Explicit solutions from residual symmetry of the Boussinesq equation*

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The Bäcklund transformation related symmetry is nonlocal, which is hard to be applied in constructing solutions for nonlinear equations. In this paper, the residual symmetry of the Boussinesq equation is localized to Lie point symmetry by introducing multiple new variables. By applying the general Lie point method, two main results are obtained: a new type of Bäcklund transformation is derived, from which new solutions can be generated from old ones; the similarity reduction solutions as well as corresponding reduction equations are found. The localization procedure provides an effective way to investigate interaction solutions between nonlinear waves and solitons.

Keywords: Boussinesq equation, localization procedure, residual symmetry, symmetry reduction solution

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1. Introduction

It is known that almost all of the fundamental equations of natural science, including physics, chemistry, biology, astrophysics, etc., are nonlinear. Nonlinear equations are usually hard to solve explicitly, so perturbation, asymptotic, and numerical methods are often used to obtain approximate solutions of these equations. However, there is also a great deal of current interest in obtaining exact analytic solutions of nonlinear equations, among which the study on integrable system plays an important role and various effective methods have been developed to study its different properties.

Since the Lie group theory was introduced by Sophus Lie to study differential equations,[1] the study of Lie group has always been an important subject in mathematics and physics. If the nonlinear equations have Lie point symmetry group, one can lower dimensions of partial differential equations (PDEs) by using both classical and non-classical Lie group approaches.[2–8] This provides a method for obtaining exact and special solutions of a given equation in terms of solutions of lower-dimensional equations, in particular, ordinary differential equations. Bäcklund transformation (BT) is another powerful method to search for new solutions of PDEs through known ones. However, the interaction solution between different waves is hard to study by the original BT because it is very difficult to solve when the seed solutions are taken as nonconstant nonlinear waves such as the cnoidal waves and Painlevé waves. In this paper, we no longer concentrate on the traditional BT itself but switch to study its infinitesimal transformation (i.e., its symmetry) and use this symmetry to construct new exact solutions. However, the BT related symmetry is nonlocal,[9] which is hard to be used for reducing nonlinear equations. To overcome this difficulty, Lou and his colleagues developed a new localization procedure. By this method, the original system is enlarged by introducing new dependent variables such that the symmetry becomes localized in the new enlarged system.[10–12]

In this paper, we focus on the residual symmetries of the well-known Boussinesq equation and its applications in obtaining new exact solutions. The Boussinesq equation

$$u_{tt} + u_{xxxx} + 6u_x^2 + 6uu_{xx} = 0 \tag{1}$$

arises in several physical applications, such as propagation of long waves in shallow water,[13–15] one-dimensional nonlinear lattice waves,[16,17] vibrations in a nonlinear string,[18] and ion sound waves in a plasma.[19,20] For this equation, a considerable number of explicit solutions, especially the multi-soliton solutions and periodic solutions, have been obtained by various methods including inverse scattering transformation, the symmetry method, the Hirota bilinear method, the Darboux transformation (DT), the Bäcklund transformation (BT), and the tanh expansion method.

The paper is organized as follows. In Section 2, the residual symmetry of the Boussinesq equation is obtained by using the Painlevé truncated expansion method, and then localized by introducing multiple new variables. By the way, the finite group transformation theorem is obtained by using Lie's first principle. In Section 3, the general form of the Lie point symmetry group for the enlarged Boussinesq system is obtained and the similarity reductions for the prolonged system is considered according to the standard Lie point symmetry approach with some special explicit solutions. Section 4 is devoted to a short summary and discussion.

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2. Localization of residual symmetry and the related Bäcklund transformation

Since the Boussinesq equation has the Painlevé property, i.e., it is Painlevé integrable, there exists a truncated transformation

$$u=\sum_{i=0}^{\alpha}u_i\phi^{i-\alpha}, \tag{2}$$

for positive integer α. By balancing the nonlinear term and dispersion term in Eq. (1), we have $\alpha=2$. Hence, the truncated Painlevé expansion reads

$$u=\frac{u_0}{\phi^2}+\frac{u_1}{\phi}+u_2. \tag{3}$$

It can be easily verified that the residue u_1 of the truncated Painlevé expansion (3) with respect to the singular manifold ϕ is a symmetry of Eq. (1) with the solution u_2. To prove this conclusion, we substitute Eq. (3) into Eq. (1) and obtain

