Concrete quantum tunneling spectrum of Schwarzschild black holes*

Chen Si-Na(陈斯纳) and Zhang Jing-Yi(张靖仪)†

Center for Astrophysics, Guangzhou University, Guangzhou 510006, *China*

(Received 19 June 2014; revised manuscript received 25 September 2014; published online 10 December 2014)

In this paper, a canonical ensemble model for black hole quantum tunneling radiation is introduced. We find that the probability distribution function is the same as the emission rate of a spherical shell in the Parikh–Wilczek tunneling framework. With this model, the probability distribution function corresponding to the emission shell system is calculated. Therefore, the concrete quantum tunneling spectrum of the Schwarzschild black hole is obtained.

Keywords: black hole, canonical ensemble model, quantum tunneling spectrum

PACS: 04.70.Dy **DOI:** 10.1088/1674-1056/24/2/020401

1. Introduction

Hawking radiation, which is named after the physicist Stephen Hawking who described this radiation in 1974, is a black body radiation that is predicted to be released by black holes due to the quantum effects near the event horizon. In the condition of a classical limit $e^{\beta\omega} \gg 1$, the Hawking radiation spectrum is[1,2]

$$N_\omega^2 = \frac{1}{e^{\beta\omega} \pm 1} \approx e^{-\beta\omega}, \tag{1}$$

where $\beta = 1/T = 8\pi M$, ω is the energy of the emitted particle, T and M are the Hawking temperature and the total mass of the black hole, respectively. Hawking radiation reduces the mass and the energy of the black hole, as a result the information puzzle will arise during the emission process.

In order to solve the information puzzle, in 2000 Parikh and Wilczek came up with a semi-classical method to calculate the emission rate of particles at the event horizon of a black hole.[3–5] The main idea is that if energy conservation is taken into account, then the emission process will be a quantum tunneling and the barrier will be determined by the self-gravitation interaction of the emission particles. In order to keep the spherical symmetry of the space time during the emission process, Parikh and Wilczek treated the tunneling particle as a spherical shell (emission shell). In this way, a corrected spectrum which is accurate to the first order approximation is given out.[6–31] According to the Parikh–Wilczek tunneling framework, the corrected emission spectrum of a black hole is[32]

$$\Gamma \sim e^{\Delta S_{\rm BH}} = \exp\left[-8\pi M\omega\left(1 - \frac{\omega}{2M}\right)\right]. \tag{2}$$

Since $N_\omega^2 \propto \Gamma$, the Parikh–Wilczek tunneling framework and their tunneling radiation spectrum Eq. (2) reveal a semi-classical correction to the Hawking radiation spectrum Eq. (1).[32] However, equation (2) is not an equation, it is only a proportional relationship of Γ and ω, the exact emission rate was not given out by the Parikh–Wilczek framework. In this paper, we attempt to present the concrete emission rate (the quantum tunneling spectrum) by introducing the canonical ensemble model for the black hole quantum tunneling radiation and we will also calculate the probability distribution function corresponding to the emission shell system.

As introduced in Ref. [32], we consider the black hole as a large heat source and the emission shell as a thermodynamical system which is composed of a large number of identical particles (photons). Then we can analyze the emission shell with the cannonical ensemble theory. We suppose that the isolated system consists of black hole and emission shell, the total energy of black hole is Mc^2, and the total energy of the emission shell system is $E = \hbar\omega$. By the cannonical ensemble model, we find that the probability distribution function is the same as the emission rate of the spherical shell Γ,[32]

$$\Gamma = \frac{1}{Z} e^{\Delta S_{\rm BH}} = \frac{1}{Z} \exp\left[-8\pi M\omega\left(1 - \frac{\omega}{2M}\right)\right], \tag{3}$$

here, Z is the canonical partition function of the spherical shell system,[32]

$$Z = \sum_i \exp\left(-\beta E_i - \frac{k_{\rm B}}{2C_{\rm BH}}\beta^2 {E_i}^2\right). \tag{4}$$

In previous study, the expression of the canonical partition function Z has not been given. In this paper, we will calculate the canonical partition function Z with the canonical ensemble model and, therefore, the formula of the quantum tunneling spectrum of the black hole Γ will be obtained.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11273009 and 11303006).

