# Unified treatment of the bound states of the Schiöberg and the Eckart potentials using Feynman path integral approach

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We obtain analytical expressions for the energy eigenvalues of both the Schiöberg and Eckart potentials using an approximation of the centrifugal term. In order to determine the $\ell$-states solutions, we use the Feynman path integral approach to quantum mechanics. We show that by performing nonlinear space–time transformations in the radial path integral, we can derive a transformation formula that relates the original path integral to the Green function of a new quantum solvable system. The explicit expression of bound state energy is obtained and the associated eigenfunctions are given in terms of hypergeometric functions. We show that the Eckart potential can be derived from the Schiöberg potential. The obtained results are compared to those produced by other methods and are found to be consistent.

**Keywords:** path integrals, $\ell$-states, Schiöberg potential, Eckart potential

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## 1. Introduction

The search for exact solutions of the radial Schrödinger equation constitutes a very useful research area and is of general interest. Many efforts have been made to study the scattering and bound states of relativistic and non-relativistic cases with some physical potential models.[1–4] It is well known that exact solutions for the s-state (angular momentum $\ell = 0$) cannot be generalized to the case of the state with arbitrary angular momentum. In this case, the radial Schrödinger equation as well as relativistic Dirac and Klein–Gordon equations are solved either numerically[5] or by approximation methods.[6–12]

Among the potentials under consideration, we mention the class of exponential-type potentials, such as the Morse, Woods–Saxon, Hulthén, and Manning–Rosen potentials. For these potentials the Schrödinger equation admits an exact solution for $\ell = 0$.[13–15]

In the case of nonzero angular momentum states, accurate eigenvalues are reached by using an appropriate approximation scheme for the centrifugal term. Different methods have been used in this sense, such as the standard method,[16] the asymptotic iteration method,[17] supersymmetric quantum mechanics,[18] and the exact quantization rule method.[8]

In this paper, we study another exponential-type potential, namely an attractive hyperbolical potential of the form

$$V^{\mathrm{Sch}}(r) = D\left[1 - \sigma_0 \coth(\alpha r)\right]^2, \tag{1}$$

where $D$, $\alpha$, and $\sigma_0$ are positive parameters.

Studying the energy eigenvalues of diatomic molecules was first suggested by Schiöberg in 1986[19] as an alternative to the Morse potential. He suggested that the potential function (1) is more accurate than the Morse potential function in fitting the experimental Rydberg–Klein–Rees (RKR) potential energy curves and the rotational–vibrating levels for some diatomic molecules. Wang *et al.*[20] showed that a simple version of the Schiöberg empirical potential function and the well-known Manning–Rosen potential energy function are the same empirical function, and the Wei potential energy function and the Tietz potential energy functions are also the same solvable empirical function.[21]

Using the dissociation energy and the equilibrium bond length for a diatomic molecule as explicit parameters, Wang *et al.*[22] generated improved expressions for two versions of the Schiöberg potential energy function. Both versions of the Schiöberg potential function are Rosen–Morse and Manning–Rosen potential functions. By choosing the experimental values of the dissociation energy, equilibrium bond length, and equilibrium harmonic vibrational frequency as inputs, they calculated the average deviations of energies calculated with the potential model from the experimental data for five diatomic molecules and found that none of the six three-parameter empirical potential energy functions is superior to the other potentials in fitting the experimental data for all of the molecules examined.

The solutions of the Schrödinger equation for the Schiöberg potential have been obtained for the s-wave case.[23] Motivated by the success of the approximate bound state solutions of the Feynman propagator with the Manning–Rosen and Rosen–Morse potentials,[24,25] we attempt here to solve the general $\ell$-wave propagator of the Schiöberg potential by us-

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ing an improved approximation scheme of the centrifugal term that was proposed by Ikhdair *et al.*[26] The Eckart potential, which can be related to the Schiöberg potential by a change of parameters, is also treated. This is used in physics[27] as well as in chemical physics.[28,29] We recall that the Feynman Path Integrals formalism only applies to the s-wave and we are faced with difficulties for nonzero $\ell$-states similar to the ones encountered with the Schrödinger equation.

The organization of this paper is as follows. In Section 2, we look for the most relevant approximation scheme of the centrifugal barrier to be used in the formulation of the path integral for the Schiöberg potential. In Section 3, using the Duru–Kleinert method, we determine the $\ell$-wave eigenvalues for the Schiöberg potential. In Section 4, we study the Eckart potential. In Section 5 we present and discuss our results for both potentials. Our conclusions are summarized in Section 6.

