Generating function of product of bivariate Hermite polynomials and their applications in studying quantum optical states*

Fan Hong-Yi(范洪义)[a)b)†], Zhang Peng-Fei(张鹏飞)[c)], and Wang Zhen(王 震)[d)]

[a)] Department of Physics, Ningbo University, Ningbo 315211, China
[b)] Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China
[c)] Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China
[d)] College of Science, Changzhou Institute of Technology, Changzhou 213002, China

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By virtue of the operator-Hermite-polynomial method, we derive some new generating function formulas of the product of two bivariate Hermite polynomials. Their applications in studying quantum optical states are presented.

Keywords: operator-Hermite-polynomials (OHP) method, generating function, product of bivariate Hermite polynomials

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1. Introduction

In quantum mechanics and mathematical physics, the special function, the Hermite polynomial, is of fundamental importance. Physically, it is the eigenfunction of the harmonic oscillator.[1] In addition, by introducing a convenient complex form of the α-th two-dimensional fractional Fourier transform operation, it is found that the bivariate Hermite polynomial is a new eigenmode that propagates in quadratic graded index media.[2] Mathematically, the single-variable Hermite polynomial $H_n(x)$ can be defined in terms of its generating function,

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)=\mathrm{e}^{2tx-t^2},\quad H_n(x)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{2tx-t^2}|_{t=0}. \tag{1}$$

In Ref. [3], by combining the operator-Hermite-polynomials (OHP) method[4] and the technique of integration within an ordered product (IWOP) of operators,[5] we derived the generating functions of even and odd Hermite polynomials

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n+l}(x)\\&=\frac{1}{(\sqrt{1+4t})^{l+1}}H_l\left(\sqrt{\frac{1}{4t+1}}x\right)\mathrm{e}^{4tx^2/(1+4t)},\end{aligned} \tag{2}$$

when $l=0,1$. Since $H_0(x)=1$, $H_1(x)=2x$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n}(x)=\frac{1}{\sqrt{1+4t}}\mathrm{e}^{\frac{4t}{1+4t}x^2},\\&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n+1}(x)=\frac{1}{(1+4t)^{3/2}}x\mathrm{e}^{4tx^2/(1+4t)},\end{aligned} \tag{3}$$

and these formulas are not only accumulated in mathematical handbooks but can also be useful in constructing some optical quantum states.[3] Furthermore, by introducing the Hermite-polynomial-operator $H_n(X)$, where X is the coordinate operator (or the quadrature operator in quantum optics theory), and combining the IWOP technique, some new operator identities about quantum squeezing are derived, which are useful for studying the squeezed number state.[6] On the other hand, by replacing the arguments of the special function by quantum mechanical operators, a binomial theorem involving Hermite polynomials and a negative-binomial theorem involving Laguerre polynomials have been derived. These two theorems will have essential applications in quantum optics calculations, and the method is concise and helpful in deducing many operator identities, which may become a new branch in mathematical physics theory.[7] Using the entangled state representation, a two-mode squeezed number state can be converted into a Hermite polynomial excited squeezed vacuum state.[8] The generalized photon-added coherent state is obtained by repeatedly acting the combination of Bose creation and annihilation operations on the coherent state. It is found that it can be regarded as a Hermite excited coherent state due to its normalization factor related to Hermite polynomials.[9]

For the bivariate Hermite polynomials $H_{m,n}(x,y)$ (note that $H_{m,n}$ is not a direct product of two independent single-variable Hermite polynomials)[10]

$$H_{m,n}(x,y)=\sum_{l=0}^{\min[m,n]}\frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}x^{m-l}y^{n-l}, \tag{4}$$

with the generating function in the form of

$$\sum_{n,m=0}^{\infty}\frac{t^m\tau^n}{m!n!}H_{m,n}(x,y)=\exp\left(tx+\tau y-t\tau\right). \tag{5}$$

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†Corresponding author. E-mail: fhym@ustc.edu.cn