$$\begin{aligned}
&6u_{2x}^2+6u_2u_{2xx}+u_{2xxxx}+u_{2tt}\\
&+\phi^{-1}(6u_1u_{2xx}+12u_{1x}u_{2x}+u_{1xxxx}+u_{1tt}+6u_2u_{1xx})\\
&+\phi^{-2}(-4u_{1x}\phi_{xxx}+6u_0u_{2xx}-12u_2u_{1x}\phi_x\\
&-12u_1\phi_xu_{2x}-4u_{1xxx}\phi_x-6u_{1xx}\phi_{xx}+6u_1u_{1xx}\\
&+u_{0xxxx}+6u_{1x}^2+u_{0tt}-u_1\phi_{xxxx}+12u_{0x}u_{2x}\\
&-2u_{1t}\phi_t-u_1\phi_{tt}-6u_2u_1\phi_{xx}+6u_2u_{0xx})\\
&+\phi^{-3}(2u_1\phi_t^2+12u_{0x}u_{1x}+6u_0u_{1xx}-2u_0\phi_{xxxx}\\
&+12u_2u_1\phi_x^2-24u_0\phi_xu_{2x}-12u_2u_0\phi_{xx}\\
&-4u_{0t}\phi_t-12u_{0xx}\phi_{xx}+6u_1u_{0xx}-8u_{0x}\phi_{xxx}\\
&+6u_1\phi_{xx}^2-8u_{0xxx}\phi_x-24u_1u_{1x}\phi_x-24u_2u_{0x}\phi_x\\
&+8u_1\phi_x\phi_{xxx}-6u_1^2\phi_{xx}+12u_{1xx}\phi_x^2\\
&+24u_{1x}\phi_x\phi_{xx}-2u_0\phi_{tt})+\phi^{-4}(6u_0u_{0xx}-36u_0u_{1x}\phi_x\\
&+6u_0\phi_t^2-18u_0u_1\phi_{xx}-36u_1\phi_x^2\phi_{xx}+72u_{0x}\phi_x\phi_{xx}\\
&+6u_{0x}^2+36u_2u_0\phi_x^2-36u_1u_{0x}\phi_x+18u_0\phi_{xx}^2\\
&+18u_1^2\phi_x^2+36u_{0xx}\phi_x^2-24u_{1x}\phi_x^3+24u_0\phi_x\phi_{xxx})\\
&+\phi^{-5}(-12u_0^2\phi_{xx}-96u_{0x}\phi_x^3+72u_0u_1\phi_x^2+24u_1\phi_x^4\\
&-48u_0u_{0x}\phi_x-144u_0\phi_x^2\phi_{xx})\\
&+60\phi^{-6}u_0\phi_x^2(2\phi_x^2+u_0)=0.
\end{aligned} \tag{4}$$

Vanishing the coefficients of ϕ^0 and ϕ^{-1} in Eq. (4), we see that u_1 satisfies the linearized Boussinesq equation with u_2 being the solution. Then, our conclusion is proved.

In order to determine the residual symmetry $\sigma^{\rm u}=u_1$ and the Schwarzian form of Boussinesq equation, we proceed to vanish the coefficients of ϕ^{-6}, ϕ^{-5}, and ϕ^{-4} in Eq. (4), then we have

$$u_0=-2\phi_x^2,\quad u_1=2\phi_{xx}, \tag{5}$$

and

$$u_2=-\frac{1}{6}\phi_x^{-2}(-3\phi_{xx}^2+4\phi_x\phi_{xxx}+\phi_t^2). \tag{6}$$

Now, substituting u_0, u_1, and u_2 in Eqs. (5) and (6) into Eq. (4), meanwhile vanishing the coefficient of ϕ^{-3}, we obtain the Schwarzian form of Boussinesq equation

$$\left[\{\phi;x\}+\frac{1}{2}\frac{\phi_t^2}{\phi_x^2}\right]_x+\left(\frac{\phi_t}{\phi_x}\right)_t=0, \tag{7}$$

where $\{\phi;x\}=\phi_{xxx}/\phi_x-3/2\phi_{xx}^2/\phi_x^2$. Equation (7) is invariant under the Möbious transformation

$$\phi\to\frac{a_1\phi+b_1}{a_2\phi+b_2},\ a_1a_2\neq b_1b_2, \tag{8}$$

which means equation (7) possesses three symmetries $\sigma_\phi=d_1$, $\sigma_\phi=d_2\phi$, and

$$\sigma_\phi=d_3\phi^2, \tag{9}$$

with arbitrary constants d_1, d_2, and d_3. Apparently, the residual symmetry $\sigma^{\rm u}=u_1$ is related to the Möbious transformation symmetry (9) by the linearized equation (6).

It is clear that the residual symmetry

$$\sigma_{\rm u}=2\phi_{xx}, \tag{10}$$

where ϕ satisfying Eq. (7) is nonlocal and this symmetry is just the generator of the Bäcklund transformation (3), which has also been given in Ref. [21]. In order to study the transformation properties of the Boussinesq equation by this symmetry, we have to localize it to Lie point symmetry by introducing the following new variables:

$$g\equiv\phi_x, \tag{11}$$

$$h\equiv g_x, \tag{12}$$

$$m\equiv\phi_t. \tag{13}$$

All the symmetries corresponding to different variables are related to each other by the linearized equations (1), (6), (7), (11), (12), and (13), i.e.,

$$\sigma_{\phi,x}-\sigma_g=0, \tag{14a}$$

$$\sigma_{g,x}-\sigma_h=0, \tag{14b}$$

$$\sigma_{\phi,t}-\sigma_m=0, \tag{14c}$$

$$\begin{aligned}
&-3\phi_x\phi_{xx}\sigma_{\phi,xx}-2\phi_x\sigma_{\phi,x}\phi_{xxx}+2\phi_x^2\sigma_{\phi,xxx}+\phi_x\phi_t\sigma_{\phi,t}\\
&+3\sigma_{\phi,x}\phi_{xx}^2-\sigma_{\phi,x}\phi_t^2+3\sigma_{\rm u}\phi_x^3=0,
\end{aligned} \tag{14d}$$

$$\sigma_{u,tt}+\sigma_{u,xxxx}+12u_x\sigma_{u,x}+6\sigma_{\rm u}u_{xx}+6u\sigma_{u,xx}=0, \tag{14e}$$

$$\begin{aligned}
&-\sigma_{\phi,xx}\phi_t^2-2\sigma_{\phi,t}\phi_{xx}\phi_t+2\phi_x\phi_{xxxx}\sigma_{\phi,x}\\
&+\phi_x^2\sigma_{\phi,xxxx}+9\phi_{xx}^2\sigma_{\phi,xx}-4\sigma_{\phi,x}\phi_{xx}\phi_{xxx},\\
&-4\phi_x\sigma_{\phi,xx}\phi_{xxx}-4\phi_x\phi_{xx}\sigma_{\phi,xxx}\\
&+2\phi_x\phi_{tt}\sigma_{\phi,x}+\phi_x^2\sigma_{\phi,tt}=0.
\end{aligned} \tag{14f}$$

