A local energy-preserving scheme for Klein–Gordon–Schrödinger equations*

Cai Jia-Xiang(蔡加祥)[a)b)], Wang Jia-Lin(汪佳玲)[a)], and Wang Yu-Shun(王雨顺)[a)†]

[a)] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, China
[b)] School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China

(Received 28 November 2014; revised manuscript received 9 December 2014; published online 27 March 2015)

A local energy conservation law is proposed for the Klein–Gordon–Schrödinger equations, which is held in any local time–space region. The local property is independent of the boundary condition and more essential than the global energy conservation law. To develop a numerical method preserving the intrinsic properties as much as possible, we propose a local energy-preserving (LEP) scheme for the equations. The merit of the proposed scheme is that the local energy conservation law can hold exactly in any time–space region. With the periodic boundary conditions, the scheme also possesses the discrete change and global energy conservation laws. A nonlinear analysis shows that the LEP scheme converges to the exact solutions with order $\mathcal{O}(\tau^2+h^2)$. The theoretical properties are verified by numerical experiments.

Keywords: Klein–Gordon–Schrödinger equations, energy conservation law, local structure, convergence analysis

PACS: 02.60.Cb, 02.70.Bf, 02.60.Lj **DOI:** 10.1088/1674-1056/24/5/050205

1. Introduction

The Klein–Gordon–Schrödinger (KGS) equations

$$\mathrm{i}\psi_t+\frac{1}{2}\psi_{xx}+\psi\phi=0,\quad \phi_{tt}-\phi_{xx}+\phi-|\psi|^2=0 \tag{1}$$

describe the interaction between conservative complex neutron field and neutral meson Yukawa in quantum field theory, where $\psi(x,t)$ represents a complex scalar nucleon field, $\phi(x,t)$ is a real scalar meson field, and $\mathrm{i}=\sqrt{-1}$. With initial and periodic conditions

$$\begin{aligned}&\psi(x,0)=\psi_0(x),\quad \phi(x,0)=\phi_0(x),\quad \phi_t(x,0)=\phi_1(x),\\&\psi(a,t)=\psi(b,t),\quad \phi(a,t)=\phi(b,t),\end{aligned} \tag{2}$$

there exist a charge conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b|\psi|^2\mathrm{d}x=0 \tag{3}$$

and a global energy conservation law (GECL)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b\frac{1}{2}(-\phi^2-\phi_t^2-\phi_x^2-|\psi_x|^2+2\phi|\psi|^2)\mathrm{d}x\\&=0.\end{aligned} \tag{4}$$

Massive works[1–3] with respect to mathematical and theoretical analyses of the KGS equations have been done in the past decades. Studying its numerical approximation has recently become a hot topic. Zhang[4] derived a charge- and energy-preserving finite difference (C-C) scheme for the KGS equations. During the last decade, many researchers have paid attention to geometrical numerical integrators. Kong *et al.*[5] noted that the KGS equations have a multi-symplectic structure and then investigated the multi-symplectic midpoint (M-M) scheme. The explicit multi-symplectic schemes were proposed in Refs. [6] and [7]; while the implicit symplectic Fourier pseudo-spectral schemes were given in Refs. [8] and [6]. Hong *et al.*[9] also proposed three schemes, which are called M-C, T-C, and T-T schemes, for the KGS equations. Some other methods can be found in Refs. [10]–[14].

As is well known, the GECL (4) depends on the boundary conditions, such as periodic or homogeneous boundary conditions. However, in practice, the complex boundary conditions lead to the invalidity of the traditional GECL algorithms. Actually, there exists a local energy conservation law, which is more essential than the global energy conservation law. Let $\psi(x,t)=p(x,t)+\mathrm{i}q(x,t)$, where $p(x,t)$ and $q(x,t)$ are real-valued functions. By further introducing some variables, the KGS system (1) can be reformed into a combination of ordinary differential equations (ODEs)

$$\begin{cases}q_t=\dfrac{1}{2}m+\phi p,\\-p_t=\dfrac{1}{2}n+\phi q,\end{cases} \tag{5a}$$

$$\begin{cases}v_t=\alpha-\phi+(p^2+q^2),\\\phi_t=v,\end{cases} \tag{5b}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11201169, 11271195, and 41231173) and the Project of Graduate Education Innovation of Jiangsu Province, China (Grant No. CXLX13_366).