## 2. Approximation scheme

To deal with the centrifugal term some authors have used the approximation proposed by Greene and Aldrich,[11] which provides analytical results for arbitrary $\ell$-wave bound states, scattering states of the Schrödinger equation, and relativistic wave equations for different forms of exponential potentials.[16,30,31] In this approximation scheme, here labeled as $F_1$, the centrifugal term is replaced by

$$\frac{1}{r^2} \approx 4\alpha^2 \left[\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right]. \tag{2}$$

As expected, this schematic approximation is more relevant when $\alpha$ is weaker. Other approximations have been proposed. For instance, Qiang[32] has suggested:

$$\frac{1}{r^2} \approx 4\alpha^2 \left[\frac{\mathrm{e}^{2\alpha}\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} + \left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)^2\right], \tag{3}$$

which we label as $F_2$. A third scheme is due to Ikhdair in Ref. [26], which is labeled as $F_3$

$$\frac{1}{r^2} \approx 4\alpha^2 \left[C_0 + \frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}} + \left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)^2\right], \tag{4}$$

where the parameter $C_0 = 0.0823058167837972$ is a shift determined by the expansion procedures. This value of $C_0$ is an approximation. Indeed, the exact value of $C_0$ is $1/12$, as reported in Ref. [33]. This has been obtained using a power series decomposition (see Eq. (6) of Ref. [34]). For bound state solutions of the Schrödinger equation with the Manning–Rosen potential and the Eckart potential, the numerical experiments showed that the analytical results obtained by employing the improved Greene–Aldrich approximation scheme are in better agreement with those obtained by using the numerical integration approach than the analytical results obtained by using the original Greene–Aldrich approximation scheme to deal with the centrifugal term.[35,36]

Moreover, in order to compare the degree of validity of the three approximations, we display in Fig. 1 the graphs of the three expressions. This clearly shows that the approximation $F_3$ is most suitable for the low values of $\alpha$ that are considered in this work. Consequently, this approximation $F_3$ will be used throughout the present study.

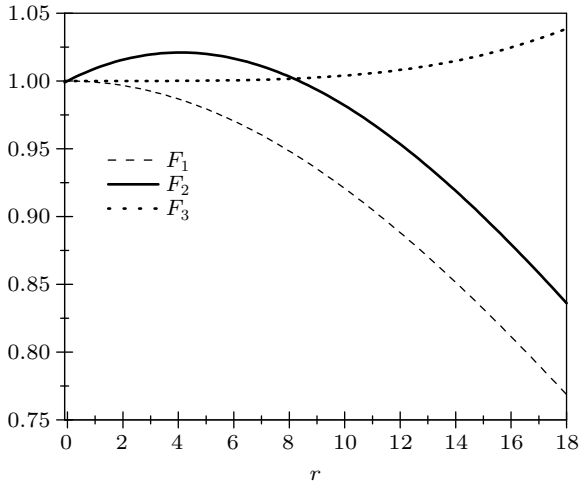



**Fig. 1.** The plot of $r^2F_i, i = 1,2,3$, with $\alpha = 0.05$.

## 3. Path integral for the Schiöberg propagator in spherical coordinates

In spherical coordinates, the time-sliced representation of the Feynman propagator takes the form[37]

$$K(\boldsymbol{r}'',t'';\boldsymbol{r}',t') = \frac{1}{4\pi r''r'}\sum_{l=0}^{\infty}(2\ell+1)K_\ell(r'',t'';r',t')\mathrm{P}_\ell(\cos\theta). \tag{5}$$

$\mathrm{P}_\ell(\cos\theta)$ denotes the Legendre polynomials with $\theta \equiv (\boldsymbol{r}'',\boldsymbol{r}')$ and

$$K_\ell(r'',t'';r',t') = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left[\frac{\mathrm{i}}{\hbar}S_j\right]\prod_{j=1}^{N}\left[\frac{m}{2\pi\mathrm{i}\hbar\varepsilon}\right]^{1/2}\prod_{j=1}^{N-1}\mathrm{d}r_j \tag{6}$$

with

$$S_j = \frac{m}{2\varepsilon}(\Delta r_j)^2 - \varepsilon V_{\mathrm{eff}}(r_j,\ell).$$

Here, $\Delta r_j = r_j - r_{j-1}$, $\varepsilon = t_j - t_{j-1}$, $t' = t_0$, and $t'' = t_N$, and

$$V_{\mathrm{eff}}(r_j,\ell) = \frac{\hbar^2 l(l+1)}{2mr_jr_{j-1}} - D\left[1-\sigma_0\coth(\alpha r_j)\right]^2. \tag{7}$$