It can be easily verified that the solution of Eq. (14) has the form of

$$\sigma_u=-h, \tag{15a}$$

$$\sigma_g = \phi g, \tag{15b}$$
$$\sigma_h = g^2 + \phi h, \tag{15c}$$
$$\sigma_\phi = \frac{\phi^2}{2}, \tag{15d}$$
$$\sigma_m = \phi m, \tag{15e}$$

if $d_3 = 1/2$ and $d_1 = d_2 = 0$ are fixed for σ_ϕ.

The result (15) indicates that the residual symmetry (10) is localized in the properly prolonged systems (1), (7), (11), (12), and (13) with the Lie point symmetry vector

$$V = -h\partial_u + g\phi\partial_g + (g^2 + h\phi)\partial_h + m\phi\partial_m + \frac{\phi^2}{2}\partial_\phi. \tag{16}$$

In other words, symmetry related to the truncated Painlevé expansion is just a special Lie point symmetry of the prolonged system.

Now, let us study the finite transformation group corresponding to the Lie point symmetry (16), which can be stated in the following theorem.

Theorem 1 If $\{u, g, h, m, \phi\}$ is a solution of the prolonged system (1), (7), (11), (12), and (13), then so is $\{\hat{u}, \hat{g}, \hat{h}, \hat{m}, \hat{\phi}\}$ with

$$\hat{u} = u + \frac{2h\varepsilon}{\varepsilon\phi - 2} - \frac{2\varepsilon^2 g^2}{(\varepsilon\phi - 2)^2}, \tag{17a}$$
$$\hat{g} = \frac{4g}{(\varepsilon\phi - 2)^2}, \tag{17b}$$
$$\hat{h} = \frac{4h}{(\varepsilon\phi - 2)^2} - \frac{8\varepsilon g^2}{(\varepsilon\phi - 2)^3}, \tag{17c}$$
$$\hat{m} = \frac{4m}{(\varepsilon\phi - 2)^2}, \tag{17d}$$
$$\hat{\phi} = -\frac{2\phi}{\varepsilon\phi - 2}, \tag{17e}$$

with arbitrary group parameter ε.

Proof Using Lie's first theorem on vector (16) with the following corresponding initial condition:

$$\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon} = -\hat{h}(\varepsilon), \quad \hat{u}(0) = u, \tag{18}$$
$$\frac{\mathrm{d}\hat{g}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{\phi}(\varepsilon)\hat{g}(\varepsilon), \quad \hat{g}(0) = g, \tag{19}$$
$$\frac{\mathrm{d}\hat{h}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{g}(\varepsilon)^2 + \hat{\phi}(\varepsilon)\hat{h}(\varepsilon), \quad \hat{h}(0) = h, \tag{20}$$
$$\frac{\mathrm{d}\hat{\phi}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{\hat{\phi}(\varepsilon)^2}{2}, \quad \hat{\phi}(0) = \phi, \tag{21}$$
$$\frac{\mathrm{d}\hat{m}(\varepsilon)}{\mathrm{d}\varepsilon} = \hat{\phi}(\varepsilon)\hat{m}(\varepsilon), \quad \hat{m}(0) = m, \tag{22}$$

we can obtain the solutions of the above equations stated in Theorem 1. Thus, the theorem is proved.

3. New symmetry reductions of the Boussinesq equation

Because the nonlocal symmetries cannot be used directly to construct exact solutions of differential equations, we seek the Lie point symmetry of the prolonged Boussinesq system in the general form of

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + G\frac{\partial}{\partial g} + H\frac{\partial}{\partial h} + M\frac{\partial}{\partial m} + \Phi\frac{\partial}{\partial \phi}, \tag{23}$$

which means that the prolonged system (1), (7), (11), (12), and (13) is invariant under the following transformation:

$$\{x, t, u, g, h, m, \phi\} \to \{x + \varepsilon X, t + \varepsilon T, u + \varepsilon U, g + \varepsilon G, h + \varepsilon H, m + \varepsilon M, \phi + \varepsilon\Phi\}, \tag{24}$$

with ε being the infinitesimal parameter. Equivalently, the symmetry in the form (23) can be written as

$$\sigma_u = Xu_x + Tu_t - U, \tag{25a}$$
$$\sigma_g = Xg_x + Tg_t - G, \tag{25b}$$
$$\sigma_h = Xh_x + Th_t - H, \tag{25c}$$
$$\sigma_m = Xm_x + Tm_t - M, \tag{25d}$$
$$\sigma_\phi = X\phi_x + T\phi_t - \Phi. \tag{25e}$$

