Recursion-transform approach to compute the resistance of a resistor network with an arbitrary boundary*

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We consider a profound problem of two-point resistance in the resistor network with a null resistor edge and an arbitrary boundary, which has not been solved before because the Green's function technique and the Laplacian matrix approach are invalid in this case. Looking for the exact solutions of resistance is important but difficult in the case of the arbitrary boundary since the boundary is a wall or trap which affects the behavior of a finite network. In this paper, we give a general resistance formula that is composed of a single summation by using the recursion-transform method. Meanwhile, several interesting results are derived by the general formula. Further, the current distribution is given explicitly as a byproduct of the method.

Keywords: exact solution, resistor network, matrix equation, recursion-transform, boundary conditions

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1. Introduction

The computation of two-point resistances in networks is a classical problem in electric circuit theory and graph theory.[1–5] Nowadays, research on the resistor network is expanded into the basic model in the field of science and engineering.[6–20] A lot of abstract and complex scientific problems have been solved by modeling resistor networks[11–20] since the German scientist Kirchhoff (1824–1887) founded the node current law and the circuit voltage law in 1845. The previous research on resistor networks has focused on the infinite networks due to the poor technology,[3] and little attention has been paid to finite networks, even though the latter occur in real life. Looking for the exact solution of resistance between any two nodes in a resistor network in the finite case is important but difficult. The construction of, and research on, the models of resistor networks therefore are of significance in theory and application.

In 2000, Cserti[3] evaluated a two-point resistance of a resistor network by using the lattice Green's function which is confined to regular lattices of infinite size. In 2004, Wu[4] formulated a different approach and derived an expression for the two-point resistance in arbitrary finite and infinite lattices with a normative boundary (such as free, periodic boundary, etc.) in terms of the eigenvalues and eigenvectors of the Laplacian matrix. Laplacian analysis has also been extended to impedance networks after a slight modification of the formulation in Ref. [5], and recently it has been applied to a cobweb network.[8] However, the Laplacian approach cannot be applied to the matrix with arbitrary elements. The actual computation depends crucially on the geometry of the network and, for nonregular lattices such as a resistor network with arbitrary boundaries, it is incapable of solving the eigenvalue problem. However, the boundary condition is important because it is a real case occurring in real life. The boundary is a wall or trap which affects the behavior of a finite network.

Recently we[9,10] proposed a new method to calculate the two-point resistance in a network. We used this method to compute the equivalent resistance, which relies on just one matrix along one direction, and expressed the exact resistance by single summation. Using the description in Refs. [9] and [10] we refer to this method as the recursion-transform approach (or the RT method). When just one boundary or a pair of opposite boundaries are arbitrary, the RT method is still valid for the computation of resistance. The RT method has been used to solve many problems of the resistor network. For example, in 2013 a cobweb model and a conjecture were proposed by us,[9,10] soon afterwards a globe network was solved.[11] Almost at the same time, the cobweb conjecture was solved in Ref. [12]. Very recently a cobweb network with $2r$ boundary was solved.[13]

We consider an $m \times n$ resistor network with resistances r and r_0 in the respective horizontal and vertical directions, except for the arbitrary right edge and the bottom boundary, as shown in Fig. 1. This paper focuses on the computation of the two-point resistance between any two nodes in the resistor network with an arbitrary boundary, which has not been solved before because the Green's function technique and the Laplacian matrix approach are invalid in this case.

The rest of this paper is organized as follows. In Section 2, we present several exact formulae of the resistance between any two nodes in a resistor network with an arbitrary boundary. In Section 3, we prove the equivalent resistance by the recursion-transform method, and test the formula by an ex-

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ample. In Section 4 a summary and discussion of the method and results are presented.

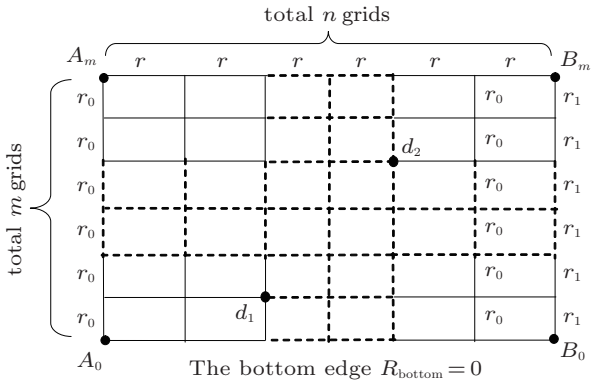


Fig. 1. An $m\times n$ resistor network with arbitrary right boundary and null resistors at the bottom. The bonds in the horizontal and vertical directions are resistors r and r_0, except for the bottom and right boundaries.