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The bivariate Hermite polynomials $H_{m,n}$ have their own applications in studying quantum optics. For example, in reference to the Weyl ordering

$$x^m p^n \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l, \tag{6}$$

where X and P are coordinate and momentum operator, respectively. In Ref. [11], we examined the operators' parameterized ordering and its classical correspondence, and found the fundamental function-operator correspondence

$$\left(\frac{1-s}{2}\right)^{(n+m)/2} H_{m,n}\left(\sqrt{\frac{2}{1-s}}\alpha^*, \sqrt{\frac{2}{1-s}}\alpha\right) \to a^{\dagger m} a^n, \tag{7}$$

and its complementary relation

$$\alpha^n \alpha^{*m} \to (-\mathrm{i})^{n+m} \left(\frac{1-s}{2}\right)^{(n+m)/2} : H_{m,n}\left(\mathrm{i}\sqrt{\frac{2}{1-s}}a^\dagger, \mathrm{i}\sqrt{\frac{2}{1-s}}a\right) : , \tag{8}$$

where the symbol : : denotes normal ordering.

In Ref. [12] we derived

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_{n,2m}(x,y) = \mathrm{i}^n \sqrt{t^n} H_n\left(\frac{x-2yt}{2\mathrm{i}\sqrt{t}}\right) \mathrm{e}^{ty^2}, \tag{9}$$

and

$$\sum_{n,m=0}^{\infty} \frac{s^n t^m}{n!m!} H_{n,2m}(x,y) = \exp\left[ts^2 + s(x-2yt)\right]. \tag{10}$$

These two equations are the generating functions of $H_{n,2m}(x,y)$ and the more complicated one is

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{s^n t^m}{n!m!} H_{2n+l,2m+k}(x,y) \\
&= A^{-1/2} (-1)^l \left(\mathrm{i}\sqrt{\frac{s}{A}}\right)^k \left(\mathrm{i}\sqrt{\frac{t}{A}}\right)^l \\
&\quad \times \sum_{m=0} \left(\frac{-1}{\sqrt{st}}\right)^m \frac{k!l!}{m!(k-m)!(l-m)!} \\
&\quad \times H_{k-m}\left(\frac{x-2sy}{2\mathrm{i}\sqrt{sA}}\right) H_{m-k}\left(\frac{2tx-y}{2\mathrm{i}\sqrt{fA}}\right) \mathrm{e}^{\frac{-4stxy+sx^2+ty^2}{1-4st}},
\end{aligned} \tag{11}$$

where $A = 1-4st$.

When $k = l = 0$, we have

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} \frac{s^n t^m}{n!m!} H_{2n,2m}(x,y) \\
&= A^{-1/2} \exp\left(\frac{-4stxy+sx^2+ty^2}{1-4st}\right).
\end{aligned} \tag{12}$$

In this work we shall derive some new generating function formula of the product of bivariate Hermite polynomials (PBHP), and we shall employ the OHP method to realize our goal. By the OHP method in this work, we aim to replace the classical Hermite polynomials $H_{m,n}(x,y)$ by the operator $\vdots H_{m,n}(a^\dagger, a)\vdots$ first, where $a^\dagger$ and a are the Bose creation and annihilation operator, respectively, obeying $[a, a^\dagger] = 1$, $\vdots\ \vdots$ denotes anti-normal ordering. By using

$$\vdots H_{m,n}(a^\dagger, a)\vdots = a^{\dagger m} a^n, \tag{13}$$

we can derive some operator identities in definite operator ordering (proof is given in Appendix A). Then we convert them back to the classical case. We also present some applications of these new generating function formulas of PBHP in quantum optics. Next, we shall briefly review the OHP method and exhibit its convenience and simplicity.