To solve Eq. (6) for nonzero angular momentum states, we apply the approximation $F_3$ (included in $S_j$ via $\varepsilon V_{\mathrm{eff}}$ in Eq. (7)). Therefore, equation (7) becomes ($r_jr_{j-1} \simeq r_j^2$)

$$V_{\mathrm{eff}}(r_j,\ell) = \frac{2\alpha^2\hbar^2 l(l+1)}{m}\left[C_0 + \frac{\mathrm{e}^{-2\alpha r_j}}{1-\mathrm{e}^{-2\alpha r_j}} + \left(\frac{\mathrm{e}^{-2\alpha r_j}}{1-\mathrm{e}^{-2\alpha r_j}}\right)^2\right] - D\left[1-\sigma_0\coth(\alpha r_j)\right]^2. \tag{8}$$

In a more compact form, it reads

$$V_{\rm eff}(r_j,\ell) = -A\coth(\alpha r_j) + \frac{B(\ell)}{\sinh^2(\alpha r_j)} + C(\ell), \quad (9)$$

where

$$\begin{aligned} A &= 2D\sigma_0, \\ B(\ell) &= \frac{\hbar^2\alpha^2}{2m}\ell(\ell+1) + D\sigma_0^2, \\ C(\ell) &= 2\frac{\hbar^2\alpha^2 C_0}{m}\ell(\ell+1) + D\left(1+\sigma_0^2\right). \end{aligned} \quad (10)$$

### 3.1. Space–time transformation

In the following, we study the potential (9) in the framework of the Duru–Kleinert method.[38] The basic idea for solving this path integral problem is to find an appropriate space–time transformation to reformulate the path integral in terms of a well-known and already solved problem. This requires a relationship between the radial kernel of the Schiöberg potential and the radial kernel of a solvable quantum system.

The path integral for the potential (9) is

$$K_\ell(r'',r';T) = \int \mathscr{D}r(t)\exp\left[\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V_{\rm eff}(r,\ell)\right)\mathrm{d}t\right]. \quad (11)$$

A convenient way to derive the corresponding transformation formula is to use the energy-dependent Green function $G$ of the propagator $K_\ell$. By carrying out the point canonical transformation and the time transformation,[39]

$$\begin{cases} r = f(q), \\ \mathrm{d}t = [f'(q)]^2\,\mathrm{d}\mathfrak{s}, \end{cases} \quad (12)$$

we obtain the following Fourier transform formula

$$K_\ell(r'',r';T) = \frac{1}{2\pi i}\int G(r'',r',E)\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\mathrm{d}E. \quad (13)$$

In Eq. (13) the Green function reads

$$G(r'',r',E) = \frac{\mathrm{i}}{\hbar}\left[f'(q')f'(q'')\right]^{1/2}\int_0^\infty \widehat{K}_\ell(q'',q';s'')\,\mathrm{d}s'', \quad (14)$$

where

$$\begin{aligned} &\widehat{K}_\ell(q'',q';s'') \\ &= \int \mathscr{D}q(\mathfrak{s})\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - f'^2(q)\left[V(f(q))-E\right] - \Delta V(q)\right)\mathrm{d}\mathfrak{s}\right], \end{aligned} \quad (15)$$

and

$$\Delta V(q) = \frac{\hbar^2}{8m}\left[3\frac{[f''(q)]^2}{[f'(q)]^2} - 2\frac{f'''(q)}{f'(q)}\right]. \quad (16)$$

The functions $f'$, $f''$, and $f'''$ denote the successive derivatives of $f$ with respect to the variable $q$.

In our case, we choose

$$r = \left(\frac{1}{\alpha}\right)\mathrm{arccoth}(2\coth^2 q - 1), \quad (17)$$

so as to obtain

$$\Delta V(q) = \frac{\hbar^2}{8m}\left[\frac{1}{\cosh^2 q} + \frac{3}{\sinh^2 q}\right]. \quad (18)$$

Note that the transformation (17) leads to the following expression of the potential (9)

$$\begin{aligned} V_{\rm eff}(f(q),\ell) = &-A(2\coth^2 q - 1) + 2B(\ell)(2\coth^2 q - 2)\coth^2 q \\ &+ C(\ell)). \end{aligned} \quad (19)$$

Collecting all these terms, we have

$$\begin{aligned} &f'^2(q)\left[V_{\rm eff}(f(q),\ell) - E\right] + \Delta V(q) \\ &= \frac{\frac{2m}{\alpha^2}(E - C(\ell) - A) + \frac{\hbar^2}{4}}{2m\cosh^2 q} + \frac{\frac{8mB(\ell)}{\alpha^2} + \frac{3\hbar^2}{4}}{2m\sinh^2 q} \\ &\quad - \frac{E - C(\ell) + A}{\alpha^2}. \end{aligned} \quad (20)$$