Substituting Eq. (25) into Eq. (14) and eliminating u_t, g_x, g_t, h_x, h_t, m_t, ϕ_x, and ϕ_t in terms of the prolonged system, we obtain more than 200 determining equations for the functions X, T, U, G, H, M, and Φ. Calculated by computer algebra, we finally obtain the desired result

$$X = \frac{c_1}{2}x + c_3, \quad T = c_1 t + c_2, \quad U = -c_1 u + c_4 h,$$
$$G = -c_4 g\phi + \frac{1}{2}(2c_5 - c_1)g,$$
$$H = -c_4 h\phi + (c_5 - c_1)h - c_4 g^2,$$
$$M = -c_4 m\phi + (c_5 - c_1)m, \quad \Phi = -\frac{c_4}{2}\phi^2 + c_5\phi + c_6, \tag{26}$$

with arbitrary constants c_1, c_2, c_3, c_4, c_5, c_6. When $c_1 = c_2 = c_3 = c_5 = c_6 = 0$, the obtained symmetry that degenerates into the special form (Eq. (16)) includes residual symmetry of the Boussinesq equation. By setting $c_2 = c_3 = c_4 = c_5 = c_6 = 0$, we obtain the scaling transformation symmetry as

$$\sigma_u = \frac{1}{2}c_1 x u_x + c_1 t u_t + c_1 u,$$
$$\sigma_g = \frac{1}{2}c_1 x g_x + c_1 t g_t + \frac{c_1}{2}g,$$
$$\sigma_h = \frac{1}{2}c_1 x h_x + c_1 t h_t + c_1 h,$$
$$\sigma_m = \frac{1}{2}c_1 x m_x + c_1 t m_t + c_1 m,$$
$$\sigma_\phi = \frac{1}{2}c_1 x\phi_x + c_1 t\phi_t + c_1\phi. \tag{27}$$

Other well-known symmetries, like x and t translation invariance symmetries, are also included in the symmetry group.

Consequently, the symmetries in Eq. (25) can be written explicitly as

$$\sigma_u = \frac{c_1}{2}xu_x + (c_1 t + c_2)u_t + c_1 u - c_4 h,$$

$$\begin{aligned}
\sigma_g &= \frac{c_1}{2}xg_x+(c_1t+c_2)g_t+c_4\phi g+\frac{1}{2}(c_1-2c_5)g,\\
\sigma_h &= \frac{c_1}{2}xh_x+(c_1t+c_2)h_t+c_4h\phi-(c_5-c_1)h+c_4g^2,\\
\sigma_m &= \frac{c_1}{2}xm_x+(c_1t+c_2)m_t+c_4m\phi-(c_5-c_1)m,\\
\sigma_\phi &= \frac{c_1}{2}x\phi_x+(c_1t+c_2)\phi_t+\frac{c_4}{2}\phi^2-c_5\phi. \qquad (28)
\end{aligned}$$

To derive the group invariant solutions of the enlarged system, we have to solve Eq. (28) under symmetry constraints $\sigma_u=\sigma_g=\sigma_h=\sigma_m=\sigma_\phi=0$, which is equivalent to solving the corresponding characteristic equation

$$\begin{aligned}
\frac{\mathrm{d}x}{c_1x/2+c_3} &= \frac{\mathrm{d}t}{c_1t+c_2}=\frac{\mathrm{d}u}{-c_1u+c_4h}\\
&= \frac{\mathrm{d}g}{-c_4g\phi+(2c_5-c_1)g/2}\\
&= \frac{\mathrm{d}h}{-c_4h\phi+(c_5-c_1)h-c_4g^2}\\
&= \frac{\mathrm{d}m}{-c_4m\phi+(c_5-c_1)m}\\
&= \frac{\mathrm{d}\phi}{-c_4\phi^2/2+c_5\phi+c_6}. \qquad (29)
\end{aligned}$$

Without loss of generality, we consider the symmetry reductions which can be categorized as the following four subcases.

Case 1 $c_i\neq 0$ $(i=1,2,3,4,5,6)$.

Firstly, we denote $\delta=\sqrt{2c_6c_4+c_5^2}$. Solving Eq. (29) leads to

$$\begin{aligned}
\phi &= \frac{c_5+\delta\tanh[\delta(\ln(\Delta)+\Phi c_1)/c_1]}{c_4},\\
\Delta &= \frac{\sqrt{c_1\xi(c_1t+c_2)}}{c_1\xi}, \qquad (30)\\
g &= \frac{2G\exp(-\Phi c_1)}{\Delta[\cosh(2\delta(\ln(\Delta)+\Phi c_1)/c_1)+1]}, \qquad (31)
\end{aligned}$$

$$h = \frac{\exp(4\delta\Phi)\Delta^{-2(c_1-\delta)/c_1}H}{[\Delta^{2\delta/c_1}\exp(2\delta\Phi)+1]^2}+\frac{16\exp[-2\Phi(c_1-\delta)]c_4G^2[\Delta^{2\delta}/c_1\exp(2\delta\Phi)+2]\Delta^{(2\delta/c_1-2)}}{\delta(\Delta^{2\delta/c_1}\exp(2\delta\Phi)+1)^3}, \qquad (32)$$

$$\begin{aligned}
m &= \frac{2M\exp(-2\Phi c_1)}{\Delta^2[\cosh(2\delta(\ln(\Delta)+\Phi c_1)/c_1)+1]}, \qquad (33)\\
u &= \frac{U}{\Delta^2}-\frac{c_4\exp(2\delta\Phi)H}{\delta\Delta^2[\Delta^{2\delta/c_1}\exp(2\delta\Phi)+1]}\\
&\quad -\frac{8c_4^2G^2[2\Delta^{2\delta/c_1}\exp(2\delta\Phi)+3]\exp(-2\Phi c_1)}{\delta^2\Delta^2[\Delta^{2\delta/c_1}\exp(2\delta\Phi)+1]^2}. \qquad (34)
\end{aligned}$$

Here, Φ, G, H, M, and U represent five group invariant functions with group invariant variable $\xi=(c_1t+c_2)/[c_1(c_1^2x^2+4c_1xc_3+4c_3^2)]$.