†Corresponding author. E-mail: thomasjeer@sohu.com, wangyushun@njnu.edu.cn

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$$\begin{cases} f_x = m, \\ p_x = f, \end{cases} \quad \begin{cases} g_x = n, \\ q_x = g, \end{cases} \quad \begin{cases} w_x = \alpha, \\ \phi_x = w. \end{cases} \tag{5c}$$

Proposition 1 The system (5) possesses a local energy conservation law (LECL)

$$\partial_t \left(-\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi^2 - \frac{1}{2}\phi_x^2 - \frac{1}{2}|\psi_x|^2 + \phi|\psi|^2 \right) + \partial_x(f p_t + g q_t + w\phi_t) = 0. \tag{6}$$

Proof In Eq. (5a), multiplying the first and the second lines by p_t and q_t, respectively, we have

$$\begin{aligned} 0 &= f_x p_t + g_x q_t + \phi \partial_t |\psi|^2 \\ &= \partial_x(f p_t + g q_t) - (f p_{xt} + g q_{xt}) + \phi \partial_t |\psi|^2 \\ &= \partial_x(f p_t + g q_t) - \frac{1}{2}\partial_t(|\psi_x|^2) + \phi \partial_t |\psi|^2. \end{aligned} \tag{7}$$

Next, multiplying the first line of Eq. (5b) by ϕ_t gives

$$\begin{aligned} 0 &= -v_t v + w_x \phi_t - \frac{1}{2}\partial_t \phi^2 + \phi_t |\psi|^2 \\ &= -\frac{1}{2}\partial_t v^2 + \partial_x(w\phi_t) - w\phi_{xt} - \frac{1}{2}\partial_t \phi^2 + \phi_t |\psi|^2 \\ &= \partial_t \left(-\frac{1}{2}v^2 - \frac{1}{2}\phi^2 - \frac{1}{2}\phi_x^2 \right) + \partial_x(w\phi_t) + \phi_t |\psi|^2. \end{aligned} \tag{8}$$

Adding Eqs. (7) and (8) leads to the LECL (6).

The LECL (6) is a local property. In other words, it is independent of the boundary conditions. Thus, the LECL is more essential than the GECL in physics. Naturally, when constructing algorithms for system (5), we expect that the LECL can still hold. Hereafter, we call these numerical methods the local energy-preserving (LEP) algorithms. For the concept of LEP algorithm and its applications, please refer to Refs. [15]–[18]. The aim of this paper is to construct a LEP algorithm for the KGS equations, and then analyze its convergence. Finally, we will check the performance of the proposed scheme.

2. LEP algorithm

As usual, we introduce some notations: $x_j = a + (j-1)h$, $j = 1, 2, \ldots, N+1$; $t_k = k\tau$, $k = 0, 1, 2, \ldots K$, where $h = (b-a)/N$ and τ are the spatial length and the temporal step span, respectively. Let u_j^k be an approximation of $u(x_j, t_k)$. We also define the vector 2-norm $\| u^k \| = (h\sum_{j=1}^{N} |u_j^k|^2)^{1/2}$. To derive the algorithms conveniently, the following operators are used: the finite difference operators $\delta_t f_j^k = (f_j^{k+1} - f_j^k)/\tau$, $\delta_x f_j^k = (f_{j+1}^k - f_j^k)/h$; the averaging operators $A_t f_j^k = (f_j^{k+1} + f_j^k)/2$, $A_x f_j^k = (f_{j+1}^k + f_j^k)/2$; and the operators $\delta_x(f\cdot g)_j^k = (a f_{j+1}^k + (1-a) f_j^k) \cdot \delta_x g_j^k + \delta_x f_j^k \cdot ((1-a) g_{j+1}^k + a g_j^k)$, $\forall 0 \le a \le 1$. Especially, we have

$$\begin{aligned} &a = 0, \quad \delta_x(f\cdot g)_j^k = f_j^k \cdot \delta_x g_j^k + \delta_x f_j^k \cdot g_{j+1}^k, \\ &a = \frac{1}{2}, \quad \delta_x(f\cdot g)_j^k = A_x f_j^k \cdot \delta_x g_j^k + \delta_x f_j^k \cdot A_x g_j^k, \\ &a = 1, \quad \delta_x(f\cdot g)_j^k = f_{j+1}^k \cdot \delta_x g_j^k + \delta_x f_j^k \cdot g_j^k. \end{aligned} \tag{9}$$

Taking $f = g$ gives

$$\begin{aligned} &\delta_x \left[\frac{1}{2}(f_j^k)^2 \right] = \delta_x f_j^k \cdot A_x f_j^k, \\ &\delta_x \left(\frac{1}{2} f_{j-1}^k \cdot f_j^k \right) = f_j^k \cdot A_x \delta_x f_{j-1}^k. \end{aligned} \tag{10}$$

Similarly, we can obtain a series of analogous discrete Leibnite rules in the time direction.