By substituting Eqs. (18) and (20) into Eq. (15), the effective potential $V_{\rm eff}$ is now expressed in terms of the modified Pöschl–Teller (MPT) potential and the propagator expression becomes

$$\begin{aligned} &\widehat{K}_\ell(q'',q';s'') \\ &= \exp\left[\frac{\mathrm{i}}{\hbar}s''\frac{E - C(\ell) + A}{\alpha^2}\right] \\ &\quad \times \int \mathscr{D}q(\mathfrak{s})\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - \frac{\hbar^2}{2m}\left[\frac{\eta(\eta-1)}{\sinh^2 q} - \frac{\upsilon(\upsilon-1)}{\cosh^2 q}\right]\right)\mathrm{d}\mathfrak{s}\right] \\ &= \exp\left[\frac{\mathrm{i}}{\hbar}s''\frac{E - C(\ell) + A}{\alpha^2}\right]K_\ell^{\rm MPT}(q'',q';s'') \end{aligned} \quad (21)$$

with

$$\eta = \frac{1}{2} \pm \sqrt{1 + \frac{8mB(\ell)}{\alpha^2\hbar^2}}, \quad \upsilon = \frac{1}{2} \pm \sqrt{-\frac{2m(E - C(\ell) - A)}{\alpha^2\hbar^2}}$$

and $K_\ell^{\rm MPT}$ is the path integral of the modified Pöschl–Teller potential which is a known solved problem.[40–42] The path integral solution reads

$$\begin{aligned} &K_\ell^{\rm MPT}(q'',q';s'') \\ &= \sum_{n=0}^{N_m}\exp(-\mathrm{i}s''E_n)\chi_{n,\ell}^{(k_1,k_2)}(q'')\chi_{n,\ell}^{*(k_1,k_2)}(q') \\ &\quad + \int_0^\infty \mathrm{d}K\exp\left(-\mathrm{i}s''\frac{\hbar k^2}{2m}\right)\chi_K^{(k_1,k_2)}(q'')\chi_K^{*(k_1,k_2)}(q'). \end{aligned} \quad (22)$$

Here, we use the following notations:[43,44] $2s = \eta(\eta-1)$, $-2c = \upsilon(\upsilon-1)$, where $c \le 1/8$, $s \ge -1/8$. The numbers $k_1$, $k_2$ are defined in terms of $c$ and $s$ as $k_1 = \frac{1}{2}[1 \pm (\frac{1}{4} - 2c)^{1/2}]$, $k_2 = \frac{1}{2}[1 \pm (\frac{1}{4} + 2s)^{1/2}]$. Notice that one of the solutions for both $k_1$, $k_2$ will be eliminated further by applying the Cauchy conditions, namely the eigenfunctions being zero at $q = 0$ and decreasing exponentially at $q \to \infty$.

The upper limit $N_m$ denotes the maximal number of states with $0,1,...,n \leq N_m < k_1 - k_2 - 1/2$. The signs, in the definition of $k_1, k_2$, depend on the boundary conditions for $q \to 0$ and $q \to \infty$, respectively.

The propagator $K_\ell^{\rm MPT}$, Eq. (22), includes both discrete and continuous components. As far as the discrete part is concerned, the bound states wave functions are explicitly given by:[43,44]

$$\begin{aligned}\chi_{n,\ell}^{(k_1,k_2)}(q) &= N_n^{(k_1,k_2)}(\sinh q)^{2k_2-1/2}(\cosh q)^{-2k_1+3/2} \\ &\times {}_2{\rm F}_1(-k_1+k_2+k,-k_1+k_2-k+1;2k_2;-\sinh^2 q) \\ &= \left[\frac{2n!\,(2k_1-1)\,\Gamma(2k_1-n-1)}{\Gamma(2k_2+n)\,\Gamma(2k_1-2k_2-n)}\right]^{1/2} \\ &\times (\sinh q)^{2k_2-1/2}(\cosh q)^{2n-2k_1+3/2} \\ &\times {\rm P}_n^{(2k_2,2(k_1-k_2-n)-1)}\left(\frac{1-\sinh^2 q}{\cosh^2 q}\right), \end{aligned} \tag{23}$$

where $\mathrm{P}_n^{(\alpha,\beta)}$ denotes the Jacobi polynomials and the normalisation constant $N_n^{(k_1,k_2)}$ is given by

