

A quantum system is exactly solvable (ES) if all the eigenvalues and the corresponding eigenfunctions can be calculated in an exact analytical manner. ES models have attracted much attention because of their important applications in various branches of physics and mathematics. However, such systems are very rare and the Schrödinger equation can be solved exactly only for a few number of potentials. In contrast, a quantum system is quasi-exactly solvable (QES) if only a finite number of eigenvalues and eigenfunctions can be determined exactly.[1–6] The QES models have received great attention due to their interesting mathematical structure and their role in applied fields.[7–13] These models are distinguished by the fact that their Hamiltonian preserves a finite-dimensional vector space of polynomials which can be written as a block diagonal matrix with at least one finite block. The eigenvalues and eigenfunctions corresponding to this block can always be calculated in an exact manner. During the past decades, the perturbed Coulomb potential has been the subject of much research in both relativistic and nonrelativistic regimes.[14–28] A very important example in this field is the Killingbeck potential

$$V(r) = -\frac{Z}{r} + gr + \lambda r^2, \tag{1}$$

which plays an important role in many branches of physics including particle physics,[29,30] quantum field theory,[31] molecular,[32–34] and solid state physics.[35,36] Until now, a variety of techniques including the supersymmetric quantum mechanics,[37] the moment method,[22] the shifted $1/N$ expansion,[15,16] the Hill determinant,[17,18] the differential quadratic,[38] and the asymptotic iteration techniques[39] have been used to deal with the potential. On the other hand, considerable efforts have been made to study the D-dimensional quantum mechanical systems with spherically symmetric potentials.[40–42] In this work, using the QES Lie-algebraic approach, we present a simple formalism for the D-dimensional Schrödinger equation with the Killingbeck potential and find a finite number of exact solutions in the representation space of $sl(2)$.

The rest of this paper is organized as follows. In Section 2, we briefly recall the general definition of QES differential equations. Section 3 is devoted to the algebraization of the D-dimensional Schrödinger equation with the Killingbeck potential where we demonstrate that the system possesses a hidden $sl(2)$ algebraic structure responsible for the QES structure. In addition, using the representation theory of $sl(2)$, we obtain the exact expressions for the energies and the corresponding wavefunctions as well as the allowed values of the potential parameters. We finish the work with our concluding remarks in Section 4.

2. Quasi-exactly solvable differential equations

A linear differential operator H in a Hilbert space $\boldsymbol{H}$ is called QES if it leaves invariant a nontrivial finite-dimensional subspace $P \subset \boldsymbol{H}$

$$HP \subset P, \quad P = \langle \varphi_1, \varphi_2, \ldots, \varphi_n \rangle, \quad \varphi_i \in \boldsymbol{H}. \tag{2}$$

Then, the first n eigenvalues and the corresponding eigenfunctions can be obtained exactly by diagonalizing the corresponding matrix of the restricted action of H to the subspace P. Two spaces P and $\tilde{P}$ are equivalent if there exists a smooth non-

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vanishing gauge function $G(x)$ such that

$$P(r) = G(x)\cdot\tilde{P}(r(x)),$$
$$\tilde{P}_{n+1} = \langle 1, r, r^2, \ldots, r^n\rangle. \quad (3)$$

In this case, $\tilde{H}$ is equivalent to the H under the transformation given by

$$\tilde{H} = G^{-1}\cdot H\cdot G. \quad (4)$$

Since the transformed operator $\tilde{H}$ preserves the eigenvalue of $\tilde{H}\tilde{P} = E\tilde{P}$ and leaves the polynomial space $\tilde{P}_{n+1}$ invariant, it is therefore an element of the universal enveloping algebra $U_{sl(2)}$ according to the Burnside's theorem.[43] In one dimension, the only Lie algebra of the first-order differential operators, which possesses finite-dimensional representations, is the $sl(2)$ algebra, whose generators

$$J_n^+ = r^2 d_r - nr,$$
$$J_n^0 = rd_r - \frac{n}{2},$$
$$J_n^- = d_r, \quad (5)$$

leave invariant in the $(n+1)$-dimensional linear space of polynomials in r (with at most degree n). Hence, the operator $\tilde{H}$ can be represented as a quadratic combination of the $sl(2)$ generators as

$$\tilde{H} = \sum_{a,b=0,\pm} C_{ab}J^aJ^b + \sum_{a=0,\pm} C_aJ^a + C. \quad (6)$$

In general, this operator does not have the form of a Schrödinger operator but can be always reduced to a Schrödinger-type operator by making a change of variable and a gauge transformation. In the next section, we apply the above method to obtain the eigenvalues and eigenfunctions as well as the allowed parameters of the Killingbeck potential for any given non-negative integer quantum number n.