LEP scheme We discrete the ODEs (5) as follows:

$$\begin{cases} \delta_t q_j^k = \frac{1}{2} A_t m_j^k + A_t \phi_j^k \cdot A_t p_j^k, \\ -\delta_t p_j^k = \frac{1}{2} A_t n_j^k + A_t \phi_j^k \cdot A_t q_j^k, \end{cases} \tag{11a}$$

$$\begin{cases} \delta_t v_j^k = A_t \alpha_j^k - A_t \phi_j^k + A_t[(p_j^k)^2 + (q_j^k)^2], \\ \delta_t \phi_j^k = A_t v_j^k, \end{cases} \tag{11b}$$

$$\begin{cases} \delta_x f_j^k = m_j^k, \\ \delta_x p_j^k = f_{j+1}^k, \end{cases} \quad \begin{cases} \delta_x g_j^k = n_j^k, \\ \delta_x q_j^k = g_{j+1}^k, \end{cases} \quad \begin{cases} \delta_x w_j^k = \alpha_j^k, \\ \delta_x \phi_j^k = w_{j+1}^k. \end{cases} \tag{11c}$$

Eliminating the auxiliary variables m, n, v, α, f, g, and w, we obtain a scheme

$$\delta_t q_j^k = \frac{1}{2} A_t \delta_x^2 p_{j-1}^k + A_t \phi_j^k \cdot A_t p_j^k, \tag{12a}$$

$$-\delta_t p_j^k = \frac{1}{2} A_t \delta_x^2 q_{j-1}^k + A_t \phi_j^k \cdot A_t q_j^k, \tag{12b}$$

$$\delta_t^2 \phi_j^k = A_t^2 \delta_x^2 \phi_{j-1}^k - A_t^2 \phi_j^k + A_t^2[(p_j^k)^2 + (q_j^k)^2], \tag{12c}$$

or an equivalent scheme

$$\mathrm{i}\delta_t \psi_j^k + \frac{1}{2} A_t \delta_x^2 \psi_{j-1}^k + A_t \phi_j^k \cdot A_t \psi_j^k = 0, \tag{13a}$$

$$\delta_t^2 \phi_j^k - A_t^2 \delta_x^2 \phi_{j-1}^k + A_t^2 \phi_j^k - A_t^2 |\psi_j^k|^2 = 0. \tag{13b}$$

Now, we analyze the local and global conservative properties of the proposed LEP scheme.

Theorem 1 The LEP scheme is a local energy conservative algorithm, which admits the discrete LECL

$$\begin{aligned} \mathcal{E}(x_j, t_k) = \ & \delta_t \Big[-\frac{1}{2}|\delta_x \psi_j^k|^2 - \frac{1}{2}(v_j^k)^2 - \frac{1}{2}(\delta_x \phi_j^k)^2 \\ & - \frac{1}{2}(\phi_j^k)^2 + \phi_j^k |\psi_j^k|^2 \Big] + \delta_x \big(A_t f_j^k \cdot \delta_t p_j^k + A_t g_j^k \cdot \delta_t q_j^k \\ & + A_t w_j^k \cdot \delta_t \phi_j^k \big) = 0. \end{aligned} \tag{14}$$

Proof In Eq. (11a), multiplying the first and the second lines by $\delta_t p_j^k$ and $\delta_t q_j^k$, respectively, and then adding them together give

$$\begin{aligned} 0 = \ & \frac{1}{2} A_t m_j^k \cdot \delta_t p_j^k + \frac{1}{2} A_t n_j^k \cdot \delta_t q_j^k \\ & + A_t \phi_j^k (A_t p_j^k \cdot \delta_t p_j^k + A_t q_j^k \cdot \delta_t q_j^k). \end{aligned} \tag{15}$$