$$N_n^{(k_1,k_2)} = \frac{1}{\Gamma(2k_2)}\left(\frac{(2k-1)\Gamma(k_1+k_2-k)\Gamma(k_1+k_2+k-1)}{\Gamma(k_1-k_2+k)\Gamma(k_1-k_2-k+1)}\right)^{1/2}, \tag{24}$$

where $k = k_1 - k_2 - n$. The corresponding eigenvalues are given by

$$E_n^{\rm MPT} = -\frac{\hbar^2}{2m}\left[2(k_1-k_2-n)-1\right]^2. \tag{25}$$

Note that equation (23) has the appropriate Dirichlet conditions at $q = 0$ and $q$ infinite when

$$k_1 = \frac{1}{2}\left[1+\left(\frac{1}{4}-2c\right)^{1/2}\right], \quad k_2 = \frac{1}{2}\left[1+\left(\frac{1}{4}+2s\right)^{1/2}\right].$$

In the next section, we use the wave functions (23) and eigenvalues (25) of the modified Pöschl–Teller propagator to derive those corresponding to the Schiöberg potential.

### 3.2. Energy spectrum and wave functions

Calculating $\widehat{K}_\ell(q'',q';s'')$ allows us to obtain the Green function $G$. Then, the whole energy spectrum is derived from the poles, and the corresponding wave functions from the residues at the poles.

By substituting Eq. (22) into Eq. (21), we get

$$\begin{aligned}\widehat{K}_\ell(q'',q';s'') &= \sum_{n=0}^{N_m} \exp\left\{\frac{\rm i}{\hbar}s''\left[\frac{E-C(\ell)+A}{\alpha^2}-E_n^{\rm MPT}\right]\right\} \\ &\times \chi_{n,\ell}^{(k_1,k_2)}(q'')\chi_{n,\ell}^{*(k_1,k_2)}(q') \\ &+ \int_0^\infty {\rm d}K \exp\left\{\frac{\rm i}{\hbar}s''\left[\frac{E-C(\ell)+A}{\alpha^2}-\frac{\hbar^2K^2}{2m}\right]\right\} \\ &\times \chi_K^{(k_1,k_2)}(q'')\chi_K^{*(k_1,k_2)}(q'). \end{aligned} \tag{26}$$

Integrating Eq. (26) over the pseudo-time parameters $s''$, the Green function $G$, equation (14), becomes

$$\begin{aligned}G_\ell(r'',r';E) &= \sum_{n=0}^{N_m}\frac{\chi_{n,\ell}^{{\rm Sch}(k_1,k_2)}(r'')\chi_{n,\ell}^{{\rm Sch}*(k_1,k_2)}(r')}{E_{n,\ell}^{\rm Sch}-E} \\ &+ \int_0^\infty {\rm d}K \frac{\chi_{\ell,K}^{\rm Sch}(r'')\chi_{\ell,K}^{{\rm Sch}*}(r')}{\dfrac{\hbar^2K^2\alpha^2}{2m}-A+C(\ell)-E}. \end{aligned} \tag{27}$$

Following Refs. [24] and [39], the values of $k_1, k_2$, satisfying the appropriate boundary conditions for $r \to 0$ and $r \to \infty$, are, respectively,

$$\begin{aligned}k_1 &= \frac{1}{2}\left[\left(1+\frac{1}{2}(s+2n+1)\right)+\frac{2mA}{\alpha^2\hbar^2(s+2n+1)}\right], \\ k_2 &= \frac{1}{2}\left(1+\sqrt{1+\frac{8mB(\ell)}{\alpha^2\hbar^2}}\right) \equiv \frac{1}{2}(1+s). \end{aligned} \tag{28}$$

Substituting these values of $k_1$, $k_2$ in Eqs. (23) and (25), and introducing $u = \frac{1}{2}\left[1-\tanh(2\alpha r)\right]$, we obtain

$$\begin{aligned}\chi_{n,\ell}^{{\rm Sch}(k_1,k_2)}(r) &= \sqrt{\alpha}N_n^{(k_1,k_2)}(1-u)^{1/2-k_1+n}u^{k_1-1-s/2-n} \\ &\times {}_2{\rm F}_1\left(-n,2k_1-n-1;s+1;\frac{1}{1-u}\right) \\ &= \left[\frac{\alpha(2k_1-1)n!\Gamma(2k_1-n-1)}{\Gamma(n+s+1)\Gamma(2k_1-s-n-1)}\right]^{1/2}(1-{\rm e}^{-2r})^{k_2} \\ &\times \exp\left\{-2r\left[k_1-\frac{s}{2}-n-1\right]\right\} \\ &\times {\rm P}_n^{(2k_1-2n-s-2,s)}(1-2\,{\rm e}^{-2r}). \end{aligned} \tag{29}$$