†Corresponding author. E-mail: zhangjy@gzhu.edu.cn

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2. Calculating the quantum tunneling spectrum of a black hole using the canonical ensemble method

Let us take the Schwarzschild black hole as an example.

We should choose appropriate coordinates to extend the theory of phase space in the canonical ensemble into curved space-time. We introduce the Painleve coordinates in which the singularity at the event horizon disappeared and the space-time slice is Euclidean. In this coordinate system, the three-dimensional space is flat, thus the theory of flat phase space is applicable.[33]

The line element of Schwarzschild black hole in the Painleve coordinates can be written as[33]

$$\begin{aligned} \mathrm{d}s^2 &= -f\,\mathrm{d}t^2 + 2\sqrt{1-f}\,\mathrm{d}t\,\mathrm{d}r + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \\ &\quad + r^2\sin^2\theta\,\mathrm{d}\psi^2, \end{aligned} \tag{5}$$

here

$$f = 1 - \frac{2GM}{c^2 r}. \tag{6}$$

The energy and the horizon of the Schwarzschild black hole is changed by the tunneling of particles. The energy of the black hole Mc^2 reduces to $Mc^2 - E$ after the particles tunnel from the black hole. The horizon of the black hole changes from r_i to r_f, where[34]

$$r_\mathrm{i} = \frac{2GM}{c^2}, \tag{7}$$

$$r_\mathrm{f} = \frac{2G}{c^2}\left(M - \frac{E}{c^2}\right). \tag{8}$$

Then, the thickness of the spherical shell d is

$$d = r_\mathrm{i} - r_\mathrm{f} = \frac{2GE}{c^4}. \tag{9}$$

Because the shell is very thin, the volume of the emission shell in Painleve coordinates is

$$V = 4\pi {r_\mathrm{i}}^2 d = \frac{32\pi G^3 M^2 E}{c^8}. \tag{10}$$

We analyze the emission spherical shell system by the semi-classical statistics. The canonical partition function Z is[35]

$$Z = \frac{1}{h^{3N}}\int \exp\left(-\beta E - \frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2\right)\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}z_1\ldots\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}z_N\,\mathrm{d}p_{x_1}\,\mathrm{d}p_{y_1}\,\mathrm{d}p_{z_1}\ldots\mathrm{d}p_{x_N}\,\mathrm{d}p_{y_N}\,\mathrm{d}p_{z_N}. \tag{11}$$

As the emission shell system consists of a vast majority of identical particles, we have

$$Z = \frac{1}{h^{3N}}\int \exp\left(-\beta E - \frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2\right)\mathrm{d}p_{x_1}\,\mathrm{d}p_{y_1}\,\mathrm{d}p_{z_1}\ldots\mathrm{d}p_{x_N}\,\mathrm{d}p_{y_N}\,\mathrm{d}p_{z_N}\left(\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^N, \tag{12}$$

therefore

$$Z = \frac{1}{h^{3N}}\int \exp\left(-\beta E - \frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2\right)\mathrm{d}p_{x_1}\,\mathrm{d}p_{y_1}\,\mathrm{d}p_{z_1}\ldots\mathrm{d}p_{x_N}\,\mathrm{d}p_{y_N}\,\mathrm{d}p_{z_N}V^N, \tag{13}$$

where $V = \int \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the volume of the emission shell in Painleve coordinates, which is just the same as Eq. (10).

Compare the first order βE and the second order $k_\mathrm{B}/(2C_\mathrm{BH})\beta^2E^2$ in Eq. (13).