Equation (15) can be reduced as follows:

$$0 = A_t \delta_x f_j^k \cdot \delta_t p_j^k + A_t \delta_x g_j^k \cdot \delta_t q_j^k + A_t \phi_j^k \cdot \delta_t |\psi_j^k|^2$$

$$
\begin{aligned}
&= \delta_x(A_t f_j^k \cdot \delta_t p_j^k + A_t g_j^k \cdot \delta_t q_j^k) - A_t f_{j+1}^k \cdot \delta_x \delta_t p_j^k \\
&\quad - A_t g_{j+1}^k \cdot \delta_x \delta_t q_j^k + A_t \phi_j^k \cdot \delta_t |\psi_j^k|^2 \\
&= \delta_x(A_t f_j^k \cdot \delta_t p_j^k + A_t g_j^k \cdot \delta_t q_j^k) \\
&\quad - \frac{1}{2}\delta_t[(f_{j+1}^k)^2 + (g_{j+1}^k)^2] + A_t \phi_j^k \cdot \delta_t |\psi_j^k|^2 \\
&= \delta_x(A_t f_j^k \cdot \delta_t p_j^k + A_t g_j^k \cdot \delta_t q_j^k) \\
&\quad - \frac{1}{2}\delta_t |\delta_x \psi_j^k|^2 + A_t \phi_j^k \cdot \delta_t |\psi_j^k|^2. \qquad (16)
\end{aligned}
$$

Multiplying the first line of Eq. (11b) by $\delta_t \phi_j^k$ yields

$$
\begin{aligned}
0 &= -\delta_t v_j^k \cdot A_t v_j^k + A_t \delta_x w_j^k \cdot \delta_t \phi_j^k - \frac{1}{2}\delta_t (\phi_j^k)^2 + A_t |\psi_j^k|^2 \cdot \delta_t \phi_j^k \\
&= -\frac{1}{2}\delta_t (v_j^k)^2 + \delta_x (A_t w_j^k \cdot \delta_t \phi_j^k) - A_t w_{j+1}^k \cdot \delta_x \delta_t \phi_j^k \\
&\quad - \frac{1}{2}\delta_t (\phi_j^k)^2 + A_t |\psi_j^k|^2 \cdot \delta_t \phi_j^k \\
&= -\frac{1}{2}\delta_t (v_j^k)^2 + \delta_x (A_t w_j^k \cdot \delta_t \phi_j^k) - \frac{1}{2}\delta_t (\delta_x \phi_j^k)^2 \\
&\quad - \frac{1}{2}\delta_t (\phi_j^k)^2 + A_t |\psi_j^k|^2 \cdot \delta_t \phi_j^k. \qquad (17)
\end{aligned}
$$

Adding Eqs. (16) and (17), and noting that $A_t \phi_j^k \cdot \delta_t |\psi_j^k|^2 + A_t |\psi_j^k|^2 \cdot \delta_t \phi_j^k = \delta_t (\phi_j^k \cdot |\psi_j^k|^2)$, we finish the proof.

Remark 1 In the above derivation, we use the Leibnite rules (9) and (10). The discrete LECL is consistent with the continuous LECL (6) and independent of the boundary conditions.

Corollary 1 For the initial and periodic boundary conditions (2), the LEP-I scheme preserves the discrete GECL

$$
\mathcal{E}^{k+1} = \mathcal{E}^k = \cdots = \mathcal{E}^1 = \mathcal{E}^0, \qquad (18)
$$

where

$$
\begin{aligned}
\mathcal{E}^k &= -\frac{1}{2} \parallel \delta_x \psi^k \parallel^2 - \frac{1}{2} \parallel v^k \parallel^2 - \frac{1}{2} \parallel \delta_x \phi^k \parallel^2 \\
&\quad - \frac{1}{2} \parallel \phi^k \parallel^2 + h \sum_{j=1}^{N} \phi_j^k |\psi_j^k|^2.
\end{aligned}
$$

To prove it, we just need to sum the LECL (14) over all space index j and utilize the periodic boundary conditions.

Remark 2 The discrete GECL contains variable v_j^k ($j = 1, 2, \cdots, N$), which are determined by Eq. (11b) with boundary conditions (2).

In general, as a quadratic invariant, the charge conservation law (3) plays an important role in quantum physics. Therefore, it is necessary to discuss whether it can be captured.

Theorem 2 For the initial and periodic boundary conditions (2), the LEP scheme conserves the charge conservation law exactly,

$$
\mathcal{Q}^{k+1} = \parallel \psi^{k+1} \parallel^2 = \cdots = \parallel \psi^0 \parallel^2 = \mathcal{Q}^0, \qquad (19)
$$

namely, the scheme is unconditionally stable with respect to the initial values.