The energy spectrum is obtained from the poles of the Green function, Eq. (27), namely

$$\begin{aligned}E_{n,\ell}^{\rm Sch} &= \alpha^2 E_n^{\rm MPT} - A + C(\ell) \\ &= -\frac{\hbar^2\alpha^2}{2m}\left[2(k_1-k_2-n)-1\right]^2 - A + C(\ell) \\ &= -\left[\frac{\hbar^2\alpha^2(s+2n+1)^2}{8m}+\frac{2mA^2}{\hbar^2\alpha^2(s+2n+1)^2}\right]+C(\ell). \end{aligned} \tag{30}$$

We note that, in Eqs. (29) and (30), the angular momentum is present through the variable $B$, Eq. (10) and in the equations for $k_1$, $k_2$, and $s$, Eq. (28).

## 4. The Eckart potential

As mentioned above, we also consider the Eckart potential

$$V^{\rm E}(r) = -\alpha'\frac{{\rm e}^{-r/a}}{1-{\rm e}^{-r/a}} + \beta'\frac{{\rm e}^{-r/a}}{(1-{\rm e}^{-r/a})^2}, \quad \alpha', \ \beta' > 0, \tag{31}$$

which is often used in chemical physics. The Eckart potential has been investigated, especially by Dong,[45] Zhang *et al.*[46] by means of the Nikiforov Uvarov (N.U.) method,[26] and Stanek,[47] who has used an improved approximation scheme to the centrifugal term. Here, the Eckart potential is combined with the approximation $F_3$, to derive the corresponding path integral.

It should be noted that the Eckart potential reduces to the Schiöberg potential by setting $\alpha' = 4D\sigma_0$, $\beta' = 4D\sigma_0^2$, and $a = 1/2\alpha$. Consequently, for these parameter values, the energy spectrum as well as the associated wavefunctions are written as

$$E^{\mathrm{E}}_{n,\ell} = -\left[\frac{\hbar^2(s+2n+1)^2}{32ma^2} + \frac{8ma^2A}{\hbar^2(s+2n+1)^2}\right] + C(\ell) \quad (32)$$

and

$$\begin{aligned}\chi^{\mathrm{E}(k_1,k_2)}_{n,\ell}(r) &= \frac{N^{(k_1,k_2)}_n}{\sqrt{2a}}(1-u)^{1/2-k_1+n}u^{k_1-1-s/2-n} \\ &\times {}_2\mathrm{F}_1\left(-n, 2k_1-n-1; s+1; \frac{1}{1-u}\right) \\ &= \left[\frac{(2k_1-1)n!\Gamma(2k_1-n-1)}{(2a)\Gamma(n+s+1)\Gamma(2k_1-s-n-1)}\right]^{1/2}(1-\mathrm{e}^{-2r})^{k_2} \\ &\times \exp\left\{-2r\left[k_1 - \frac{s}{2} - n - 1\right]\right\} \\ &\times \mathrm{P}^{(2k_1-2n-s-2,s)}_n(1-2\mathrm{e}^{-2r}). \quad (33)\end{aligned}$$

The parameters appearing in Eqs. (32) and (33) have been defined in Eqs. (10) and (28) and depend on $\ell$.

## 5. Numerical results

Energies calculated by using the path integral method that is described above are reported in the following tables. The energy eigenvalues $E_{n',\ell}$ have been calculated for principal quantum numbers $n' = n+\ell+1$ and various angular quantum number $\ell$, as in Ref. [47]. Two different values of the parameter $\sigma_0$ and a set of parameters $\alpha$ are considered. The present results for the Schiöberg potential Eq. (30) are displayed in Tables 1 and 2, and compared to the values given by Ikhdair and Sever,[26] who used the N.U. method with the same approximation of the barrier. All of the results are checked against the numerical data of Lucha *et al.*,[5] who calculated the energies by using MATHEMATICA. The method of Ref. [5] gives the harmonic oscillator energies with a precision of $10^{-5}$. We found that our calculated energy eigenvalues are improved as compared to the values of Ikhdair and Sever[26] and closer to the numerical results of Lucha *et al.*[5] At higher values of $\alpha$ both approaches are of similar quality. The relative error between our calculations and the numerical results is of about $2\times10^{-4}$ for all values of $\ell$.