2. Main results

In order to facilitate the computation of the equivalent resistance of a resistor network, we define variables $\alpha_x^{(i)}$, $F_k^{(i)}$, and $S_{k,i}$ by

$$\alpha_x^{(i)}=\Delta F_x^{(i)}+(h_1-1)\Delta F_{x-1}^{(i)},\ \ \beta_{s,x}^{(i)}=\Delta F_s^{(i)}\alpha_{n-x}^{(i)},$$
$$F_k^{(i)}=(\lambda_i^k-\bar\lambda_i^k)/(\lambda_i-\bar\lambda_i),\ \ \Delta F_k^{(i)}=F_{k+1}^{(i)}-F_k^{(i)}, \quad (1)$$
$$h_k=r_k/r_0,\ \ h=r/r_0,\ \ S_{k,i}=\sin(y_k\theta_i),$$
$$\theta_i=(2i-1)\pi/(2m+1),\ \ i=1,2\ldots,m, \quad (2)$$

where $\lambda_i\cdot\bar\lambda_i=1$, and λ_i, $\bar\lambda_i$ are, for later uses, expressed as

$$\lambda_i=1+h-h\cos\theta_i+\sqrt{(1+h-h\cos\theta_i)^2-1}. \quad (3)$$

We consider an $m\times n$ resistor network as shown in Fig. 1. Assume that A_0 is the origin of the rectangular coordinate system, then the left edge will act as a Y axis and the bottom edge as an X axis. Denoting nodes of the network by coordinate $\{x, y\}$, the equivalent resistance between any two nodes $d_1(x_1,y_1)$ and $d_2(x_2,y_2)$ in the $m\times n$ resistor network can be written as

$$R_{m\times n}(d_1,d_2)=\frac{2r_0}{2m+1}\sum_{i=1}^{m}\frac{\beta_{x_1,x_1}^{(i)}S_{1,i}^2-2\beta_{x_1,x_2}^{(i)}S_{1,i}S_{2,i}+\beta_{x_2,x_2}^{(i)}S_{2,i}^2}{(1-\cos\theta_i)[F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}]}. \quad (4)$$

In particular, we have the following special cases.

Cases 2.1 When $n\to\infty$, $x_1,x_2\to\infty$ with x_1-x_2 being finite, we have

$$\frac{R_{m\times\infty}(d_1,d_2)}{r}=\frac{2}{2m+1}\sum_{i=1}^{m}\frac{S_{1,i}^2+S_{2,i}^2-2\bar\lambda_i^{x_2-x_1}S_{1,i}S_{2,i}}{\sqrt{(1+h-h\cos\theta_t)^2-1}}. \quad (5)$$

Case 2.2 When $m,n\to\infty$, but y_1,y_2, and x_1-x_2 are finite, we have

$$\frac{R_{\infty\times\infty}(d_1,d_2)}{r}=\frac{1}{\pi}\int_0^\pi\frac{S_1^2+S_2^2-2\bar\lambda_\theta^{|x_2-x_1|}S_1S_2}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \quad (6)$$

where $S_k=\sin(y_k\theta)$, and $\bar\lambda_\theta=1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}$.

Case 2.3 When $m,n\to\infty$ but x_1-x_2 and y_1-y_2 are finite, we have

$$\frac{R_{\infty\times\infty}(d_1,d_2)}{r}=\frac{1}{\pi}\int_0^\pi\frac{1-\bar\lambda_\theta^{|x_2-x_1|}\cos(y_1-y_2)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta. \quad (7)$$

Case 2.4 When $d_1=(x_1,0)$ is at bottom edge and $d_2=(x_2,m)$ is at top edge, we have

$$R_{m\times n}(d_1,d_2)=\frac{r_0}{2m+1}\sum_{i=1}^{m}\left(\frac{\Delta F_{x_2}^{(i)}\alpha_{n-x_2}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}\right)\cot^2\left(\frac{1}{2}\theta_i\right). \quad (8)$$

Case 2.5 When d_1 and d_2 are both at the left edge at $d_1(0,y_1)$ and $d_2(0,y_2)$, we have

$$R_{m\times n}(d_1,d_2)=\frac{2r_0}{2m+1}\sum_{i=1}^{m}\left(\frac{\Delta F_n^{(i)}+(h_1-1)\Delta F_{n-1}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}\right)\times\frac{(S_{1,i}-S_{2,i})^2}{1-\cos\theta_i}. \quad (9)$$

Case 2.6 When d_1 and d_2 are both on the left boundary, and $m,n\to\infty$, we have

$$R_{\infty\times\infty}(x_1=x_2=0)=\frac{r_0}{\pi}\int_0^\pi\frac{1-\cos(y_1-y_2)\theta}{1-\cos\theta}\left(1-\bar\lambda_\theta\right)\mathrm{d}\theta. \quad (10)$$

Case 2.7 When d_1=(0, y_1) is at the left edge and $d_2=(n,y_2)$ is at the right edge, we have

$$R_{m\times n}(d_1,d_2)=\frac{2r_0}{2m+1}\sum_{i=1}^{m}\frac{\alpha_n^{(i)}S_{1,i}^2-2h_1S_{1,i}S_{2,i}+h_1\Delta F_n^{(i)}S_{2,i}^2}{(1-\cos\theta_i)[F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}]}. \quad (11)$$

Case 2.8 When $h_1=0$, we have

$$R_{m\times n}(d_1,d_2)=\frac{4r}{2m+1}\sum_{i=1}^{m}\times\frac{\Delta F_{x_1}^{(i)}F_{n-x_1}^{(i)}S_{1,i}^2-2\Delta F_{x_1}^{(i)}F_{n-x_2}^{(i)}S_{1,i}S_{2,i}+\Delta F_{x_2}^{(i)}F_{n-x_2}^{(i)}S_{2,i}^2}{F_{n+1}^{(i)}-F_n^{(i)}}. \quad (12)$$

The general formula (4) and the above eight cases are given for the first time. From expression (4) we can deduce a lot of simple results because the r_1 is an arbitrary resistor and m, n are two arbitrary natural numbers.