2. Brief review of the OHP method

One of the fundamental operator identities in the OHP method is

$$H_n(X) = 2^n : X^n : , \tag{14}$$

where $X = (a+a)^\dagger/\sqrt{2}$. We can use it to search for the generating function of the product $H_n(x)H_n(y)$; namely, we replace $H_n(x)$ by $H_n(X)$. By using Eqs. (1) and (14), then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n H_n(X) H_n(y)}{n!} &= \sum_{n=0}^{\infty} \frac{t^n 2^n : X^n : H_n(y)}{n!} \\
&= : \mathrm{e}^{4tXy - 4t^2X^2} : .
\end{aligned} \tag{15}$$

Again by using[13]

$$\begin{aligned}
\mathrm{e}^{\lambda X^2} &= \frac{1}{\sqrt{1-\lambda}} : \mathrm{e}^{\lambda X^2/(1-\lambda)} : , \\
: \mathrm{e}^{fX^2} : &= \sqrt{1/(1+f)}\, \mathrm{e}^{X^2 f/(1+f)},
\end{aligned} \tag{16}$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n H_n(X) H_n(y)}{n!} \\
&= \mathrm{e}^{2\sqrt{2}a^\dagger ty} : \mathrm{e}^{-4t^2X^2} : \mathrm{e}^{2\sqrt{2}aty} \\
&= \frac{1}{\sqrt{1-4t^2}} \mathrm{e}^{2\sqrt{2}a^\dagger ty} \mathrm{e}^{-4t^2X^2/(1-4t^2)} \mathrm{e}^{2\sqrt{2}aty} \\
&= \frac{1}{\sqrt{1-4t^2}} \mathrm{e}^{2\sqrt{2}a^\dagger ty} \mathrm{e}^{2\sqrt{2}aty} \mathrm{e}^{-2\sqrt{2}aty} \mathrm{e}^{-4t^2X^2/(1-4t^2)} \mathrm{e}^{2\sqrt{2}aty} \\
&= \frac{1}{\sqrt{1-4t^2}} \mathrm{e}^{4Xty} \mathrm{e}^{-4t^2y^2} \mathrm{e}^{-4t^2(X-2yt)^2/(1-4t^2)} \\
&= \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{-4t^2y^2 + 4Xyt - 4t^2X^2}{1-4t^2}\right).
\end{aligned} \tag{17}$$

Converting X back to x, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n H_n(x) H_n(y)}{n!} \\
&= \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{-4t^2y^2 + 4xyt - 4t^2x^2}{1-4t^2}\right).
\end{aligned} \tag{18}$$

Thus, the merit of using the OHP method is fully displayed.

3. One-index-summation of the product of bivariate Hermite polynomials

Now we prove the following one-index-summation of the product of two bivariate Hermite polynomials:

$$\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y)H_{m,n}(x',y') = (-s)^n H_{n,n}\left[\mathrm{i}\left(\frac{y}{s}-x'\right),\mathrm{i}\left(y'-sx\right)\right]\mathrm{e}^{sx'x}. \tag{19}$$

In this case, the OHP method we use is to replace $H_{m,n}(x',y')$ by $:H_{m,n}(a^\dagger,a):$.

Proof Rewrite Eq. (5) as

$$\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(x,y) = \exp\left[t\left(y-s\right)+sx\right] = \mathrm{e}^{sx}\sum_{n=0}^{\infty}\frac{t^n\left(y-s\right)^n}{n!}. \tag{20}$$

By comparing the same power term of $t^n/n!$ on the two sides we conclude

$$\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y) = \left(y-s\right)^n\mathrm{e}^{sx}. \tag{21}$$

Then we employ the OHP method to consider the summation

$$\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y):H_{m,n}(a^\dagger,a):=?, \tag{22}$$

where $:H_{m,n}(a^\dagger,a):$ is in the anti-normal ordering form. Noting that a and $a^\dagger$ are permuted within the anti-normal ordering symbol $:\ :$ and using Eqs. (13) and (21) we can re-express Eq. (22) as

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y):H_{m,n}(a^\dagger,a):\\ &=\sum_{m=0}^{\infty}\frac{:s^m a^{\dagger m}a^n:}{m!}H_{m,n}(x,y)\\ &=:\left(y-sa^\dagger\right)^n\mathrm{e}^{sa^\dagger x}a^n:\\ &=\left(y-sa^\dagger\right)^n\left(a-sx\right)^n\mathrm{e}^{sa^\dagger x}.\end{aligned} \tag{23}$$