3. Algebraic solutions of the Schrödinger equation with the Killingbeck potential

In atomic units ($m = \hbar = c = 1$), the D-dimensional Schrödinger equation with spherically symmetric potential $V(r)$ is given by

$$\left(-\frac{\nabla_D^2}{2} + V(r)\right)\Psi_{n,l,m}(r,\Omega) = E\Psi_{n,l,m}(r,\Omega), \quad (7)$$

where the D-dimensional Laplacian operator ∇_D^2 is

$$\nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{L_D^2(\Omega)}{r^2},$$
$$L_D^2(\Omega)\mathrm{Y}_{l,m}(\Omega) = l(l+D-2)\mathrm{Y}_{l,m}(\Omega). \quad (8)$$

Using the well-known technique of separation of variables and via

$$\Psi_{n,l,m}(r,\Omega) = R_{n,l}(r)\mathrm{Y}_{l,m}(\Omega),$$
$$R_{n,l}(r) = r^{(D-1)/2}\psi_{n,l}(r), \quad (9)$$

the radial part of the Schrödinger equation becomes

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(D+2l-1)(D+2l-3)}{8r^2} + V(r)\right)\psi_{n,l}(r) = E\psi_{n,l}(r). \quad (10)$$

Now, considering the potential in Eq. (1), equation (10) reduces to the following operator equation:

$$H\psi_{n,l}(r) = 0,$$
$$H = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(D+2l-1)(D+2l-3)}{4r^2} - \frac{2Z}{r} + 2gr + 2\lambda r^2 - 2E. \quad (11)$$

Making the gauge transformation

$$\psi_{n,l}(r) = G\cdot\tilde{\psi}_n(r),$$
$$G = \exp\left(-\sqrt{\frac{\lambda}{2}}r^2 - \frac{g}{\sqrt{2\lambda}}r + (l'+1)\log(r)\right), \quad (12)$$

where

$$l' = \frac{-1+\sqrt{1+(D+2l-1)(D+2l-3)}}{2}, \quad (13)$$

equation (4) gives

$$\tilde{H} = -r\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(2\sqrt{2\lambda}r^2 + \frac{2g}{\sqrt{2\lambda}}r - 2(l'+1)\right)\frac{\mathrm{d}}{\mathrm{d}r} + \left(-2E - \frac{g^2}{2\lambda} + 2\sqrt{2\lambda}(l'+\frac{3}{2})\right)r + \left(\frac{2g(l'+1)}{\sqrt{2\lambda}} - 2Z\right), \quad (14)$$

where we have multiplied the equation $\tilde{H}\tilde{\psi}(r) = 0$ by r on the left for later convenience. We further simplify the equation by letting $\tilde{H} = r\tilde{H}$. In order to show how $sl(2)$ Lie algebra underlies the QES of the problem, we consider the action of the operator $\tilde{H}$ on monomials of r. It is easy to verify that

$$\tilde{H}r^n = \left(-2E - \frac{g^2}{2\lambda} + 2\sqrt{2\lambda}(n+l'+\frac{3}{2})\right)r^{n+1} + \left(\frac{2g(n+l'+1)}{\sqrt{2\lambda}} - 2Z\right)r^n + \left(-n(2l'+1)\right)r^{n-1}, \quad (15)$$

which implies that the operator $\tilde{H}$ is not ES due to the existence of the r^{n+1} term. If this term vanishes from the right-hand side of Eq. (15), the operator $\tilde{H}$ preserves a finite-dimensional subspace $\tilde{P}_{n+1}(r)$ which is related to quasi-exact solvability. Hence, the operator $\tilde{H}$ is QES with the invariant subspace $\tilde{P}_{n+1}(r)$. On the other hand, from Eq. (6), any QES differential operator can be represented as

$$\tilde{H} = C_{++}J_n^+J_n^+ + C_{+0}J_n^+J_n^0 + C_{+-}J_n^+J_n^- + C_{0-}J_n^0J_n^- + C_{--}J_n^-J_n^- + C_+J_n^+ + C_0J_n^0 + C_-J_n^- + C, \quad (16)$$

where $C_{\alpha\beta}$, C_α, $C \in R$. Substitution of Eq. (5) into Eq. (16) yields the following differential form:

$$\left\{P_4(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + P_3(r)\frac{\mathrm{d}}{\mathrm{d}r} + P_2(r)\right\}\tilde{\psi}(r) = 0, \quad (17)$$

where P_i are the polynomials of degree i;

$$\begin{aligned}
P_4(r) &= C_{++}r^4 + C_{+0}r^3 + C_{+-}r^2 + C_{0-}r + C_{--},\\
P_3(r) &= C_{++}(2-2n)r^3 + \left(C_+ + C_{+0}(1-\frac{3n}{2})\right)r^2\\
&\quad + (C_0 - nC_{+-})\,r + \left(C_- - \frac{n}{2}C_{0-}\right),\\
P_2(r) &= C_{++}n(n-1)r^2 + \left(\frac{n^2}{2}C_{+0} - nC_+\right)r + \left(C - \frac{n}{2}C_0\right). \quad (18)
\end{aligned}$$

Comparing Eq. (17) with Eq. (14), we find

$$\begin{aligned}
&C_{++} = C_{+0} = C_{+-} = C_{--} = 0,\\
&C_{0-} = -1,\ \ C_+ = 2\sqrt{2\lambda},\ \ C_0 = \frac{2g}{\sqrt{2\lambda}},\\
&\left(C_- - \frac{n}{2}C_{0-}\right) = -2(l'+1),\\
&(-nC_+) = -2E - \frac{g^2}{2\lambda} + 2\sqrt{2\lambda}\left(l' + \frac{3}{2}\right),\\
&\left(C - \frac{n}{2}C_0\right) = \frac{2g(l'+1)}{\sqrt{2\lambda}} - 2Z. \quad (19)
\end{aligned}$$

The exact energy eigenvalues for the Killingbeck potential are immediately obtained from the fourth equation in Eqs. (19) as

$$E = \sqrt{2\lambda}\left(n + \frac{\sqrt{1+(D+2l-1)(D+2l-3)}}{2} + 1\right) - \frac{g^2}{4\lambda}, \quad (20)$$

which reduces to the result of Ref. [15] for the three-dimensional space. It is seen that the eigenvalues depend on five parameters λ, g, n, l, and D. It should be noted that these parameters are not independent of each other; rather they are connected by certain relations which will be discussed later. From Eqs. (16) and (19), the Lie-algebraic differential operator $\tilde{H} \in U_{sl(2)}$ is written as follows:

$$\begin{aligned}
\tilde{H} &= -J_n^0 J_n^- + 2\sqrt{2\lambda}J_n^+ + \frac{2g}{\sqrt{2\lambda}}J_n^0 - \left(2(l'+1) + \frac{n}{2}\right)J_n^-\\
&\quad + \left(\left(\frac{n}{2} + l' + 1\right)\frac{2g}{\sqrt{2\lambda}} - 2Z\right). \quad (21)
\end{aligned}$$

Therefore, we have succeeded in obtaining the QES Schrödinger equation with the Killingbeck potential which preserves the finite-dimensional subspace $\tilde{P}_{n+1}(r)$. As mentioned above, there exist important constraints on the potential parameters Z, λ, and g which differ for different values of n. Bessis *et al.*[22] and Roychoudhury *et al.*[15] obtained these constraints by means of the moment and shifted $1/N$ expansion methods, respectively. Now, with the help of the representation theory, we demonstrate that these constraints can be obtained in the framework of QES. For this purpose, we seek solutions of the operator $\tilde{H}$ restricted to the subspace $\tilde{P}_{n+1}$. From Eq. (21), the restriction of $\tilde{H}$ to $\tilde{P}_{n+1}$ is represented by an $(n+1)$-dimensional matrix. Therefore, an $(n+1)$-set of constraints on the potential parameters can be calculated algebraically by diagonalizing this matrix, which, together with Eq. (20), gives the exact solutions of the problem. Since the operator $\tilde{H}$ in Eq. (21) satisfies $\tilde{H}\tilde{P}_{n+1} \subset \tilde{P}_{n+1}$, the corresponding eigenfunctions can be written as

$$\tilde{\psi}_n(r) = \sum_{m=0}^{n} a_m r^m, \quad n = 0,1,2,\ldots, \quad (22)$$

which allows us to determine the $(n+1)$-algebraic eigenfunctions. In the following, through $sl(2)$ algebraization, we first determine the exact solutions for $n = 0,1,2$ and then generalize them to arbitrary n.