For the Eckart potential, the results, Eq. (32) are displayed in Table 3 and 4, where a comparison is made with the calculations of Stanek,[47] who analytically solved the Schrödinger equation, but using another approach for the centrifugal barrier. All of the results are compared to the numerical data.[5] Our results are almost indistinguishable from the numerical ones.[5] Nevertheless, except in a few cases, we get results closer to the numerical estimates compared to the method of Stanek.[47] This is due to our choice of the centrifugal barrier.

**Table 1.** Eigenvalues $-E_{n',\ell}$ of the Schiöberg potential in atomic units ($\hbar = m = 1$) with $D = 10$ and $\sigma_0 = 0.1$.

| States | $\alpha$ | Present | Ikhdair *et al.*[26] | Lucha *et al.*[5] |
|---|---|---|---|---|
| 2p | 0.1 | 2.61889 | 2.61874 | 2.61935 |
| | 0.15 | 3.90579 | 3.90544 | 3.90645 |
| | 0.2 | 5.00395 | 5.00331 | 5.00457 |
| | 0.25 | 5.88694 | 5.88594 | 5.88725 |
| 3p | 0.1 | 4.73556 | 4.73540 | 4.73638 |
| | 0.15 | 6.04578 | 6.04543 | 6.04649 |
| | 0.2 | 6.91727 | 6.91663 | 6.91733 |
| | 0.25 | 7.48500 | 7.48400 | 7.48358 |
| 3d | 0.1 | 3.62746 | 3.62699 | 3.62769 |
| | 0.15 | 5.29512 | 5.29404 | 5.29510 |
| | 0.2 | 6.47684 | 6.47492 | 6.47598 |
| 4p | 0.1 | 6.00302 | 6.00287 | 6.00390 |
| | 0.15 | 7.11562 | 7.11526 | 7.11589 |
| | 0.2 | 7.71967 | 7.71903 | 7.71826 |
| 4d | 0.1 | 5.33176 | 5.33129 | 5.33216 |
| | 0.15 | 6.73691 | 6.73583 | 6.73642 |
| | 0.2 | 7.54672 | 7.54480 | 7.54331 |
| 4f | 0.1 | 4.69061 | 4.68965 | 4.69058 |
| | 0.15 | 6.43208 | 6.42992 | 6.43112 |
| | 0.2 | 7.43781 | 7.43397 | 7.43334 |
| 5p | 0.1 | 6.80360 | 6.80345 | 6.80432 |
| 5d | 0.1 | 6.37810 | 6.37762 | 6.37842 |
| 5f | 0.1 | 5.98134 | 5.98063 | 5.98147 |
| 5g | 0.1 | 5.62964 | 5.62805 | 5.62926 |
| 6p | 0.1 | 7.32432 | 7.32416 | 7.32476 |
| 6d | 0.1 | 7.04871 | 7.04824 | 7.04873 |
| 6f | 0.1 | 6.79574 | 6.79479 | 6.79528 |
| 6g | 0.1 | 6.57537 | 6.57377 | 6.57452 |

**Table 2.** Eigenvalues $-E_{n',\ell}$ of the Schiöberg potential in atomic units ($\hbar = m = 1$) with $D = 10$ and $\sigma_0 = 0.2$.

| States | $\alpha$ | Present | Ikhdair *et al.*[26] | Lucha *et al.*[5] |
|---|---|---|---|---|
| 2p | 0.1 | 1.20892 | 1.20876 | 1.20903 |
| | 0.15 | 1.86671 | 1.86636 | 1.86689 |
| | 0.2 | 2.52064 | 2.52000 | 2.52080 |
| | 0.25 | 3.14766 | 3.14666 | 3.14766 |
| 3p | 0.1 | 2.68323 | 2.68308 | 2.68358 |
| | 0.15 | 3.67163 | 3.67127 | 3.67198 |
| | 0.2 | 4.46579 | 4.46516 | 4.46579 |
| | 0.25 | 5.09330 | 5.09231 | 5.09235 |
| 3d | 0.1 | 1.57920 | 1.57873 | 1.57920 |
| | 0.15 | 2.54880 | 2.54773 | 2.54859 |
| | 0.2 | 3.48311 | 3.48119 | 3.48228 |
| 4p | 0.1 | 3.75708 | 3.75692 | 3.75758 |
| | 0.15 | 4.81251 | 4.81215 | 4.81274 |
| | 0.2 | 5.53175 | 5.53111 | 5.53087 |
| 4d | 0.1 | 2.95305 | 2.95257 | 2.95317 |
| | 0.15 | 4.10518 | 4.10410 | 4.10470 |
| | 0.2 | 5.00371 | 5.00179 | 5.00137 |
| 4f | 0.1 | 2.07438 | 2.07342 | 2.07417 |
| | 0.15 | 3.35838 | 3.35622 | 3.35742 |
| | 0.2 | 4.47792 | 4.47408 | 4.47486 |
| 5p | 0.1 | 4.54961 | 4.54946 | 4.55015 |
| 5d | 0.1 | 3.95725 | 3.95677 | 3.95740 |
| 5f | 0.1 | 3.31592 | 3.31497 | 3.31567 |
| 5g | 0.1 | 2.64177 | 2.64017 | 2.64124 |
| 6p | 0.1 | 5.13778 | 5.13763 | 5.13824 |
| 6d | 0.1 | 4.69977 | 4.69929 | 4.69979 |
| 6f | 0.1 | 4.22750 | 4.22654 | 4.22706 |
| 6g | 0.1 | 3.73461 | 3.73301 | 3.73378 |