Then we need to change the right-hand side of Eq. (23) into anti-normal ordering for comparing with the left-hand side. Using the identity[14] which is responsible for converting an operator $\rho(a,a^\dagger)$ into its anti-normally ordered form

$$\rho(a,a^\dagger) = \int\frac{\mathrm{d}^2\beta}{\pi}:\langle-\beta|\rho(a,a^\dagger)|\beta\rangle \times\exp\left(|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right):, \tag{24}$$

where $|\beta\rangle = \exp\left(-|\beta|^2/2+\beta a^\dagger\right)|0\rangle$ is a coherent state, $\langle-\beta|\,\beta\rangle = \mathrm{e}^{-2|\beta|^2}$, we try to put $\left(y-sa^\dagger\right)^n\left(a-sx\right)^n$ into its anti-normal ordering

$$\begin{aligned}&\left(y-sa^\dagger\right)^n\left(a-sx\right)^n\\ &=\int\frac{\mathrm{d}^2\beta}{\pi}:\langle-\beta|\left(y-sa^\dagger\right)^n\left(a-sx\right)^n|\beta\rangle\\ &\quad\times\exp\left(|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right):\\ &=s^n\int\frac{\mathrm{d}^2\beta}{\pi}\left(\frac{y}{s}+\beta^*\right)^n\left(\beta-sx\right)^n\\ &\quad\times\exp\left(-|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right):\\ &=s^n\mathrm{e}^{xy}\int\frac{\mathrm{d}^2\beta'}{\pi}|\beta'|^{2n}:\exp\left[-|\beta'|^2+\beta'\left(\frac{y}{s}-a^\dagger\right)\right.\\ &\quad\left.+\left(a-sx\right)\beta'^*-sxa^\dagger-\frac{y}{s}a+a^\dagger a\right]:\\ &=(-s)^n:H_{n,n}\left[\mathrm{i}\left(\frac{y}{s}-a^\dagger\right),\mathrm{i}\left(a-sx\right)\right]:,\end{aligned} \tag{25}$$

where in the last step we have used the integration formula

$$\int\frac{\mathrm{d}^2 z}{\pi}z^m z^{*n}\exp\left(-|z|^2+\xi z+\eta z^*\right) = (-\mathrm{i})^{m+n}\,\mathrm{e}^{\eta\xi}H_{m,n}(\mathrm{i}\eta,\mathrm{i}\xi). \tag{26}$$

Substituting Eq. (25) into Eq. (23) we obtain

$$\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y):H_{m,n}(a^\dagger,a): = (-s)^n:H_{n,n}\left[\mathrm{i}\left(\frac{y}{s}-a^\dagger\right),\mathrm{i}\left(a-sx\right)\right]:\mathrm{e}^{sa^\dagger x}. \tag{27}$$

Since both sides in Eq. (18) are in anti-normal ordering and $a^\dagger$, a are commutable within $:\ :$, we can replace $a^\dagger\to x', a\to y'$, leading to Eq. (19).

Based on the relation to the Laguerre polynomials

$$H_{m,m}(x,y) = (-1)^m m!L_m(xy), \tag{28}$$

where

$$L_m(x) = \sum_{l=0}^{m}(-1)^l\binom{m}{m-l}\frac{x^l}{l!}, \tag{29}$$

we can obtain

$$\sum_{m=0}^{\infty}\frac{s^m}{m!}H_{m,n}(x,y)H_{m,n}(x',y') = s^n n!L_n\left[-\left(\frac{y}{s}-x'\right)\left(y'-sx\right)\right]\mathrm{e}^{sx'x}. \tag{30}$$