Case (i) $n = 0$

In this case, the operator $\tilde{H}$ possesses the invariant subspace $\tilde{P}_1(r)$ spanned by the basis $\{1\}$. Thus, the eigenfunction has the form $\tilde{\psi}_0(r) = a_0$ and the corresponding matrix equation is given by

$$\left(\frac{2g(l'+1)}{\sqrt{2\lambda}} - 2Z\right)a_0 = 0. \quad (23)$$

The nontrivial solution of this equation exists if the following relation among the potential parameters is satisfied:

$$Z = \frac{g(l'+1)}{\sqrt{2\lambda}}. \quad (24)$$

It is seen that when $n = 0$, there is a single relation between potential parameters. In other words, for any two of λ, Z, and g, there exists a unique value of the third parameter. Thus, from Eq. (12), we obtain the ground-state wavefunction as

$$\begin{aligned}
\psi_{0,l}(r) &= \mathrm{e}^{-A(r)}\tilde{\psi}_0(r)\\
&= a_0 \exp\left(-\sqrt{\frac{\lambda}{2}}r^2 - \frac{g}{\sqrt{2\lambda}}r + (l'+1)\log(r)\right). \quad (25)
\end{aligned}$$

Case (ii) $n = 1$

In this case, the operator $\tilde{H}$ leaves the subspace $\tilde{P}_2(r)$ spanned by $\{1, r\}$ invariant and the corresponding eigenfunction becomes $\tilde{\psi}_1(r) = a_0 + a_1 r$. The corresponding matrix equation is therefore given by

$$\begin{pmatrix} Z - \dfrac{g(l'+1)}{\sqrt{2\lambda}} & (l'+1) \\ \sqrt{2\lambda} & Z - \dfrac{g(l'+2)}{\sqrt{2\lambda}} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0. \quad (26)$$

The nontrivial solutions are obtained provided that

$$Z = \left(l' + \frac{3}{2}\right)\frac{g}{\sqrt{2\lambda}} \pm \sqrt{\frac{g^2}{8} + (l'+1)\sqrt{2\lambda}}. \quad (27)$$

It is seen that for any two of λ, Z, and g, there exist two possible values of the third parameter. From Eqs. (26) and (27), we obtain

$$a_1 = \frac{1}{l'+1}\left(\frac{g(l'+1)}{\sqrt{2\lambda}} - Z\right)a_0. \quad (28)$$

Therefore, by using Eq. (12), the wavefunction of the first excited-state can be written as

$$\begin{aligned}
\psi_{1,l}(r) &= \mathrm{e}^{-A(r)}\tilde{\psi}_1(r)\\
&= \exp\left(-\sqrt{\frac{\lambda}{2}}r^2 - \frac{g}{\sqrt{2\lambda}}r + (l'+1)\log(r)\right)
\end{aligned}$$

$$\times\left(1+\left(\frac{1}{l'+1}\left(\frac{g(l'+1)}{\sqrt{2\lambda}}-Z\right)r\right)\right)a_0. \quad (29)$$

Case (iii) $n=2$

In this case, the operator $\tilde{H}$ possesses the invariant subspace $\tilde{P}_3(r)$ with polynomial basis $\{1,r,r^2\}$ and the corresponding eigenfunction has the form $\tilde{\psi}_2(r)=a_0+a_1r+a_2r^2$. The corresponding matrix equation therefore becomes

$$\begin{pmatrix} Z-\frac{g(l'+1)}{\sqrt{2\lambda}} & (l'+1) & 0 \\ 2\sqrt{2\lambda} & Z-\frac{g(l'+2)}{\sqrt{2\lambda}} & (2l'+3) \\ 0 & \sqrt{2\lambda} & Z-\frac{g(l'+3)}{\sqrt{2\lambda}} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=0. \quad (30)$$

From the condition of nontrivial solution of Eq. (30), the potential parameters satisfy the constraint condition

$$Z^3+A_1Z^2+A_2Z+A_3=0, \quad (31)$$

where

$$A_1=-(3l'+6)\frac{g}{\sqrt{2\lambda}},$$

$$A_2=(3l'^2+12l'+11)\frac{g^2}{2\lambda}-(4l'+5)\sqrt{2\lambda},$$

$$A_3=-(l'^3+6l'^2+11l'+6)\frac{g^3}{(2\lambda)^{3/2}}+(3l'^2+9l'+6)g. \quad (32)$$