In order to obtain the corresponding reduction equations, we first substitute Eqs. (30)–(33), into Eqs. (11)–(13)

$$\delta^2\Phi_\xi-M\exp(-2\Phi c_1)c_4=0, \qquad (35)$$

$$-2c_1\xi\delta^2\Phi_\xi+\delta^2-c_4\exp(-c_1\Phi)G=0, \qquad (36)$$

and

$$\begin{aligned}
&96c_1^2\xi^2\delta^3\Phi_\xi^2-24c_1\xi\delta^2(4\delta+c_1)\Phi_\xi+\exp(2\delta\Phi)Hc_4\\
&+4\delta^2c_1+24\delta^3-16c_1^2\delta^2\xi^2\Phi_{\xi\xi}=0. \qquad (37)
\end{aligned}$$

Then, by substituting Eqs. (30) and (34) into Eqs. (7) and (6) with u_2 replaced by u, meanwhile substituting M, G, and H by Eqs. (35)–(37), we obtain the reduction equations for U and Φ

$$\begin{aligned}
&-64\delta^2c_1^4\xi^4\Phi_\xi^4+32c_1^3\delta\xi^3(4\delta+9c_1)\Phi_\xi^3\\
&+\big[-1+192\delta c_1^4\xi^4\Phi_{\xi\xi}\\
&-12c_1^2(7c_1^2+28c_1\delta+8\delta^2+2U)\xi^2\big]\Phi_\xi^2\\
&+\big[-48c_1^3\xi^3(3c_1+4\delta)\Phi_{\xi\xi}-64c_1^4\xi^4\Phi_{\xi\xi\xi}\\
&+4c_1\xi(19c_1^2+6U+8\delta^2+30c_1\delta)\big]\Phi_\xi\\
&+48c_1^4\xi^4\Phi_{\xi\xi}^2+24c_1^2\xi^2(2\delta+5c_1)\Phi_{\xi\xi}-4\delta^2\\
&-5c_1^2-12c_1\delta+32c_1^3\xi^3\Phi_{\xi\xi\xi}-6U=0, \qquad (38)\\
&-384c_1^6\xi^5\delta^2\Phi_\xi^5-64c_1^5\xi^4\delta^2(4c_1\Phi_{\xi\xi}\xi^2-13)\Phi_\xi^4\\
&+\big[512c_1^5\xi^5\delta^2\Phi_{\xi\xi}-2c_1^2\xi(352\xi^2c_1^2\delta^2\\
&+12c_1^4\xi^2+3)\big]\Phi_\xi^3+\big[-384c_1^4\xi^4\delta^2\Phi_{\xi\xi}+192c_1^6\xi^5\Phi_{\xi\xi\xi}\\
&+c_1(1+64c_1^5\xi^6\Phi_{\xi\xi\xi\xi}+288\xi^2c_1^2\delta^2+36c_1^4\xi^2)\big]\Phi_\xi^2\\
&+\big[-288c_1^6\xi^5\Phi_{\xi\xi}^2-4c_1\xi(64c_1^5\xi^5\Phi_{\xi\xi\xi}-32\xi^2c_1^2\delta^2\\
&+8c_1^4\xi^2+1)\Phi_{\xi\xi}-320c_1^5\xi^4\Phi_{\xi\xi\xi}-2c_1^2\xi(32c_1^3\Phi_{\xi\xi\xi\xi}\xi^4\\
&-19c_1^2+28\delta^2)\big]\Phi_\xi+192c_1^6\Phi_{\xi\xi}^3\xi^6+432c_1^5\Phi_{\xi\xi}^2\xi^4\\
&+\big[1+128c_1^5\xi^5\Phi_{\xi\xi\xi}+16c_1^2(10c_1^2-\delta^2)\xi^2\big]\Phi_{\xi\xi}\\
&+112c_1^4\Phi_{\xi\xi\xi}\xi^3+c_1(-c_1^2+4\delta^2+16c_1^3\Phi_{\xi\xi\xi\xi}\xi^4)=0. \qquad (39)
\end{aligned}$$

We can see that once Φ is solved from Eq. (39), new group invariant solutions u of Eq. (1) would be immediately obtained through Eqs. (34), (36)–(38). The entrance of exp part in Eq. (34) indicates that the interaction solution between nonlinear waves and solitons can thus be obtained. This type of interaction solutions are hard to be derived by other approaches.

To reach a concrete solution for the Boussinesq equation, we derive a simple solution for Eq. (39)

$$\Phi=\frac{1}{2c_1}\ln\left(\frac{\sqrt{3}+6c_1^2\xi}{6c_1^2}\right) \qquad (40)$$

with the constraint condition $c_1^2=2c_6c_4+c_5^2$. Then, by solving H, G, and U through Eqs. (36)–(38), and substituting them into Eq. (34), we finally obtain

$$u = \frac{36c_1^3\xi}{[6c_1^2(c_1+c_1t+c_2)\xi+\sqrt{3}(c_1t+c_2)]^2(1+2\sqrt{3}c_1^2\xi)^2(\sqrt{3}+6c_1^2\xi)^2} \times\big[288\sqrt{3}(c_1+c_1t+c_2)c_1^{10}\xi^5+144c_1^8(3c_2+4c_1+3c_1t)\xi^4+48\sqrt{3}(3c_1+c_1t+c_2)c_1^6\xi^3 -24c_1^4(c_1t-2c_1+c_2)\xi^2-2\sqrt{3}(-c_1+3c_1t+3c_2)c_1^2\xi-c_1t-c_2\big], \quad (41)$$

which can be easily verified by substituting Eq. (41) into Eq. (1).