The Hawking temperature of the black hole is[34]

$$T = \frac{\hbar c^3}{8\pi GMk_\mathrm{B}}, \tag{14}$$

and

$$k_\mathrm{B}\beta = \frac{1}{T}. \tag{15}$$

From Eqs. (14) and (15) we have

$$\beta = \frac{8\pi GM}{\hbar c^3}. \tag{16}$$

The thermal capacity of the black hole is[34]

$$C_\mathrm{BH} = \frac{\partial M}{\partial T} = -\frac{\hbar c^3}{8\pi G k_\mathrm{B}T^2} = -\frac{8\pi GM^2k_\mathrm{B}}{\hbar c^3}. \tag{17}$$

Then, divide the second order by the first one,

$$\frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2/(\beta E) = \frac{k_\mathrm{B}\beta E}{2C_\mathrm{BH}}, \tag{18}$$

substituting Eqs. (16) and (17) into Eq. (18), we obtain

$$\frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2/(\beta E) = -\frac{c^2}{2}\frac{E}{Mc^2}. \tag{19}$$

As the energy of the emission shell is more smaller than the energy of the black hole, $E/(Mc^2) \to 0$, we consider

$$\frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2/(\beta E) \to 0. \tag{20}$$

Therefore,

$$\beta E \gg \frac{k_\mathrm{B}}{2C_\mathrm{BH}}\beta^2E^2, \tag{21}$$

we can ignore the second order terms. As a result,

$$Z=\frac{1}{h^{3N}}\int \mathrm{e}^{-\beta E}\,\mathrm{d}p_{x_1}\,\mathrm{d}p_{y_1}\,\mathrm{d}p_{z_1}\ldots\mathrm{d}p_{x_N}\,\mathrm{d}p_{y_N}\,\mathrm{d}p_{z_N}V^N. \quad (22)$$

In curved space time nearby the black hole, the relation between energy and momentum of the particle satisfies the Klein–Gordon equation,[33]

$$p^2=\frac{4\omega^2}{f^2}, \quad (23)$$

then

$$\omega=\frac{pf}{2}. \quad (24)$$

Therefore, the total energy of the emission shell is

$$E=\hbar\omega_1+\cdots+\hbar\omega_N=\frac{\hbar f}{2}(p_1+\cdots+p_N). \quad (25)$$

By substituting Eq. (25) into Eq. (22), the canonical partition function of the emission shell will become

$$\begin{aligned}Z&=\frac{1}{h^{3N}}\int\exp\left[-\beta\frac{\hbar f}{2}(p_1+\cdots+p_N)\right]\\&\quad\times\mathrm{d}p_{x_1}\,\mathrm{d}p_{y_1}\,\mathrm{d}p_{z_1}\ldots\mathrm{d}p_{x_N}\,\mathrm{d}p_{y_N}\,\mathrm{d}p_{z_N}V^N\\&=\frac{1}{h^{3N}}\left(\int\exp\left(-\beta\frac{\hbar f}{2}p\right)\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z\right)^N V^N. \quad (26)\end{aligned}$$

We then switch it into spherical coordinates,

$$\begin{aligned}Z&=\frac{1}{h^{3N}}(4\pi\int_0^\infty\exp\left(-\beta\frac{\hbar f}{2}p\right)p^2\,\mathrm{d}p)^N V^N\\&=\frac{1}{h^{3N}}\left(4\pi\frac{-2\exp\left(-\frac{1}{2}fp\beta\hbar\right)(8+4fp\beta\hbar+f^2p^2\beta^2\hbar^2)}{f^3\beta^3\hbar^3}\Bigg|_0^\infty\right)^N V^N=\frac{1}{h^{3N}}\left(\frac{64\pi}{f^3\beta^3\hbar^3}\right)^N V^N. \quad (27)\end{aligned}$$