Proof Multiplying Eq. (13a) by $2hA_t \overline{\psi_j^k}$ and summing it over all space index j gives

$$
\begin{aligned}
&\mathrm{i} \sum_{j=1}^{N} \delta_t \psi_j^k \cdot 2hA_t \overline{\psi_j^k} + \sum_{j=1}^{N} A_t \delta_x^2 \psi_{j-1}^k \cdot hA_t \overline{\psi_j^k} \\
&+ \sum_{j=1}^{N} A_t \phi_j^k \cdot A_t \psi_j^k \cdot 2hA_t \overline{\psi_j^k} = 0. \qquad (20)
\end{aligned}
$$

The first term of Eq. (20) becomes

$$
\frac{\mathrm{i}}{\tau}(\parallel \psi^{k+1} \parallel^2 - \parallel \psi^k \parallel^2) - \frac{2h}{\tau} \sum_{j=1}^{N} \mathrm{Im}(\psi_j^{k+1} \overline{\psi_j^k}), \qquad (21)
$$

where Im stands for the imaginary part. The second term

$$
\begin{aligned}
&\sum_{j=1}^{N} A_t \delta_x^2 \psi_{j-1}^k \cdot hA_t \overline{\psi_j^k} \\
&= \frac{1}{h}\left(\sum_{j=1}^{N} A_t \psi_{j-1}^k \cdot A_t \overline{\psi_j^k} + \sum_{j=1}^{N} A_t \psi_j^k \cdot A_t \overline{\psi_{j-1}^k} - 2 \sum_{j=1}^{N} |A_t \psi_j^k|^2 \right) \\
&= \frac{1}{h}\left[\sum_{j=1}^{N} 2\mathrm{Re}(A_t \psi_{j-1}^k \cdot A_t \overline{\psi_j^k}) - 2 \sum_{j=1}^{N} |A_t \psi_j^k|^2 \right] \qquad (22)
\end{aligned}
$$

is a real-valued function, where Re stands for the real part. The last term

$$
\sum_{j=1}^{N} A_t \phi_j^k \cdot A_t \psi_j^k \cdot 2hA_t \overline{\psi_j^k} = 2h \sum_{j=1}^{N} A_t \phi_j^k \cdot |A_t \psi_j^k|^2 \qquad (23)
$$

is also a real-valued function. Therefore, the imaginary part of Eq. (20) implies the charge conservation law (19).

By using the discrete Sobolev inequality and the Gronwall inequality,[19] we have the following convergence results.

Theorem 3 Assume $\psi \in C^{4,3}$ and $\phi \in C^{4,4}$, then the solution of the LEP scheme converges to the solution of problem (1)–(2) with order $\mathcal{O}(h^2 + \tau^2)$ in the $\parallel \cdot \parallel$ norm.

Since the proof is very trivial, the details are not presented here. It should be pointed out that, in order to make the convergence analysis, we must prove $\parallel \psi^k \parallel_\infty \leq C$ and $\parallel \phi^k \parallel_\infty \leq C$, and analyze the truncation error of the LEP scheme first.

3. Numerical experiments

In this section, some numerical experiments are carried out to show the performance of the LEP scheme. The performance of the proposed method will be exhibited in the following aspects: (i) the accuracy of single soliton solutions; (ii) the performance in preserving the local energy, charge, and global energy conservation laws. The accuracy of migration of a soliton at $t_k = k\tau$ is measured by

$$
L_\infty(\psi) = \max_{1 \leq j \leq N} | \psi(x_j, k\tau) - \psi_j^k |
$$

and

$$
L_2(\psi) = \left(h \sum_{j=1}^{N} | \psi(x_j, k\tau) - \psi_j^k |^2 \right)^{1/2}.
$$

In order to show the variation of the local energy at the k-th time level against time, we define $\mathcal{E}^k_{\text{local}} = \max_{j=1,\ldots,N} |\mathcal{E}(x_j,t_k)|$. The errors in charge and global energy can be scaled by $|\mathcal{Q}^n - \mathcal{Q}^0|$ and $|\mathcal{E}^n - \mathcal{E}^0|$, respectively.