Case 2 $c_i \neq 0$ $(i=1,2,4,5)$ and $c_3=c_6=0$.

In this case, by solving Eq. (29), we obtain

$$\phi = \frac{2c_5}{2c_5\Delta^{-\frac{2c_5}{c_1}}\Phi+c_4}, \quad (42)$$

$$g = \frac{\Delta^{-(\frac{2c_5}{c_1}+1)}G}{\left[2c_5\Delta^{-\frac{2c_5}{c_1}}\Phi+c_4\right]^2}, \quad (43)$$

$$h = \frac{4c_5^2\Delta^{(\frac{2c_5}{c_1}-2)}\Phi^2H}{\left[c_4\Delta^{\frac{2c_5}{c_1}}+2c_5\Phi\right]^2} - \frac{c_4\Delta^{(\frac{4c_5}{c_1}-2)}G^2}{2c_5^2\left[c_4\Delta^{\frac{2c_5}{c_1}}+2c_5\Phi\right]^3\Phi}, \quad (44)$$

$$m = \frac{\Delta^{(-\frac{2c_5}{c_1}-2)}M}{\left[2c_5\Delta^{-\frac{2c_5}{c_1}}\Phi+c_4\right]^2}, \quad (45)$$

$$u = \frac{U}{\Delta^2} + \frac{2c_4\Delta^{(\frac{2c_5}{c_1}-2)}\Phi H}{\left[c_4\Delta^{\frac{2c_5}{c_1}}+2c_5\Phi\right]} - \frac{c_4^2\Delta^{(\frac{4c_5}{c_1}-2)}G^2}{8c_5^4\left[c_4\Delta^{\frac{2c_5}{c_1}}+2c_5\Phi\right]^2\Phi^2}, \quad (46)$$

where the group invariant variable is $\xi=(c_1t+c_2)/c_1x^2$, while U, G, H, M, and Φ are all group invariant functions of ξ.

To derive the symmetry reduction equations for the group invariant functions U, G, H, M, and Φ, we first substitute Eqs. (42)–(45) into Eqs. (11–(13), then we have

$$4c_5^2\Phi_\xi+M=0, \quad (47)$$

$$8c_5^3\Phi+8c_5^2c_1\xi\Phi_\xi-c_1G=0, \quad (48)$$

and

$$-8c_1^2\Phi_\xi^2\xi^2+2c_1(3c_1-4c_5)\xi\Phi\Phi_\xi+4c_1^2\xi^2\Phi\Phi_{\xi\xi} +(2c_1c_5-4c_5^2+c_1^2H\Phi)\Phi^2=0. \quad (49)$$

Next, we substitute Eqs. (42) and (46) into Eqs. (7) and (6) with u_2 substituted by u. Replacing M, G, H by Eqs. (47)–(49) at the same time, we finally derive the symmetry reduction equations for U and Φ as

$$192c_1^4\xi^4\Phi_\xi^4-96c_1^3(3c_1-4c_5)\xi^3\Phi\Phi_\xi^3 +\big[-192c_1^4\xi^4\Phi\Phi_{\xi\xi}+12c_1^2(16c_5^2+7c_1^2+2c_1^2U -48c_1c_5)\xi^2\Phi^2+c_1^4\Phi^2\big]\Phi_\xi^2+\big[48c_1^3(-8c_5+3c_1)\xi^3\Phi^2\Phi_{\xi\xi} +64c_1^4\xi^4\Phi^2\Phi_{\xi\xi\xi}+8c_1c_5(19c_1^2+8c_5^2 -30c_1c_5+6c_1^2U)\xi\Phi^3\big]\Phi_\xi-48c_1^4\xi^4\Phi^2\Phi_{\xi\xi}^2 +48c_1^2c_5(-2c_5+5c_1)\xi^2\Phi^3\Phi_{\xi\xi}+64c_1^3c_5\xi^3\Phi^3\Phi_{\xi\xi\xi} +4c_5^2(4c_5^2-12c_1c_5+5c_1^2+6c_1^2U)\Phi^4=0, \quad (50)$$

and

$$\big[4c_1^2(4c_5+c_1)(-8c_5^2-4c_1c_5+3c_1^2)\xi^3 +c_1^4(3c_1+4c_5)\xi\big]\Phi_\xi^3+\big[96c_1^3c_5(-2c_5+c_1)\xi^4\Phi_{\xi\xi} -32c_1^4(3c_1-4c_5)\xi^5\Phi_{\xi\xi\xi}-32c_1^5\xi^6\Phi_{\xi\xi\xi\xi}+4c_1c_5(9c_1^3-8c_5^3 +72c_5^2c_1+80c_1^2c_5)\xi^2\Phi+c_1^3c_5(c_1+2c_5)\Phi\big]\Phi_\xi^2 +\big[48(3c_1-4c_5)c_1^4\xi^5\Phi_{\xi\xi}^2+(128c_1^5\xi^6\Phi_{\xi\xi\xi} -32c_5c_1^2(c_1^2-21c_1c_5-4c_5^2)\xi^3\Phi-4c_1^4c_5\xi\Phi)\Phi_{\xi\xi} -64c_1^3c_5(-2c_5+5c_1)\xi^4\Phi\Phi_{\xi\xi\xi}-64c_1^4c_5\xi^5\Phi\Phi_{\xi\xi\xi\xi} -4c_5^2c_1(19c_1^2-28c_5^2)\xi\Phi^2\big]\Phi_\xi-96c_1^5\xi^6\Phi_{\xi\xi}^3 +48c_1^3c_5(2c_5+9c_1)\xi^4\Phi\Phi_{\xi\xi}^2+\big[128c_1^4c_5\xi^5\Phi\Phi_{\xi\xi\xi} -32c_5^2c_1(10c_1^2-c_5^2)\xi^2\Phi^2-2c_1^3c_5^2\Phi^2\big]\Phi_{\xi\xi} -224c_1^3c_5^2\xi^3\Phi^2\Phi_{\xi\xi\xi}-32c_1^3c_5^2\xi^4\Phi^2\Phi_{\xi\xi\xi\xi} -4c_5^3(-2c_5+c_1)(c_1+2c_5)\Phi^3=0. \quad (51)$$