3. Matrix equation and general solution

3.1. Overall train of thought and design

Assume that the electric current J is constant and goes from the input d_1 to the output d_2, as shown in Fig. 1. Denote the currents in all segments of the network as shown in Fig. 2. The resulting currents passing through all $m+1$ row horizontal resistors with the same resistance of r are $I_{x,k}^{(1)}, I_{x,k}^{(2)}, I_{x,k}^{(3)}, \ldots, I_{x,k}^{(n)}$ $(0 \le k \le m)$, respectively. The resulting currents passing through all $n+1$ row vertical resistors with the same resistance of r_0 are $I_k^{(1)}, I_k^{(2)}, \ldots, I_k^{(m)}$ $(0 \le k \le n)$, respectively.

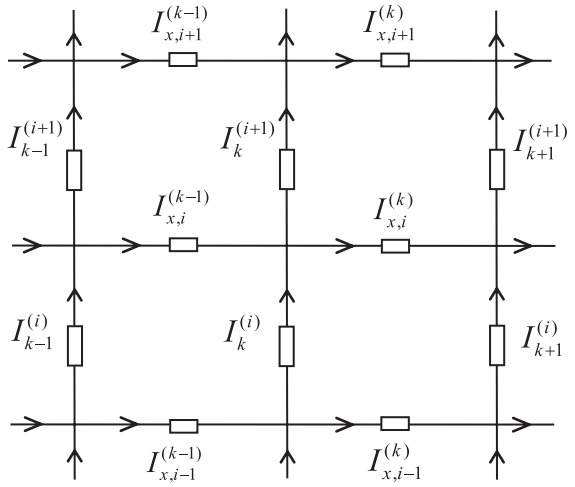


Fig. 2. Sub-network of resistors with marked parameters and directions of the currents.

We use Ohm's law to compute the resistance $R_{m\times n}(d_1,d_2)$. All of the nodes on the bottom boundary can be uniformly called O because the resistances of the bottom resistors are all zero. The voltages between two nodes O and d_1, and two nodes O and d_2 are, respectively,

$$U_{m\times n}(O,d_1) = r_0 \sum_{i=1}^{y_1} I_{x_1}^{(i)} \quad \text{and} \quad U_{m\times n}(O,d_2) = r_0 \sum_{i=1}^{y_2} I_{x_2}^{(i)}.$$

Using Ohm's law, we have

$$R_{m\times n}(d_1,d_2) = \frac{r_0}{I}\left(\sum_{i=1}^{y_2} I_{x_2}^{(i)} - \sum_{i=1}^{y_1} I_{x_1}^{(i)}\right). \tag{13}$$

How to solve the current parameters $I_{x_1}^{(i)}$ and $I_{x_2}^{(i)}$ is the key to the problem. We are going to solve the problem by the RT method.

3.2. Modeling the matrix equation

Using Kirchhoff's law (KCL and KVL) to analyze the resistor network, the node current equations and the mesh voltage equations can be achieved from Fig. 2. We focus on the four rectangular meshes and nine nodes, and give the relations in the absence of an injected current,

$$\begin{aligned} I_{k+1}^{(1)} &= (2+h)I_k^{(1)} - hI_k^{(2)} - I_{k-1}^{(1)}, \quad i=1, \\ I_{k+1}^{(i)} &= (2+2h)I_k^{(i)} - hI_k^{(i-1)} - hI_k^{(i+1)} - I_{k-1}^{(i)}, \quad 1<i<m, \\ I_{k+1}^{(m)} &= (2+2h)I_k^{(m)} - hI_k^{(m-1)} - I_{k-1}^{(m)}, \quad i=m, \end{aligned} \tag{14}$$

where $h = r/r_0$ is a ratio of two resistance. Equation (14) can be written in a matrix form after considering the input and output current

$$\boldsymbol{I}_{k+1} = \boldsymbol{A}_{m\times m}\boldsymbol{I}_k - \boldsymbol{I}_{k-1} - J\boldsymbol{H}_x\delta_{k,x}, \tag{15}$$

where $\boldsymbol{I}_k$ and $\boldsymbol{H}_x$ are, respectively, $m\times 1$ column matrix, and reads

$$\boldsymbol{I}_k = [I_k^{(1)}, I_k^{(2)}, I_k^{(3)}, \ldots, I_k^{(m)}]^{\mathrm{T}}, \tag{16}$$

$$(H_1)_i = h(-\delta_{i,y_1} + \delta_{i,y_1+1}),$$

$$\text{and} \quad (H_2)_i = h(\delta_{i,y_2} - \delta_{i,y_2+1}), \tag{17}$$

where $[\,]^{\mathrm{T}}$ denotes matrix transpose, and $(H_k)_i$ is the element of $\boldsymbol{H}_x$ with the injection of current J being at $d_1(x_1,y_1)$ and the exit of current J being at $d_2(x_2,y_2)$. And

$$\boldsymbol{A}_m = \begin{pmatrix} 2+h & -h & 0 & 0 & 0 \\ -h & 2(1+h) & -h & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -h & 2(1+h) & -h \\ 0 & 0 & 0 & -h & 2+2h \end{pmatrix}. \tag{18}$$

Equations (15)–(18) form the matrix equation model of the $m\times n$ resistor network.