Example 1 (soliton) The KGS equations have the following soliton solution:

$$\psi(x,t,\nu,x_0) = \frac{3\sqrt{2}}{4\sqrt{1-\nu^2}}\,\mathrm{sech}^2(A)\exp(\mathrm{i}B),$$
$$\phi(x,t,\nu,x_0) = \frac{3}{4(1-\nu^2)}\,\mathrm{sech}^2(A),$$

where

$$A = \frac{1}{2\sqrt{1-\nu^2}}[x-\nu t-x_0],$$
$$B = \nu x + \frac{1-\nu^2+\nu^4}{2(1-\nu^2)}t.$$

Here, ν represents the propagating velocity of the wave and x_0 is the initial phase. We take the initial conditions $\psi_0(x) = \psi(x,0,\nu,0)$, $\phi_0(x) = \phi(x,0,\nu,0)$, $\phi_1(x) = \phi_t(x,0,\nu,0)$, and the periodic boundary conditions $\psi(-40,t) = \psi(40,t)$, $\phi(-40,t) = \phi(40,t)$, $\phi_t(-40,t) = \phi_t(40,t)$.

We choose $\nu = 0.1$, $\tau = 0.01$, and $h = 0.1$. From Fig. 1, one can see that the errors in solution ψ grow nearly linearly while the ones in solution ϕ grow oscillatory. One can also find that figure 1 is very similar to that in Ref. [7]. The local energy and the errors in the global invariants are listed in Table 1. It is clear that the local energy $\mathcal{E}_{\text{local}}$ is always in the scale of 10^{-14} throughout computations. Therefore, the local energy is conserved exactly. Since the errors in charge and global energy oscillate near zero in the scale of 10^{-15}, the two global invariants are also captured.

Table 1. Local energy and the errors in global invariants for soliton at different time.

Time	$\mathcal{E}^n_{\text{local}}/10^{-14}$	$\lvert\mathcal{E}^n-\mathcal{E}^0\rvert/10^{-15}$	$\lvert\mathcal{Q}^n-\mathcal{Q}^0\rvert/10^{-16}$
10	1.42	0.333	4.44
20	2.13	2.88	17.7
40	3.55	3.99	8.88
60	2.13	5.32	62.1
80	2.48	3.10	93.2

Next, we compare the numerical performance of the LEP scheme with that of the existing methods.[4,5,9] We implement all methods with $\nu = 0.3$, various temporal and spatial step sizes. Table 2 reports the errors in the solutions, and the changes in charge and global energy at $T = 1$. From the table, we find that the solutions of LEP are better than those of C-C, M-M, T-C, and T-T schemes, while a little worse than the M-C scheme. The errors in invariants show that our method conserves the charge and the global energy much better than the others.

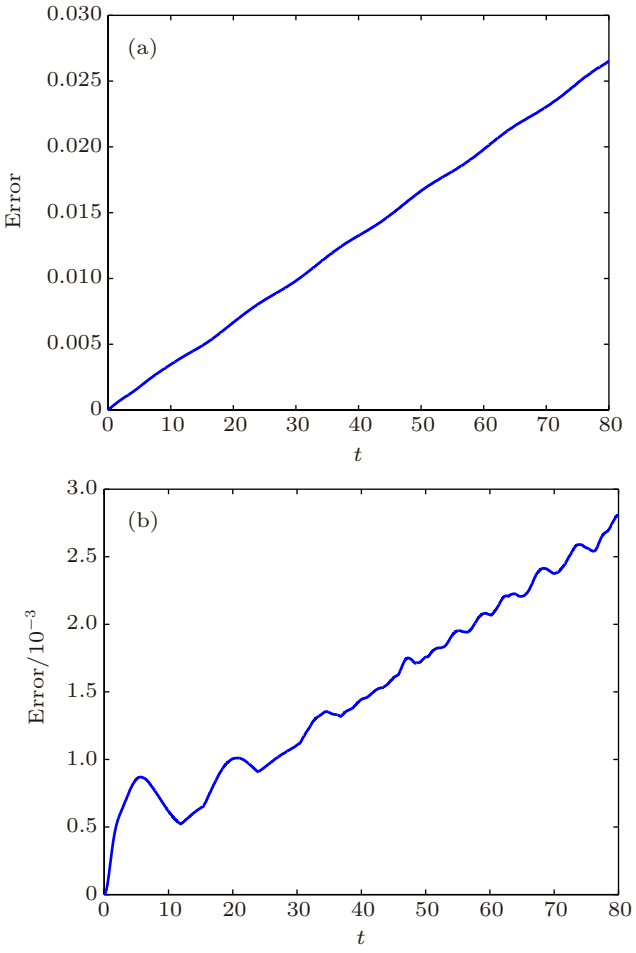


Fig. 1. (a) $L_\infty(\psi)$, (b) $L_\infty(\phi)$.