Equation (51) is a differential equation solely for Φ with variant coefficients. One can see that when Φ is solved from Eq. (51), then U, G, and H can be solved directly from Eqs. (48)–(50), respectively. The explicit solutions for the Boussinesq equation (1) are immediately obtained by substituting U, H, G, and Φ into Eq. (46).

Case 3 $c_i \neq 0$ $(i=1,2,4)$ and $c_3=c_5=c_6=0$.

Similar to Case 1, solving the characteristic equation with the assumed condition, we obtain the following symmetry reduction solutions for the enlarged Boussinesq system:

$$\phi = \frac{c_1}{c_4\ln(\Delta)+c_1\Phi}, \quad (52)$$

$$g = \frac{G}{\Delta(c_4\ln(\Delta)+c_1\Phi)^2}, \quad (53)$$

$$h = \frac{2G^2}{c_1\Delta^2(c_4\ln(\Delta)+c_1\Phi)^3} + \frac{H}{\Delta^2(c_4\ln(\Delta)+c_1\Phi)^2}, \quad (54)$$

$$m = \frac{M}{\Delta^2(c_4\ln(\Delta)+c_1\Phi)^2}, \quad (55)$$

$$u = -\frac{2G^2}{c_1^2\Delta^2(c_4\ln(\Delta)+c_1\Phi)^2}$$

$$-\frac{2H}{c_1\Delta^2(c_4\ln(\Delta)+c_1\Phi)}+\frac{U}{\Delta^2}, \tag{56}$$

with the group invariant variable ξ unchanged as in Case 1. Seeking the symmetry reduction equations for symmetry group invariant functions in this case, we substitute Eqs. (52)–(55) into Eqs. (11)–(13) and find

$$-c_1^2\Phi_\xi - M = 0, \tag{57}$$
$$-c_4c_1 + 2c_1^2\xi\Phi_\xi - G = 0, \tag{58}$$
$$2\xi G_\xi + G + H = 0. \tag{59}$$

Substituting Eq. (52) into Eq. (7), we have

$$\begin{aligned}&(24c_1^3\xi^3+6c_1^3\xi)\Phi_\xi^3-(36c_1^2c_4\xi^2+64c_1^3\xi^6\Phi_{\xi\xi\xi\xi}\\&+c_4c_1^2+192c_1^3\xi^5\Phi_{\xi\xi\xi})\Phi_\xi^2+\big[288c_1^3\xi^5\Phi_{\xi\xi}^2\\&+(32c_1^2c_4\xi^3+256c_1^3\xi^6\Phi_{\xi\xi\xi}+4c_1^2c_4\xi)\Phi_{\xi\xi}\\&+64c_1^2c_4\xi^5\Phi_{\xi\xi\xi\xi}+320c_1^2c_4\xi^4\Phi_{\xi\xi\xi}-38c_1c_4^2\xi\big]\Phi_\xi\\&-192c_1^3\xi^6\Phi_{\xi\xi}^3-432c_1^2c_4\xi^4\Phi_{\xi\xi}^2\\&-(128c_1^2c_4\xi^5\Phi_{\xi\xi\xi}+160c_1c_4^2\xi^2+c_4^2c_1)\Phi_{\xi\xi}\\&-16c_1c_4^2\xi^4\Phi_{\xi\xi\xi\xi}-112c_1c_4^2\xi^3\Phi_{\xi\xi\xi}+c_4^3=0.\end{aligned} \tag{60}$$

Now, substituting Eqs. (52) and (56) into Eq. (6) with u_2 replaced by u and vanishing G, H, M by Eqs. (57)–(59), we have

$$\begin{aligned}&\big[12c_1^2(2U+7)\xi^2+c_1^2\big]\Phi_\xi^2+\big[144c_1^2\xi^3\Phi_{\xi\xi}\\&+64c_1^2\xi^4\Phi_{\xi\xi\xi}-4c_4c_1\xi(19+6U)\big]\Phi_\xi\\&-48c_1^2\xi^4\Phi_{\xi\xi}^2-120c_4c_1\xi^2\Phi_{\xi\xi}\\&-32c_4c_1\xi^3\Phi_{\xi\xi\xi}+c_4^2(6U+5)=0.\end{aligned} \tag{61}$$

From the symmetry reduction equations (57)–(61), we obtain the exact solutions of Boussinesq equation by substituting U, G, H, and Φ into Eq. (56).