It is seen that for any two of λ, Z, and g, there exist three possible values of the third parameter. Thus, the wavefunction for the second excited-state can be written as

$$\begin{aligned}\psi_{2,l}(r) &= \mathrm{e}^{-A(r)}\tilde{\psi}_2(r)\\ &= \exp\left(-\sqrt{\frac{\lambda}{2}}r^2-\frac{g}{\sqrt{2\lambda}}r+(l'+1)\log(r)\right)\\ &\quad\times\left(a_0+a_1r+a_2r^2\right), \end{aligned} \quad (33)$$

where

$$a_1=\frac{1}{l'+1}\left(\frac{g(l'+1)}{\sqrt{2\lambda}}-Z\right)a_0,$$

$$a_2=\frac{1}{2l'+3}\left(\left(\frac{g(l'+2)}{\sqrt{2\lambda}}-Z\right)a_1-\left(2\sqrt{2\lambda}\right)a_0\right). \quad (34)$$

Following this method, an $(n+1)$-set of exact solutions can be generated for an arbitrary n, as follows.

Case (iv) arbitrary n

In this case, the operator $\tilde{H}$ has a finite-dimensional invariant subspace $\tilde{P}_{n+1}(r)$, which is spanned by the basis $\{1,r,r^2,\ldots,r^n\}$. Similarly, by considering the eigenfunction Eq. (22), one can obtain the following matrix equation:

$$\begin{pmatrix} Z-\frac{g(l'+1)}{\sqrt{2\lambda}} & (l'+1) & 0 & 0 & 0 \\ n\sqrt{2\lambda} & Z-\frac{g(l'+2)}{\sqrt{2\lambda}} & (2l'+3) & 0 & 0 \\ 0 & (n-1)\sqrt{2\lambda} & Z-\frac{g(l'+3)}{\sqrt{2\lambda}} & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \frac{n(2l'+n+1)}{2} \\ 0 & 0 & 0 & \sqrt{2\lambda} & Z-\frac{g(n+l'+1)}{\sqrt{2\lambda}} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}=0, \quad (35)$$

which is a generalization of Eqs. (23), (26), and (30). It should be noted that this relation reduces to the results obtained in Refs. [15] and [22] when $D=3$ and $l=0$, respectively. For a nontrivial solution to exist, the determinant of the coefficient matrix must vanish which yields the constraints on the potential parameters. In this case, for any given non-negative integer n and two of λ, Z, and g parameters, there exist $(n+1)$ different values of the third parameter which correspond to the radial quantum numbers $n_r=0,1,2,\ldots,n$. Thus, the wavefunction for the n-th excited-state is given by

$$\psi_{n,l}(r)=\exp\left(-\sqrt{\frac{\lambda}{2}}r^2-\frac{g}{\sqrt{2\lambda}}r+(l'+1)\log(r)\right)\sum_{m=0}^{n}a_mr^m, \quad (36)$$

where the coefficients a_m satisfy the recursion relation

$$a_{m+1}=\frac{\left(\frac{2g}{\sqrt{2\lambda}}(m+l'+1)-2Z\right)a_m+\left(-2E+2\sqrt{2\lambda}(m+l'+\frac{1}{2})-\frac{g^2}{2\lambda}\right)a_{m-1}}{(m^2+2l'(m+1)+3m+2)}, \quad (37)$$

with the boundary conditions $a_{-1}=0$ and $a_{m+1}=0$. Some of the numerical results obtained for the ground, the first, and the second excited states are displayed in Table 1. We use the parameter set given in Ref. [15] for further comparison. It is observed that our results are exactly identical with those obtained in Ref. [15]. Also, it is noted that our method reproduces the results of Ref. [18] by choosing an appropriate set of parameter values. In Figs. 1–3, the variations of energy as a function of dimension are plotted for $n=0,1,2$. The corresponding unnormalized wavefunctions with initial condition $a_0=1$ are plotted in Fig. 4. It is seen from Figs. 1–3 that when $l=0$, the energy is symmetric with respect to $D=2$ for $0<D<4$.

This is a common property of the s-wave energy spectrum irrespective of the quantum number n and can be explained by calculating the first derivative of the energy with respect to the dimension D. From Eq. (20), the first derivative $\partial E/\partial D$ involves the term proportional to $D+2l-2$ which implies that for $l=0$, equation (20) has a turning point at $D=2$.