Table 2. Comparison with some existing methods in numerical error and conserved invariants.

τ/h	Method	$L_2(\phi)/10^{-4}$	$L_2(\psi)/10^{-4}$	$\lvert\mathcal{Q}^n-\mathcal{Q}^0\rvert$	$\lvert\mathcal{E}^n-\mathcal{E}^0\rvert$
0.1/0.2	LEP	13.609	31.168	1.7764×10^{-15}	5.5511×10^{-16}
	C-C[4]	14.248	31.348	5.7731×10^{-15}	3.4417×10^{-15}
	M-M[5]	33.417	64.025	4.4409×10^{-15}	1.8917×10^{-6}
	M-C[9]	12.718	30.537	6.9201×10^{-15}	1.2436×10^{-6}
	T-C[9]	13.668	31.750	4.1392×10^{-6}	8.4530×10^{-7}
	T-T[9]	27.508	65.403	8.9399×10^{-16}	6.6946×10^{-5}
0.05/0.1	LEP	3.6181	8.1743	8.8818×10^{-16}	1.1102×10^{-16}
	C-C[4]	3.7850	8.2222	1.1102×10^{-14}	4.5519×10^{-15}
	M-M[5]	8.8406	16.938	7.5495×10^{-15}	1.3297×10^{-7}
	M-C[9]	3.3875	8.0046	1.2155×10^{-14}	7.7985×10^{-8}
	T-C[9]	3.6365	8.3281	6.7716×10^{-8}	8.8421×10^{-8}
	T-T[9]	7.2219	17.011	4.3669×10^{-7}	6.6713×10^{-6}

Example 2 (solitons collision) In the following simulations, we study head-on collisions of two solitary waves. The initial conditions are chosen as $\psi_0 = \psi(x,0,v_1,x_1) + \psi(x,0,v_2,x_2)$, $\phi_0 = \phi(x,0,v_1,x_1) + \phi(x,0,v_2,x_2)$, and $\phi_1 = \phi_t(x,0,v_1,x_1) + \phi_t(x,0,v_2,x_2)$. We consider the simulation over the region $[-40,40] \times [0,80]$ with parameters $\tau = 0.01$ and $h = 0.1$.

First, we investigate the collision of symmetric solitons, i.e., $v_1 = 0.4$, $x_1 = -20$ and $v_2 = -0.4$, $x_2 = 20$. Figure 2 displays the evolution of ψ and ϕ. The two solitons collide at about $T = 40$, and then seem to be trapped. Table 3 displays the local energy, and the errors in charge and global energy against time. The results coincide with the theoretical analysis.

Second, we consider the non-symmetric solitions collision with $v_1 = 0.4$, $x_1 = -20$ and $v_2 = -0.2$, $x_2 = 15$. The simulation results are illustrated in Fig. 3. It is clear that the soliton with the larger amplitude becomes larger and the one with the smaller amplitude becomes smaller after the collision. The local energy $\mathcal{E}_{\text{local}}$, and the errors in charge and global energy are presented in Table 4. The results are all within the roundoff error of the machine. Again the proposed LEP scheme conserves the invariants exactly.