It is seen that the symmetry reduction equation (60) of U with variant coefficient is still hard to solve. In order to give some explicit solutions of the Boussinesq equation, we make some further constraints. For instance, setting $c_4=0$ and taking integral constants as 0,we have

$$\Phi=\frac{16\sqrt{3}\Gamma(3/4)^2\text{Bessel K}(1/4,\sqrt{3}/8\xi)}{3\pi^2\text{Bessel I}(1/4,\sqrt{3}/8\xi)}, \tag{62}$$

where Bessel K and Bessel I are the corresponding Bessel functions. Then we finally obtain the special solution as

$$u=\frac{c_1\big[-9\text{Bessel K}^2(3/4,\sqrt{3}/8\xi)+5\text{Bessel K}^2(1/4,\sqrt{3}/8\xi)\big]}{24\text{Bessel K}^2(1/4,\sqrt{3}/8\xi)(c_1t+c_2)\xi}. \tag{63}$$

Case 4 $c_i\neq 0$ $(i=2,3,4)$ and $c_1=c_5=c_6=0$.

Execute the similar procedure as in Cases 1 and 2 under this condition, we can find the third group invariant solutions as

$$\phi = \frac{2c_2}{2c_2\Phi+c_4(t-\xi)}, \tag{64}$$
$$g = \frac{c_2^2G}{c_3^2(2c_2\Phi+c_4(t-\xi))^2}, \tag{65}$$
$$h = \frac{c_2^2H}{c_3^2(2c_2\Phi+c_4(t-\xi))^2}+\frac{c_2^3G^2}{c_3^4(2c_2\Phi+c_4(t-\xi))^3}, \tag{66}$$
$$m = \frac{c_2^2M}{c_3^2(2c_2\Phi+c_4(t-\xi))^2}, \tag{67}$$
$$u = -\frac{c_2^2H}{c_3^2(2c_2\Phi+c_4(t-\xi))}-\frac{c_2^2G^2}{2c_3^4(2c_2\Phi+c_4(t-\xi))^2}+U, \tag{68}$$

with group invariant variable $\xi=-(c_2x-c_3t)/c_3$. The related symmetry reduction equations are

$$-4c_3^2\Phi_\xi - M = 0, \tag{69}$$
$$4c_2^2\Phi_{\xi\xi}+H=0, \tag{70}$$
$$-2c_3c_4+4c_2c_3\Phi_\xi-G=0, \tag{71}$$
$$\begin{aligned}&2c_3^2(6c_2^2U+c_3^2)\Phi_\xi^2+(8c_2^4\Phi_{\xi\xi\xi}-12c_3^2c_4c_2U)\Phi_\xi\\&-6c_2^4\Phi_{\xi\xi}^2-4c_4c_2^3\Phi_{\xi\xi\xi}+3c_3^2c_4^2U=0,\end{aligned} \tag{72}$$
$$\begin{aligned}&-128c_2^6\Phi_\xi^2\Phi_{\xi\xi\xi\xi}+\big[(128c_3^4c_4c_2+512c_2^6\Phi_{\xi\xi\xi})\Phi_{\xi\xi}\\&+128c_4c_2^5\Phi_{\xi\xi\xi\xi}\big]\Phi_\xi-384c_2^6\Phi_{\xi\xi}^3\\&-(32c_3^4c_4^2+256c_4c_2^5\Phi_{\xi\xi\xi})\Phi_{\xi\xi}\\&-32c_4^2c_2^4\Phi_{\xi\xi\xi\xi}=0.\end{aligned} \tag{73}$$

Again, if we set $c_4=0$, then we have

$$\Phi=-\frac{2e_2}{e_1}\tanh\left(\frac{\xi+e_3}{2e_1}\right), \tag{74}$$

with e_1, e_2, and e_3 being integral constants, and

$$u=-\frac{(c_2^4+c_3^4e_1^2)\cosh^2(\xi+e_3/2e_1)+2c_2^4-c_3^4e_1^2}{6c_3^2c_2^2e_1^2\big[\cosh^2(\xi+e_3/2e_1)-1\big]} \tag{75}$$

as a special solution for the Boussinesq equation.

4. Conclusion and discussion

In summary, the Boussinesq equation is investigated from the perspective of residual symmetry, which is related to the truncated Painlevé expansion. In order to apply residual symmetry to construct exact solutions of the Boussinesq equation, we introduce multiple new variables such that the residual symmetry is localized in the new system. On the basis of the localized symmetry group, we have obtained Bäcklund transformation for the enlarged Boussinesq system by using

Lie's first theorem. After obtaining the general form of localized residual symmetry in the enlarged system, we consider the symmetry reduction in four subcases without loss of generality. In these four cases under consideration, we obtain new symmetry reduction solutions for the enlarged Boussinesq system as well as the corresponding symmetry reduction equations. Then the exact solutions for Boussinesq equation (1) can be constructed by solving these symmetry reduction equations. We also give some special explicit solutions under certain constraints.

It should be remarked that the BT related symmetry reduction method has obvious advantages as well as disadvantages. On the one hand, using the localization procedure, we can obtain new interaction solutions between nonlinear waves and solitons which are hard to get by other approaches. On the other hand, the complex reduction equations with variant coefficients are still hard to solve out, which needs to be further studied.

In this paper, we have shown that nonlocal symmetry can be used to construct new exact solutions for the Boussinesq equation, so it is meaningful to get as much as possible nonlocal symmetries. Besides Painlevé analysis to obtain nonlocal residual symmetry, there exist other ways to obtain nonlocal symmetries, such as those obtained from Bäcklund transformation, the bilinear forms and negative hierarchies, the nonlinearizations,[22,23] and self-consistent sources.[24] Applying the localization procedure to these various nonlocal symmetries to construct new exact solutions would deserve further study. Another critical question is what kind of nonlocal symmetries can be localized in a closed system, which also needs deep investigation to establish universal criterion for the localizing procedure.

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