Similarly, we can prove

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{m,n}(x,y)H_{m,n}(x',y')\\ &=(-t)^m H_{m,m}\left[\mathrm{i}\left(yt-x'\right),\mathrm{i}\left(y'-\frac{x}{t}\right)\right]\mathrm{e}^{ty'y}\\ &=t^m m!L_m\left[-\left(yt-x'\right)\left(y'-\frac{x}{t}\right)\right]\mathrm{e}^{ty'y}.\end{aligned} \tag{31}$$

4. Summation formula of $\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(x,y)H_{m,n}(x',y')$

By using the generating function of Laguerre polynomials

$$\sum_{n=0}^{\infty}L_n(x)s^n=(1-s)^{-1}\exp\left(\frac{-xs}{1-s}\right), \tag{32}$$

we can further make summation $\sum_{n=0}^{\infty}\frac{t^n}{n!}$ for Eq. (30) and obtain

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(x,y)H_{m,n}(x',y')\\&=\sum_{n=0}^{\infty}t^n s^n L_n\left[-\left(\frac{y}{s}-x'\right)\left(y'-sx\right)\right]\mathrm{e}^{sx'x}\\&=\frac{1}{1-ts}\exp\left[\frac{sxx'+tyy'-ts\left(xy+x'y'\right)}{1-ts}\right].\end{aligned} \tag{33}$$

This formula can be used to derive

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{t^n s^m}{n!m!}H_{m,n}(x,y)\vdots H_{m,n}(a^\dagger,a)\vdots\\&=\frac{1}{1-ts}\vdots\exp\left[\frac{sxa^\dagger+tya-ts\left(xy+a^\dagger a\right)}{1-ts}\right]\vdots.\end{aligned} \tag{34}$$

On the other hand, using Eq. (13) it is equal to

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{t^n s^m}{n!m!}H_{m,n}(x,y)\vdots H_{m,n}(a^\dagger,a)\vdots\\&=\sum_{n,m=0}^{\infty}\frac{:s^m a^{\dagger m}t^n a^n:}{m!n!}H_{m,n}(x,y)\\&=:\exp\left(-sta^\dagger a+sa^\dagger x+tay\right):,\end{aligned} \tag{35}$$

we obtain a new operator identity which links normal ordering to anti-normal ordering

$$\begin{aligned}&:\exp\left(-sta^\dagger a+sa^\dagger x+tay\right):\\&=\frac{1}{1-ts}\vdots\exp\left[\frac{sxa^\dagger+tya-ts\left(xy+a^\dagger a\right)}{1-ts}\right]\vdots.\end{aligned} \tag{36}$$

In particular, when $x=y=0$,

$$:\exp\left(-sta^\dagger a\right):=\frac{1}{1-ts}\vdots\exp\left(\frac{-tsa^\dagger a}{1-ts}\right)\vdots. \tag{37}$$

5. Generating function of $H_{m+l,n+k}(x,y)$

Next we examine the summation

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m+l,n+k}(x,y)=?.$$

To proceed, using the OHP method and Eq. (13) we instead consider

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{s^m t^n}{m!n!}\vdots H_{m+l,n+k}(a^\dagger,a)\vdots\\&=\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}a^{\dagger m+l}a^{n+k}\\&=a^{\dagger l}:\mathrm{e}^{sa^\dagger+ta}:a^k\\&=\mathrm{e}^{-st}a^{\dagger l}\mathrm{e}^{ta}\mathrm{e}^{sa^\dagger}a^k\\&=\mathrm{e}^{-st}\mathrm{e}^{ta}\left(a^\dagger-t\right)^l(a-s)^k\mathrm{e}^{sa^\dagger}.\end{aligned} \tag{38}$$

Using the binomial theorem regarding the bivariate Hemite polynomials[15]

$$\sum_{r,q}^{\infty}\binom{l}{r}\binom{k}{q}H_{r,q}(x,y)t^{l-r}s^{k-q}=H_{r,q}(x+t,y+s), \tag{39}$$

we have

$$\begin{aligned}&\left(a^\dagger-t\right)^l(a-s)^k\\&=\sum_{r,q}^{\infty}\binom{l}{r}a^{\dagger r}(-t)^{l-r}\binom{k}{q}a^q(-s)^{k-q}\\&=\sum_{r,q}^{\infty}\binom{l}{r}\binom{k}{q}\vdots H_{r,q}\left(a^\dagger,a\right)\vdots(-t)^{l-r}(-s)^{k-q}\\&=\vdots H_{l,k}\left(a^\dagger-t,a-s\right)\vdots.\end{aligned} \tag{40}$$