3.3. General solution of the matrix equation

How to solve the matrix equation (15) is the key to the problem. In this section we consider the solution of Eq. (15) in the absence of an injected current, namely: $J=0$. Here we rebuild a new difference equation to solve Eq. (15) indirectly. To realize the idea, first we multiply Eq. (15) from the left-hand side by $m\times m$ undetermined square matrix $\boldsymbol{P}_m$. Thus we have

$$\boldsymbol{P}_m\boldsymbol{I}_{k+1} = \boldsymbol{P}_m\boldsymbol{A}_m\boldsymbol{I}_k - \boldsymbol{P}_m\boldsymbol{I}_{k-1} - J\boldsymbol{P}_m\boldsymbol{H}_x\delta_{k,x}. \tag{19}$$

We conduct the transformation of diagonal matrix by the following identity:

$$\boldsymbol{P}_m\boldsymbol{A}_m = \boldsymbol{T}_m\boldsymbol{P}_m, \tag{20}$$

where $\boldsymbol{T}_m = \mathrm{diag}(t_1,t_2,\ldots,t_m)$ is a diagonal matrix. Solving Eq. (20) we obtain

$$t_i = 2(1+h) - 2h\cos\theta_i, \quad (i=1,2,\ldots,m), \tag{21}$$

where $\theta_i = (2i-1)\pi/(2m+1)$, and

$$\boldsymbol{P}_m = \begin{pmatrix} \cos\left(1-\frac{1}{2}\right)\theta_1 & \cos\left(2-\frac{1}{2}\right)\theta_1 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_1 \\ \cos\left(1-\frac{1}{2}\right)\theta_2 & \cos\left(2-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(1-\frac{1}{2}\right)\theta_m & \cos\left(2-\frac{1}{2}\right)\theta_m & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_m \end{pmatrix}. \tag{22}$$

A simple calculation shows that, the new matrix $\boldsymbol{P}_m$ is invertible, with the following inverse matrix:

$$\boldsymbol{P}_m^{-1} = \frac{4}{2m+1}\begin{pmatrix} \cos\left(1-\frac{1}{2}\right)\theta_1 & \cos\left(1-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(1-\frac{1}{2}\right)\theta_m \\ \cos\left(2-\frac{1}{2}\right)\theta_1 & \cos\left(2-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(2-\frac{1}{2}\right)\theta_m \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(m-\frac{1}{2}\right)\theta_1 & \cos\left(m-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_m \end{pmatrix}. \tag{23}$$

By Eqs. (19) and (20) we define

$$\boldsymbol{P}_m\boldsymbol{I}_k = \boldsymbol{X}_k \ \text{ or } \ \boldsymbol{I}_k = \boldsymbol{P}_m^{-1}\boldsymbol{X}_k, \tag{24}$$

where $\boldsymbol{X}_m$ is an $m\times 1$ column matrix, and reads

$$\boldsymbol{X}_k = [X_k^{(1)}, X_k^{(2)}, X_k^{(3)}, \ldots, X_k^{(m)}]^{\mathrm{T}}. \tag{25}$$

Using Eqs. (19) and (24), we obtain the equation

$$X_{k+1}^{(i)} = t_i X_k^{(i)} - X_{k-1}^{(i)} - Jh\delta_{k,x}\zeta_{x,i}, \tag{26}$$

where (read $\zeta_{x_1,i}$ as $\zeta_{1,i}$, $\zeta_{x_2,i}$ as $\zeta_{2,i}$)

$$\begin{aligned}\zeta_{1,i} &= \cos\left(y_1+\frac{1}{2}\right)\theta_i - \cos\left(y_1-\frac{1}{2}\right)\theta_i \\ &= -2\sin\left(\frac{1}{2}\theta_i\right)\sin(y_1\theta_i),\end{aligned} \tag{27}$$

$$\begin{aligned}\zeta_{2,i} &= \cos\left(y_2-\frac{1}{2}\right)\theta_i - \cos\left(y_2+\frac{1}{2}\right)\theta_i \\ &= 2\sin\left(\frac{1}{2}\theta_i\right)\sin(y_2\theta_i).\end{aligned} \tag{28}$$

Equation (26) is much simpler than Eq. (15) because it is a linear equation that is easily derived. Suppose that λ_i, $\bar{\lambda}_i$ are the roots of the characteristic equation for X_k, from Eq. (26) we obtain expression (3).

Next, we consider the solution of Eq. (26) when the injection of current J is at $d_1(x_1,y_1)$ and the exit of current J is at $d_2(x_2,y_2)$. We therefore need to consider the piecewise solution of Eq. (26) and obtain

$$X_k^{(i)} = X_1^{(i)}F_k^{(i)} - X_0^{(i)}F_{k-1}^{(i)}, \ \ 0\le k\le x_1, \tag{29}$$

$$X_{x_1+1}^{(i)} = t_i X_{x_1}^{(i)} - X_{x_1-1}^{(i)} - hJ\zeta_{1,i}, \tag{30}$$

$$X_k^{(i)} = X_{x_1+1}^{(i)}F_{k-x_1}^{(i)} - X_{x_1}^{(i)}F_{k-x_1-1}^{(i)}, \ \ x_1\le k\le x_2, \tag{31}$$

$$X_{x_2+1}^{(i)} = t_i X_{x_2}^{(i)} - X_{x_2-1}^{(i)} - hJ\zeta_{2,i}, \tag{32}$$

$$X_k^{(i)} = X_{x_2+1}^{(i)}F_{k-x_2}^{(i)} - X_{x_2}^{(i)}F_{k-x_2-1}^{(i)}, \ \ x_2\le k\le n-1, \tag{33}$$

where $F_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k)/(\lambda_i - \bar{\lambda}_i)$ is defined in Eq. (1) .