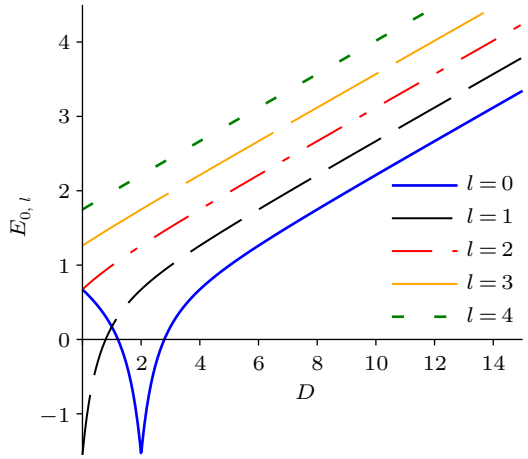


Fig. 1. (color online) Plots of energy versus D for $n=0$ and various values of l.

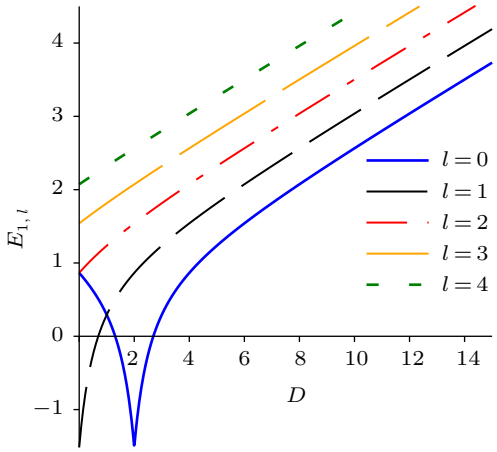


Fig. 2. (color online) Plots of energy versus D for $n=1$ and various values of l.

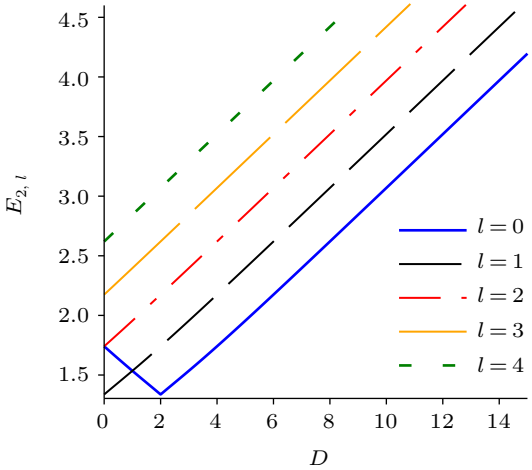


Fig. 3. (color online) Plots of energy versus D for $n=2$ and various values of l.

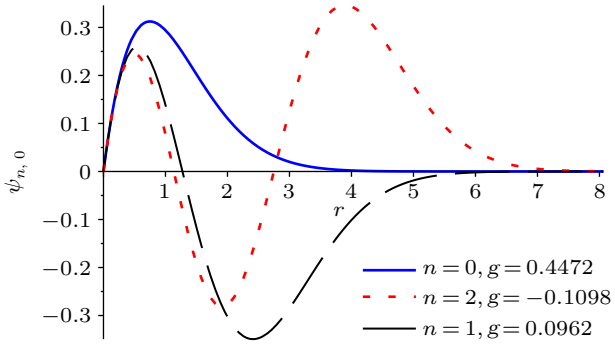


Fig. 4. (color online) Plots of wave function versus r for $l=0$ and $n=0,1,2$ in three dimensional space.

Table 1. Eigenenergies of the ground, the first, and the second excited states for $\lambda=0.1$ and $Z=1$ in three-dimensional space.

n_r	l	g	E
0	0	0.44721	0.17082
0	1	0.22361	0.99303
0	2	0.14907	1.50969
1	0	0.09620	1.09490
0		0.57462	0.29256
1	1	0.00969	1.56501
0		0.36298	1.23586
1	2	−0.02025	2.01144
0		0.28113	1.81488
2	0	−0.10976	1.53513
1		0.24531	1.41481
0		0.68434	0.39443
2	1	−0.15147	1.95511
1		0.15464	1.95268
0		0.48131	1.43331
2	2	−0.15959	2.39632
1		0.11397	2.42720
0		0.39533	2.06856

4. Conclusions

By using the Lie algebraic approach of QES, we investigate the algebraic solutions of D-dimensional Schrödinger equation with Killingbeck potential. We demonstrate that the corresponding equation is reducible to a QES differential equation with $sl(2)$ hidden algebraic structure. With the help of the representation theory, we obtain expressions for the energies and wavefunctions as well as the allowed values of the potential parameters through the $sl(2)$ algebraization. We also show that our results are identical to those obtained by other methods. It is found that the quasi-exact Lie algebraic approach is a powerful method of reproducing the exact analytical results.

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