Thus, equation (29) becomes

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{s^m t^n}{m!n!}\vdots H_{m+l,n+k}(a^\dagger,a)\vdots\\&=\vdots\mathrm{e}^{-st}\mathrm{e}^{ta}H_{l,k}\left(a^\dagger-t,a-s\right)\mathrm{e}^{sa^\dagger}\vdots.\end{aligned} \tag{41}$$

Since the both sides are in anti-normal ordering, we have

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m+l,n+k}(x,y)\\&=\mathrm{e}^{-st+sx+ty}H_{l,k}\left(x-t,y-s\right).\end{aligned} \tag{42}$$

This is another type of generating function of bivariate Hermite polynomials.

6. Applications in analyzing quantum optical states

As an application of Eq. (30), we consider the two-mode photon-subtracted squeezed vacuum state $a^m b^m S_2(\lambda)|00\rangle$, where $S_2(\lambda)=\exp[\lambda(a^\dagger b^\dagger-ab)]$ is the two-mode squeezing operator, with λ being a real squeezing parameter. The two-mode squeezed state is[15]

$$S_2(\lambda)|00\rangle=\operatorname{sech}\lambda\,\mathrm{e}^{a^\dagger b^\dagger\tanh\lambda}|00\rangle. \tag{43}$$

Using

$$a^n a^{\dagger m}=(-\mathrm{i})^{m+n}:H_{m,n}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right):, \tag{44}$$

its derivation is shown in Appendix A. From Eq. (30), we obtain

$$a^n b^n\,\mathrm{e}^{a^\dagger b^\dagger\tanh\lambda}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{\tanh^m \lambda}{m!} a^n a^{\dagger m} b^n b^{\dagger m} \\
&= (-1)^n \sum_{m=0}^{\infty} \frac{(-\tanh\lambda)^m}{m!} : H_{m,n}\left(\mathrm{i}a^\dagger, \mathrm{i}a\right) H_{m,n}\left(\mathrm{i}b^\dagger, \mathrm{i}b\right) : \\
&= n! \tanh^n \lambda \, \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} : L_n[\left(-a\coth\lambda - b^\dagger\right) \\
&\quad \times \left(b + a^\dagger \tanh\lambda\right)] : \\
&= n! \tanh^n \lambda \, \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} : L_n(-a^\dagger a - b^\dagger b \\
&\quad - ab\coth\lambda - a^\dagger b^\dagger \tanh\lambda) : . \qquad (45)
\end{aligned}
$$

Thus, the two-mode photon-subtracted squeezed vacuum state can be expressed as

$$
\begin{aligned}
|\lambda\rangle_n &\equiv a^n b^n S_2(\lambda) |00\rangle \\
&= \operatorname{sech}\lambda a^n b^n \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} |00\rangle \\
&= (\operatorname{sech}\lambda) n! \tanh^n \lambda \, \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} L_n\left(-a^\dagger b^\dagger \tanh\lambda\right) |00\rangle \\
&= n! \tanh^n \lambda L_n\left(-a^\dagger b^\dagger \tanh\lambda\right) S_2(\lambda) |00\rangle . \qquad (46)
\end{aligned}
$$