3.4. General formula of the bound current

We consider the boundary conditions of the left and right edges in the network. By applying Kirchoff's laws (KCL and KVL) to a single mesh adjacent to each of the left and right boundaries, we obtain three matrix equations to model the boundary currents, that is,

$$\boldsymbol{I}_1 = (\boldsymbol{A}_m - \boldsymbol{E}_m)\boldsymbol{I}_0, \tag{34}$$

$$\boldsymbol{I}_{n-1} = [\boldsymbol{A}_m - (2-h_1)\boldsymbol{E}_m]\boldsymbol{I}_n,$$

$$h_1\boldsymbol{I}_n + \boldsymbol{I}_{n-2} = \boldsymbol{A}_m\boldsymbol{I}_{n-1}, \tag{35}$$

where $\boldsymbol{E}_m$ is an m-dimensional unit matrix, and $\boldsymbol{A}_m$ is given by Eq. (18).

Conducting the same matrix transformation as in Eq. (19), and applying $\boldsymbol{P}_m$ to Eqs. (34) and (35) on the left-hand sides, we have

$$X_1^{(i)} = (t_i-1)X_0^{(i)}, \tag{36}$$

$$X_{n-1}^{(i)} = (t_i+h_1-2)X_n^{(i)},$$

$$h_1X_n^{(i)} + X_{n-2}^{(i)} = t_iX_{n-1}^{(i)}. \tag{37}$$

The above equations and transformations are proposals for the computation of the equivalent resistance. Next we derive two solutions of $X_{x_1}^{(i)}$ and $X_{x_2}^{(i)}$ by the above equations.

Substituting Eq. (36) into Eq. (29) to eliminate $X_1^{(i)}$, we obtain

$$X_k^{(i)} = \Delta F_k^{(i)}X_0^{(i)}, \ \ 0\le k\le x_1, \tag{38}$$

where $\Delta F_k^{(i)}$ is defined in Eq. (1). By Eq. (31), together with Eqs. (38) and (30), we obtain

$$X_k^{(i)} = \Delta F_k^{(i)} X_0^{(i)} - hJ F_{k-x_1}^{(i)} \zeta_{1,i}, \quad x_1 \le k \le x_2, \tag{39}$$

where

$$F_{k-x_1+1}^{(i)} \Delta F_{x_1}^{(i)} - F_{k-x_1}^{(i)} \Delta F_{x_1-1}^{(i)} = \Delta F_k^{(i)}$$

is used. Using Eq. (37) to eliminate $X_n^{(i)}$ yields

$$(t_i b_i - h_1) X_{n-1}^{(i)} = b_i X_{n-2}^{(i)}, \tag{40}$$

where we define $b_i = t_i + (h_1 - 2)$.

To obtain the initial conditions $X_{x_1}^{(i)}$ and $X_{x_2}^{(i)}$ needed in our resistance calculation of Eq. (13), we also need several independent equations. Since $X_{x_2-1}^{(i)}$ and $X_{x_2}^{(i)}$ satisfies Eq. (39), and $X_{n-2}^{(i)}$ and $X_{n-1}^{(i)}$ satisfies Eq. (33), we therefore obtain four independent equations. Together with Eqs. (32) and (40) we have six independent equations relating to the six unknowns $X_0^{(i)}, X_{x_2-1}^{(i)}, X_{x_2}^{(i)}, X_{x_2+1}^{(i)}, X_{n-2}^{(i)}, X_{n-1}^{(i)}$, namely,

$$\begin{pmatrix} \Delta F_{x_2-1}^{(i)} & -1 & 0 & 0 & 0 & 0 \\ \Delta F_{x_2}^{(i)} & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & t_i & -1 & 0 & 0 \\ 0 & 0 & F_{n-x_2-3} & -F_{n-x_2-2} & 1 & 0 \\ 0 & 0 & F_{n-x_2-2} & -F_{n-x_2-1} & 0 & 1 \\ 0 & 0 & 0 & 0 & -b_i & t_i b_i - h_1 \end{pmatrix}$$

$$\times \begin{pmatrix} X_0^{(i)} \\ X_{x_2-1}^{(i)} \\ X_{x_2}^{(i)} \\ X_{x_2+1}^{(i)} \\ X_{n-2}^{(i)} \\ X_{n-1}^{(i)} \end{pmatrix} = hJ \begin{pmatrix} F_{x_2-x_1-1}^{(i)} \zeta_{1,i} \\ F_{x_2-x_1}^{(i)} \zeta_{1,i} \\ \zeta_{2,i} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{41}$$

where $t_i = 2(1+h) - 2h\cos\theta_i$ is given by Eq. (21), and $b_i = t_i + (h_1 - 2)$ appearing in Eq. (40). Solving Eq. (41), we obtain after some algebra and reduction the solution of $X_0^{(i)}$ and $X_{x_2}^{(i)}$ that is needed in our process of resistance calculation,

$$X_0^{(i)} = \frac{\alpha_{n-x_1}^{(i)} \zeta_{1,i} + \alpha_{n-x_2}^{(i)} \zeta_{2,i}}{(t_i - 2)[F_{n+1}^{(i)} + (h_1 - 1) F_n^{(i)}]} hJ, \tag{42}$$

where $\alpha_x^{(i)} = \Delta F_x^{(i)} + (h_1 - 1)\Delta F_{x-1}^{(i)}$ is defined in Eq. (1) and

$$X_{x_2}^{(i)} = \frac{\alpha_{n-x_2}^{(i)} (\Delta F_{x_1}^{(i)} \zeta_{1,i} + \Delta F_{x_2}^{(i)} \zeta_{2,i})}{(t_i - 2)[F_{n+1}^{(i)} + (h_1 - 1) F_n^{(i)}]} hJ. \tag{43}$$