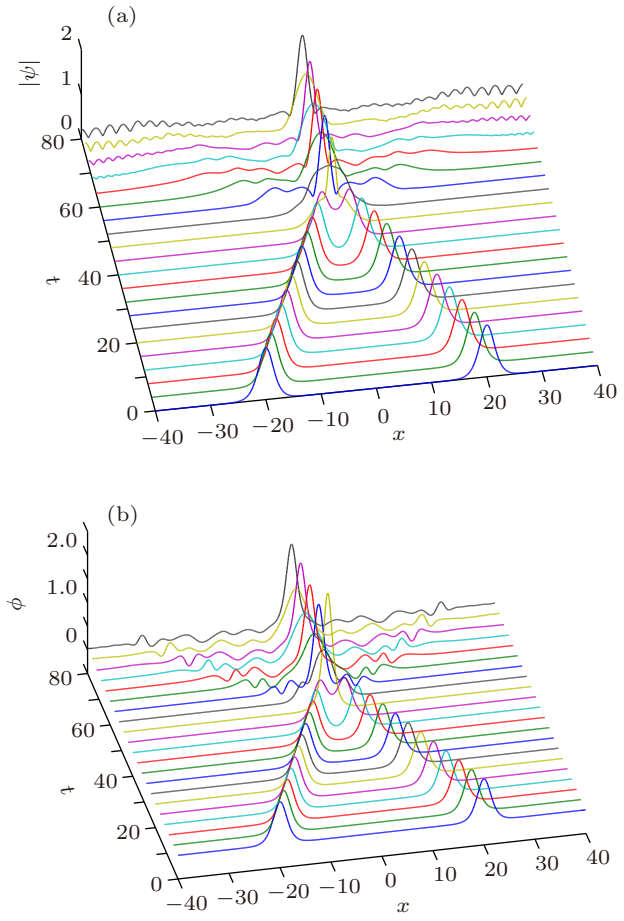


Fig. 2. (color online) Symmetric solitons collision: (a) $|\psi|$, (b) ϕ.

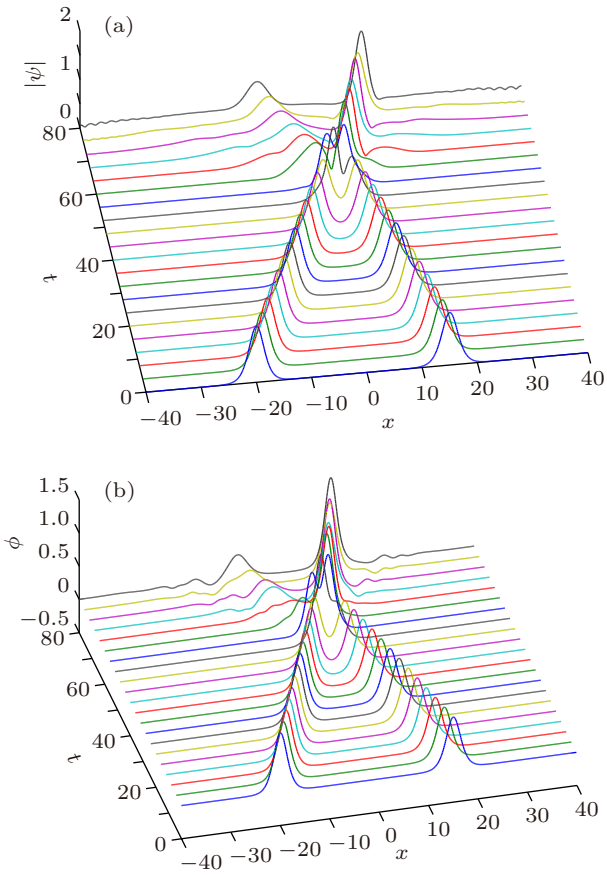


Fig. 3. (color online) Non-symmetric solitons collision: (a) $|\psi|$, (b) ϕ.

Table 3. Local energy, and the errors in global invariants for symmetric solitons collision at different time.

Time	$\mathcal{E}^n_{\text{local}}/10^{-14}$	$\lvert\mathcal{E}^n - \mathcal{E}^0\rvert/10^{-15}$	$\lvert\mathcal{Q}^n - \mathcal{Q}^0\rvert/10^{-15}$
10	4.97	2.55	5.32
20	4.26	16.8	7.10
40	3.19	66.0	17.7
60	4.26	13.0	17.7
80	17.0	19.3	17.7

Table 4. Local energy, and the errors in global invariants for non-symmetric solitons collision at different time.

Time	$\mathcal{E}^n_{\text{local}}/10^{-14}$	$\lvert\mathcal{E}^n - \mathcal{E}^0\rvert/10^{-15}$	$\lvert\mathcal{Q}^n - \mathcal{Q}^0\rvert/10^{-16}$
10	4.26	25.5	62.1
20	3.55	26.6	8.88
40	4.26	15.9	97.7
60	11.3	15.9	133
80	8.52	1.11	142

4. Conclusion

In this work, we propose a LEP scheme to simulate the KGS equations. The local property is independent of the boundary conditions. Our method preserves the discrete LECL in any time–space region. If the boundary conditions are suitable, the LEP method will be the GEP method. Therefore, the method in the present paper extends the applying

scope of the traditional GEP algorithm. Numerical results indicate that the present scheme can provide accurate solutions in long-term computations, and also show excellent performance in conserving the local energy, global energy, and charge invariants. From Eq. (5), we can also derive local momentum and global momentum conservation laws for the KGS equations. As we know, they are also important invariants in physics. Similarly, we could consider local momentum-preserving schemes for the KGS equations.

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