Therefore, $|\lambda\rangle_n$ is equivalent to Laguerre polynomial excitation on the two-mode squeezed vacuum state. In Ref. [16], we calculated its normalization factor

$$
\begin{aligned}
&\langle 00| \mathrm{e}^{ab\tanh\lambda} a^{\dagger n} b^{\dagger n} a^n b^n \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} |00\rangle \\
&= (n!)^2 \sinh^{2n}\lambda P_n(\cosh 2\lambda), \qquad (47)
\end{aligned}
$$

where $\mathrm{P}_m(x)$ is the Legendre polynomial

$$
\mathrm{P}_n(x) = \sum_{l=0}^{[n/2]} \frac{(-1)^l (2n-2l)! x^{n-2l}}{2^n l!(n-l)!(n-2l)!}. \qquad (48)
$$

Using Eqs. (46) and (47) as well as the coherent state's completeness relation $\int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} |\alpha,\beta\rangle\langle\alpha,\beta| = 1$,[17] we have

$$
\begin{aligned}
&\langle 00| \mathrm{e}^{ab\tanh\lambda} a^{\dagger n} b^{\dagger n} a^n b^n \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} |00\rangle \\
&= (n!)^2 \tanh^{2n}\lambda \langle 00| L_n(-ab\tanh\lambda) \mathrm{e}^{ab\tanh\lambda} \\
&\quad \times \int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} |\alpha,\beta\rangle\langle\alpha,\beta| \\
&\quad \times \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} L_n\left(-a^\dagger b^\dagger \tanh\lambda\right) |00\rangle , \qquad (49)
\end{aligned}
$$

so

$$
\begin{aligned}
&\int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} |L_n(-\alpha\beta\tanh\lambda)|^2 \\
&\quad \times \mathrm{e}^{-|\alpha|^2 - |\beta|^2 + (\alpha\beta + \alpha^*\beta^*)\tanh\lambda} \\
&= \mathrm{P}_n(\cosh 2\lambda) \cosh^{2n}\lambda, \qquad (50)
\end{aligned}
$$

which is a new integration formula.

As an application of Eq. (33), we obtain

$$
\begin{aligned}
&\mathrm{e}^{sab} \mathrm{e}^{ta^\dagger b^\dagger} \\
&= \sum_{m=0}^{\infty} \frac{s^m}{m!} a^m b^m \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{\dagger n} b^{\dagger n} \\
&= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} s^m t^n}{m!n!} : H_{m,n}\left(\mathrm{i}a^\dagger, \mathrm{i}a\right) H_{m,n}\left(\mathrm{i}b^\dagger, \mathrm{i}b\right) : \\
&= \frac{1}{1-ts} : \exp\left[\frac{ts\left(a^\dagger a + b^\dagger b\right) + sa^\dagger b^\dagger + tab}{1-ts}\right] : , \qquad (51)
\end{aligned}
$$

which changes the anti-normally ordered form $\mathrm{e}^{sab}\mathrm{e}^{ta^\dagger b^\dagger}$ into its normal ordering, or

$$
\mathrm{e}^{sab}\mathrm{e}^{ta^\dagger b^\dagger} = \frac{1}{1-ts} \mathrm{e}^{\frac{s}{1-ts}a^\dagger b^\dagger} \mathrm{e}^{-\left(a^\dagger a + b^\dagger b\right)\ln(1-ts)} \mathrm{e}^{\frac{t}{1-ts}ab}. \qquad (52)
$$

Then we turn to an application of Eq. (42) in quantum optics. We consider the following state vector which is generated by operating $H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)$ on the entangled state $|\xi\rangle$, where

$$
|\xi\rangle = \exp\left(-\frac{1}{2}|\xi|^2 + a^\dagger\xi + \xi^* b^\dagger - a^\dagger b^\dagger\right)|00\rangle , \qquad (53)
$$

which takes the bivariate Hermite polynomials $H_{m,n}(\xi,\xi^*)$ as its expansion function in the two-mode Fock space

$$
|\xi\rangle = \mathrm{e}^{-|\xi|^2/2} \sum_{m,n=0}^{\infty} \frac{a^{\dagger m} b^{\dagger n}}{m!n!} H_{m,n}(\xi,\xi^*)|00\rangle , \qquad (54)
$$