Substituting Eq. (42) into Eq. (38) with $k = x_1$ yields

$$X_{x_1}^{(i)} = \frac{\Delta F_{x_1}^{(i)} (\alpha_{n-x_1}^{(i)} \zeta_{1,i} + \alpha_{n-x_2}^{(i)} \zeta_{2,i})}{(t_i - 2)\left[F_{n+1}^{(i)} + (h_1 - 1) F_n^{(i)}\right]} hJ. \tag{44}$$

3.5. Derivation of the general formula

Equation (13) obviously shows the key currents $I_{x_1}^{(i)}$ and $I_{x_2}^{(i)}$ must be calculated for deducing the equivalent resistance $R_{m\times n}(d_1, d_2)$. Making use of Eqs. (23) and (24) we obtain

$$\begin{pmatrix} I_k^{(1)} \\ I_k^{(2)} \\ \cdots \\ I_k^{(m)} \end{pmatrix} = \frac{4}{2m+1} \begin{pmatrix} \cos\left(1-\frac{1}{2}\right)\theta_1 & \cos\left(1-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(1-\frac{1}{2}\right)\theta_m \\ \cos\left(2-\frac{1}{2}\right)\theta_1 & \cos\left(2-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(2-\frac{1}{2}\right)\theta_m \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(m-\frac{1}{2}\right)\theta_1 & \cos\left(m-\frac{1}{2}\right)\theta_2 & \cdots & \cos\left(m-\frac{1}{2}\right)\theta_m \end{pmatrix} \begin{pmatrix} X_k^{(1)} \\ X_k^{(2)} \\ \cdots \\ X_k^{(m)} \end{pmatrix}, \tag{45}$$

where $k = 0, 1, 2, \ldots, n$. Taking $k = x_1$, we therefore achieve the following equation:

$$\begin{aligned} \sum_{i=1}^{y_1} I_{x_1}^{(i)} &= \frac{4}{2m+1} \sum_{i=1}^{m} X_{x_1}^{(i)} \sum_{k=1}^{y_1} \cos(k - \frac{1}{2})\theta_i \\ &= \frac{2}{2m+1} \sum_{i=1}^{m} X_{x_1}^{(i)} \frac{\sin(y_1\theta_i)}{\sin(\theta_i/2)}, \end{aligned} \tag{46}$$

where the identity $\sum_{k=1}^{y_1} \cos(k - 1/2)\theta_i = \sin(y_1\theta_i)/2\sin(\theta_i/2)$ is used.

Similarly, taking $k = x_2$ in Eq. (45), we also obtain

$$\sum_{i=1}^{y_2} I_{x_2}^{(i)} = \frac{2}{2m+1} \sum_{i=1}^{m} X_{x_2}^{(i)} \frac{\sin(y_2\theta_i)}{\sin(\theta_i/2)}. \tag{47}$$

Substituting Eqs. (46) and (47) into Eq. (13) we obtain

$$\begin{aligned} &R_{m\times n}(d_1, d_2) \\ &= \frac{2r_0}{J(2m+1)} \left(\sum_{i=1}^{m} X_{x_2}^{(i)} \frac{\sin(y_2\theta_i)}{\sin(\theta_i/2)} - \sum_{i=1}^{m} X_{x_1}^{(i)} \frac{\sin(y_1\theta_i)}{\sin(\theta_i/2)} \right). \end{aligned} \tag{48}$$

Finally, we obtain the main result (4) by further substituting $X_{x_1}^{(i)}$ and $X_{x_2}^{(i)}$ from Eqs. (44) and (43) into Eq. (48). So far the general formula (4) is proved.

3.6. Several special results

In this section we will prove several special results that were proposed in Section 1 by formula (4).

Case 3.6.1 when $n\to\infty$ with m being finite, by Eqs. (1) and (2) we obtain

$$\frac{\Delta F_{x_1}^{(i)}\alpha_{n-x_2}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}=\frac{t_i-2}{\lambda_i-\bar{\lambda}_i}\times\left(\frac{a\lambda_i^{n+x_1-x_2}+\bar{a}\bar{\lambda}_i^{n+x_1-x_2}+a\lambda_i^{n-x_1-x_2-1}+\bar{a}\bar{\lambda}_i^{n-x_1-x_2-1}}{a\lambda_i^n-\bar{a}\bar{\lambda}_i^n}\right), \tag{49}$$

where $a=(\lambda_i+h_1-1)$ and $\bar{a}=(\bar{\lambda}_i+h_1-1)$. From Eq. (3) we know that $\lambda_i>1>\bar{\lambda}_i>0$, thus

$$\begin{aligned}&\lim_{n\to\infty}\frac{\Delta F_{x_1}^{(i)}\alpha_{n-x_2}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}\\&=\left(\frac{t_i-2}{\lambda_i-\bar{\lambda}_i}\right)\left(\lambda_i^{x_1-x_2}+\bar{\lambda}_i^{x_1+x_2+1}\right),\\&\lim_{\substack{n\to\infty\\x\to\infty}}\frac{\Delta F_{x_1}^{(i)}\alpha_{n-x_2}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}\\&=\left(\frac{t_i-2}{\lambda_i-\bar{\lambda}_i}\right)\bar{\lambda}_i^{|x_1-x_2|},\quad(x_1\le x_2).\end{aligned} \tag{50}$$

Similarly, we obtain

$$\lim_{\substack{n\to\infty\\x\to\infty}}\frac{\Delta F_x^{(i)}\alpha_{n-x}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}=\frac{t_i-2}{\lambda_i-\bar{\lambda}_i},\quad(x=x_1,x_2). \tag{51}$$

Substituting Eqs. (50) and (51) into Eq. (4), we obtain Eq. (5) after having used $\lambda_i-\bar{\lambda}_i=2\sqrt{(1+h-h\cos\theta_i)^2-1}$.