By substituting Eq. (10) into Eq. (27), we obtain

$$Z=\left(\frac{256G^3M^2E}{\pi f^3\beta^3\hbar^6c^8}\right)^N. \quad (28)$$

We compute the relation between the energy of the emission shell and the number of particle in the spherical system. According to thermodynamics and statistical physics, the average energy of the spherical system is[35]

$$\overline{E}=\frac{1}{Z}\sum_{\mathrm{i}}E_{\mathrm{i}}\exp\left(-\beta E_{\mathrm{i}}-\frac{k_{\mathrm{B}}}{2C_{\mathrm{BH}}}\beta^2{E_{\mathrm{i}}}^2\right). \quad (29)$$

From Eq. (21), we find that the first order is much larger than the second one, therefore the second order can be ignored,[35]

$$\begin{aligned}\overline{E}&\approx\frac{1}{Z}\sum_{\mathrm{i}}E_{\mathrm{i}}\,\mathrm{e}^{-\beta E_{\mathrm{i}}}=\frac{1}{Z}\left(-\frac{\partial}{\partial\beta}\right)\sum_{\mathrm{i}}\mathrm{e}^{-\beta E_{\mathrm{i}}}\\&=-\frac{\partial}{\partial\beta}\ln Z. \quad (30)\end{aligned}$$

By inserting Eq. (28) into Eq. (30), we have

$$E=\frac{3N}{\beta}. \quad (31)$$

Then, the relation between the energy of the emission shell and the number of particle in the spherical system is

$$N=\frac{\beta E}{3}. \quad (32)$$

In the emission process, the mass of the black hole M reduces to $M-E/c^2$, and the horizon of the black hole is determined by Eq. (7). The Schwarzschild metric in the Painleve coordinates Eq. (6) will then become

$$f=1-\frac{2G(M-E/c^2)}{c^2\frac{2GM}{c^2}}=\frac{E}{Mc^2}. \quad (33)$$

By substituting Eqs. (16), (32), and (33) into Eq. (28), we obtain the canonical partition function of the spherical shell system

$$Z=\left(\frac{M^2c^7}{2\pi^4E^2\hbar^3}\right)^{8\pi GME/(3\hbar c^3)}. \quad (34)$$

Inserting Eq. (34) and

$$\Delta S_{\mathrm{BH}}=-\frac{8\pi G^2ME}{c^6}\left(1-\frac{E}{2Mc^2}\right)$$

into Eq. (3), we come out with a result that the quantum tunneling spectrum of the black hole Γ is

$$\begin{aligned}\Gamma&=\exp\left[-\frac{8\pi G^2ME}{c^6}\left(1-\frac{E}{2Mc^2}\right)\right]\\&\quad\times\left(\frac{2\pi^4E^2\hbar^3}{M^2c^7}\right)^{8\pi GME/(3\hbar c^3)}. \quad (35)\end{aligned}$$

By transforming the expression into the Planck units $G=c=\hbar=1$, we have

$$\Gamma=\exp\left[-8\pi M\omega(1-\frac{\omega}{2M})\right]\left(\frac{2\pi^4\omega^2}{M^2}\right)^{8\pi M\omega/3}, \quad (36)$$

which is different from Eq. (2). This is an exact equation about Γ, M, and ω.

3. Conclusion

In summary, we will now discuss Eq. (36) in the following.

(i) The corrected emission spectrum of a black hole Eq. (2) is not an exact equation in the Parikh–Wilczek tunneling framework, there is a coefficient to be determined. In this paper, we have introduced a canonical ensemble model for the black hole quantum tunneling radiation and computed the probability distribution function of the emission shell system with this model. In this way, we have given out the coefficient of the $e^{\Delta S_{BH}}$ term, which has not been calculated in Eq. (2), and figured out the concrete quantum tunneling spectrum of black hole Γ, which is only related to the mass of the black hole M and the energy of the emission spherical shell system ω.

(ii) According to Eq. (36), we can see that the concrete quantum tunneling spectrum of the black hole, which is accurate to the first order approximation, is different from the black-body spectrum. This reveals that the Hawking radiation of the black hole is a non-black-body radiation. Applying this result, we are able to discuss the average radiation flux and estimate the life span of a black hole.

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