$|\xi\rangle$ obeys[18]

$$
\left(a + b^\dagger\right)|\xi\rangle = \xi|\xi\rangle , \ \left(a^\dagger + b\right)|\xi\rangle = \xi^*|\xi\rangle , \qquad (55)
$$

so

$$
\begin{aligned}
&H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|\xi\rangle \\
&= \mathrm{e}^{-|\xi|^2/2} \mathrm{e}^{a^\dagger\xi + \xi^* b^\dagger - b^\dagger a^\dagger} H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|00\rangle . \qquad (56)
\end{aligned}
$$

In comparison with Eq. (42) we see

$$
\begin{aligned}
&\mathrm{e}^{a^\dagger\xi + \xi^* b^\dagger - b^\dagger a^\dagger} H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|00\rangle \\
&= \sum_{n,m=0}^{\infty} \frac{H_{m+l,n+k}(\xi,\xi^*)}{m!n!} a^{\dagger m} b^{\dagger n}|00\rangle \\
&= \sum_{n,m=0}^{\infty} \frac{H_{m+l,n+k}(\xi,\xi^*)}{\sqrt{m!n!}} |m,n\rangle , \qquad (57)
\end{aligned}
$$

where $|m,n\rangle = a^{\dagger m} b^{\dagger n}|00\rangle / \sqrt{m!n!}$, so

$$
\begin{aligned}
&H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|\xi\rangle \\
&= \mathrm{e}^{-|\xi|^2/2} \sum_{n,m=0}^{\infty} \frac{H_{m+l,n+k}(\xi,\xi^*)}{\sqrt{m!n!}} |m,n\rangle . \qquad (58)
\end{aligned}
$$

It follows the Fock space wave function of $H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|\xi\rangle$ that

$$
\begin{aligned}
&\langle m,n| H_{l,k}\left(\xi - b^\dagger, \xi^* - a^\dagger\right)|\xi\rangle \\
&= \mathrm{e}^{-|\xi|^2/2} \frac{H_{m+l,n+k}(\xi,\xi^*)}{\sqrt{m!n!}}. \qquad (59)
\end{aligned}
$$

7. Conclusion

In summary, by virtue of the OHP method and the IWOP technique, we have derived some new generating function formulas about the bivariate Hermite polynomials. Their applications in studying quantum optics are presented as well.

Appendix A

By noticing

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}{:}H_{m,n}(a^\dagger,a){:}\\ &={:}\exp\left(-st+sa^\dagger+ta\right){:}\\ &=\mathrm{e}^{-st}\,\mathrm{e}^{ta}\,\mathrm{e}^{sa^\dagger}\\ &=\mathrm{e}^{sa^\dagger}\,\mathrm{e}^{ta}\\ &=\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}a^{\dagger m}a^n,\end{aligned}\tag{A1}$$

so we have the operator identity as in Eq. (13)

$${:}H_{m,n}(a^\dagger,a){:}=a^{\dagger m}a^n=:a^{\dagger m}a^n:=:a^n a^{\dagger m}:.\tag{A2}$$

On the other hand, from

$$\begin{aligned}\mathrm{e}^{t'a}\,\mathrm{e}^{ta^\dagger}&=\mathrm{e}^{tt'}\,\mathrm{e}^{ta^\dagger}\,\mathrm{e}^{t'a}\\ &=:\mathrm{e}^{(-\mathrm{i}t')\mathrm{i}a+(-\mathrm{i}t)\mathrm{i}a^\dagger-(-\mathrm{i}t)(-\mathrm{i}t')}:\\ &=\sum_{m,n=0}^{\infty}\frac{(-\mathrm{i}t)^m(-\mathrm{i}t')^n}{m!n!}:H_{m,n}\left(\mathrm{i}a^\dagger,ia\right):,\end{aligned}\tag{A3}$$

comparing it with

$$\mathrm{e}^{t'a}\,\mathrm{e}^{ta^\dagger}=\sum_{m,n=0}^{\infty}\frac{t'^n t^m}{n!m!}a^n a^{\dagger m},\tag{A4}$$

then we have the operator identity in Eq. (44).

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