Case 3.6.2 In order to prove Eqs. (6) and (7), we need to refer an integral theory, as follows.

If $\theta_k=(2k-1)\pi/(2m+1)$, we have $\Delta\theta_k=\theta_{k+1}-\theta_k=2\pi/(2m+1)$. In the limit of $m\to\infty$, we have

$$\lim_{m\to\infty}\frac{1}{2m+1}\sum_{k=1}^{m}g(\theta_k)=\frac{1}{2\pi}\int_0^{\pi}g(\theta)\mathrm{d}\theta, \tag{52}$$

which is an identity valid for any function $g(\theta_k)$. Equation (52) is prepared to prove Eqs. (6) and (7) in the following.

When $m,n\to\infty$, but y_1,y_2, and x_1-x_2 are finite, making use of Eqs. (5) and (52) we immediately obtain Eq. (6).

Case 3.6.3 According to the identity transform of a trigonometric function, we have

$$\begin{aligned}S_{1,i}^2+S_{2,i}^2&=1-\cos(y_1+y_2)\theta_i\cos(y_1-y_2)\theta_i,\\S_{1,i}S_{2,i}&=\frac{1}{2}[\cos(y_1-y_2)\theta_i-\cos(y_2+y_2)\theta_i].\end{aligned}$$

When $m\to\infty$ and $y_1,y_2\to\infty$, we must transform

$$(y_1,y_2)=\left(\frac{m}{2}+p,\frac{m}{2}+q\right),$$

where $p,q\ll m$ are integers, and $(p-q)=(y_1-y_2)$ are finite. As $\theta_i=(2i-1)\pi/(2m+1)$, we have

$$\lim_{m\to\infty}(y_1+y_2)\theta_i=\lim_{m\to\infty}\left(\frac{m+p+q}{2m+1}\right)(2i-1)\pi=\left(i-\frac{1}{2}\right)\pi.$$

Thus

$$\begin{aligned}&\lim_{m\to\infty}(S_{1,i}^2+S_{2,i}^2)=1-\sin(i\pi)\cos(y_1-y_2)\theta_i=1,\\&\lim_{m\to\infty}S_{1,i}S_{2,i}=\frac{1}{2}\cos(y_1-y_2)\theta_i.\end{aligned} \tag{53}$$

By Eqs. (5) and (53) we obtain

$$\begin{aligned}R_{\infty\times\infty}(d_1,d_2)&=\lim_{m\to\infty}\frac{2r}{2m+1}\sum_{i=1}^{m}\frac{1-\bar{\lambda}_i^{|x_1-x_2|}\cos(y_1-y_2)\theta_i}{\sqrt{(1+h-h\cos\theta_i)^2-1}}\\&=\frac{r}{\pi}\int_0^{\pi}\frac{1-\bar{\lambda}_i^{|x_1-x_2|}\cos(y_1-y_2)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta.\end{aligned} \tag{54}$$

Thus equation (7) is proved.

Case 3.6.4 When d_1 is at the bottom edge $(x_1,0)$ and d_2 is at the top edge $d_2(x_2,m)$, since d_1 is on the null resistor edge, we have $x_1=x_2$, thus

$$\beta_{x_1,x_1}^{(i)}=\beta_{x_1,x_2}^{(i)}=\beta_{x_2,x_2}^{(i)},$$

and

$$S_{1,i}=\sin(y_1\theta_i)=0,\quad S_{2,i}^2=\sin^2(m\theta_i)=\cos^2\left(\frac{1}{2}\theta_i\right).$$

Since $S_{2,i}^2/(1-\cos\theta_i)=\cot^2(\theta_i/2)/2$, equation (4) is immediately converted into Eq. (8).

Case 3.6.5 When d_1 and d_2 are both on the left edge, we have $x_1=x_2=0$. Since $\Delta F_0^{(i)}=1$ and $\Delta F_{-1}^{(i)}=1$, we have

$$\beta_{x_1,x_1}^{(i)}=\beta_{x_1,x_2}^{(i)}=\beta_{x_2,x_2}^{(i)}=\alpha_n^{(i)}=\Delta F_n^{(i)}+(h_1-1)\Delta F_{n-1}^{(i)}.$$

So, by Eq. (4) we obtain Eq. (9).

Case 3.6.6 When $m,n\to\infty$ with $x_1=x_2=0$, taking limit

$$\lim_{n\to\infty}\frac{\Delta F_n^{(i)}+(h_1-1)\Delta F_{n-1}^{(i)}}{F_{n+1}^{(i)}+(h_1-1)F_n^{(i)}}=1-\bar{\lambda}_i. \tag{55}$$

Substituting Eq. (55) into Eq. (9), and using Eqs. (52) and (53), we obtain Eq. (10).

Case 3.6.7 When d_1 is at the left edge $(0,y_1)$ and d_2 is at the right edge (n,y_2), we have $x_1=0$ and $x_2=n$. Thus $\beta_{x_1,x_2}^{(i)}=\Delta F_{x_1}^{(i)}\alpha_{n-x_2}^{(i)}=h_1$, and

$$\begin{aligned}\beta_{x_1,x_1}^{(i)}&=\Delta F_{x_1}^{(i)}\alpha_{n-x_1}^{(i)}=\alpha_n^{(i)}=\Delta F_n^{(i)}+(h_1-1)\Delta F_{n-1}^{(i)},\\\beta_{x_2,x_2}^{(i)}&=\Delta F_{x_2}^{(i)}\alpha_{n-x_2}^{(i)}=\Delta F_n^{(i)}\alpha_0^{(i)}=h_1\Delta F_n^{(i)}.\end{aligned} \tag{56}$$

Substituting Eq. (56) into Eq. (4), we obtain Eq. (11).

Case 3.6.8 When $h_1=0$, we have $\alpha_{n-x}^{(i)}=\Delta F_{n-x}^{(i)}-\Delta F_{n-x-1}^{(i)}=2h(1-\cos\theta_i)F_{n-x}^{(i)}$, thus equation (4) immediately turns into Eq. (12).

4. Testing of a simple example

Because the main formulae are too complicated to understand, we will verify their correctness by an example.

When $m = 1$, figure 1 degrades into a $1\times n$ resistor network with an arbitrary right boundary, as shown in Fig. 3. When we compute the resistance between nodes A and B, then we have $y_1 = 0$, $y_2 = 1$. Thus by Eq. (2) we have $\theta_i = \theta_1 = \pi/3$, and $S_{1,i} = \sin(0) = 0$, $S_{2,i} = \sin(\theta_1) = \sqrt{3}/2$, and

$$\lambda = 1 + \frac{1}{2}h + \sqrt{\left(1+\frac{1}{2}h\right)^2 - 1}. \tag{57}$$

From Eq. (9) we obtain the resistance

$$R_{A,B}(n) = r_0\left(\frac{\Delta F_n^{(1)} + (h_1-1)\Delta F_{n-1}^{(1)}}{F_{n+1}^{(1)} + (h_1-1)F_n^{(1)}}\right), \tag{58}$$

where $h = r/r_0$, $h_1 = r_1/r_0$, and $\lambda_k \cdot \bar{\lambda}_k = 1$. Simplifying Eq. (58) leads to

$$R_{A,B}(n) = r_0\left(1 - \frac{\lambda^n - \bar{\lambda}^n + (h_1-1)(\lambda^{n-1} - \bar{\lambda}^{n-1})}{\lambda^{n+1} - \bar{\lambda}^{n+1} + (h_1-1)(\lambda^n - \bar{\lambda}^n)}\right). \tag{59}$$

Formula (59) is fully consistent with the result in Ref. [9]. In particular, when $n = 1$, from Eq. (59) we have

$$R_{A,B}(1) = r_0\left(1 - \frac{1}{\lambda + \bar{\lambda} + (h_1-1)}\right) = r_0\left(\frac{r+r_1}{r_0+r+r_1}\right). \tag{60}$$

Obviously, formula (60) is completely accordant with the actual circuit. In fact, our results are bound to be perfect since all of the processes of algebra and reduction are strict and self-consistent.

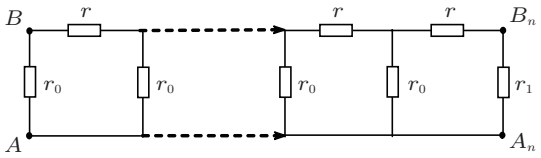


Fig. 3. The $1 \times n$ resistor network with arbitrary right boundary.

5. Summary and discussion

For the explicit computation of two-point resistance, Cserti[3] evaluated the two-point resistance using the lattice Green's function and restricted his study to regular lattices of infinite size. In 2004, Wu[4] established a theorem to compute the equivalent resistance between two nodes in a resistor network using the Laplacian approach, and he restricted his study to a regular boundary. The arbitrary boundary is a wall or trap which affects the behavior of a finite network. When the network is of finite size and with arbitrary boundary, Green's function technique and the Laplacian matrix approach are both invalid.

In the present article, we solve a complicated problem of a resistor network with an arbitrary boundary by the RT method. Our results are given in the form of a single summation. This method splits the derivation into three parts. The first part is to create a recursion relation (expressed by matrix) between the current distributions on three successive vertical lines. The second part is to diagonalize the matrix relation to produce a recurrence relation involving only variables on the same transverse line. The third part is to derive a recursion relation between the current distributions on the boundary. Basically, the method reduces the problem from two dimensions to one dimension.

The RT method can be extended to impedance networks because it is formulated on Ohm's law and this is also applicable to impedances. In addition, the grid elements r and r_0 can be either resistors or impedances, we can therefore study an $m \times n$ complex impedance network, such as

$$r = \mathrm{j}\omega L + \frac{R}{1+\mathrm{j}\omega CR} \quad \text{and} \quad r_0 = \mathrm{j}\omega L_0 + \frac{R_0}{1+\mathrm{j}\omega C_0 R_0}.$$

Based on the plural analysis, the formula of the equivalent impedances of the $m \times n$ RLC network with an arbitrary boundary can be obtained.

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