**Table 3.** Eigenvalues $-E_{n',\ell}$ of the Eckart potential in atomic units ($\hbar = m = 1$) with $\alpha' = 1/a$ and $\beta' = 0.0001$.

| States | $\alpha = 1/(2a)$ | Present | Stanek[47] | Lucha *et al.*[5] |
|---|---|---|---|---|
| 2p | 0.025 | 0.1008358 | 0.1008306 | 0.1008358 |
| | 0.050 | 0.0978350 | 0.0978345 | 0.0978358 |
| | 0.075 | 0.0884143 | 0.0884174 | 0.0884183 |
| | 0.100 | 0.0783726 | 0.0783776 | 0.0783854 |
| | 0.150 | 0.0590391 | 0.0590651 | 0.0591059 |
| 3p | 0.025 | 0.0401247 | 0.0401232 | 0.0401250 |
| | 0.050 | 0.0322432 | 0.0322435 | 0.0322482 |
| | 0.075 | 0.0235301 | 0.0235330 | 0.0235553 |
| | 0.100 | 0.0157766 | 0.0157843 | 0.0158588 |
| | 0.150 | 0.0039335 | 0.0039725 | 0.0044091 |
| 3d | 0.025 | 0.0413635 | 0.0413638 | 0.0413642 |
| | 0.050 | 0.0321870 | 0.0321936 | 0.0321973 |
| | 0.075 | 0.0227453 | 0.0227792 | 0.0227991 |
| | 0.100 | 0.0141878 | 0.0142976 | 0.0143675 |
| 4p | 0.025 | 0.0184621 | 0.0184616 | 0.0184632 |
| | 0.050 | 0.0106990 | 0.0106995 | 0.0107159 |
| | 0.075 | 0.0044141 | 0.0044170 | 0.0045059 |
| | 0.100 | 0.0003795 | 0.0003883 | 0.0007212 |
| 4d | 0.025 | 0.0189189 | 0.0189196 | 0.0189216 |
| | 0.050 | 0.0104172 | 0.0104280 | 0.0104603 |
| | 0.075 | 0.0035230 | 0.0035778 | 0.0037658 |
| 4f | 0.025 | 0.0190182 | 0.0190212 | 0.0189216 |
| | 0.050 | 0.0098510 | 0.0098979 | 0.0104603 |
| | 0.075 | 0.0021369 | 0.0023963 | 0.0037658 |
| 5p | 0.025 | 0.0086822 | 0.0086820 | 0.0086850 |
| 5d | 0.025 | 0.0088503 | 0.0088512 | 0.0088576 |
| 5f | 0.025 | 0.0088178 | 0.0088225 | 0.0088297 |
| 5g | 0.025 | 0.0086797 | 0.0086919 | 0.0086943 |

**Table 4.** Eigenvalues $-E_{n',\ell}$ of the Eckart potential in atomic units ($\hbar = m = 1$) with $\alpha' = 1/a$ and $\beta' = 5 \times 10^{-5}$.

| States | $\alpha = 1/(2a)$ | Present | Stanek[47] | Lucha *et al.*[5] |
|---|---|---|---|---|
| 2p | 0.025 | 0.1064736 | 0.1064616 | 0.1064737 |
| | 0.050 | 0.0994153 | 0.0994116 | 0.0994162 |
| | 0.075 | 0.0891243 | 0.0891252 | 0.0891284 |
| | 0.100 | 0.0787681 | 0.0787732 | 0.0787809 |
| | 0.150 | 0.0592067 | 0.0592326 | 0.0592734 |
| 3p | 0.025 | 0.0418397 | 0.0418361 | 0.0418400 |
| | 0.050 | 0.0326962 | 0.0326956 | 0.0327011 |
| | 0.075 | 0.0237215 | 0.0237237 | 0.0237464 |
| | 0.100 | 0.0158740 | 0.0158817 | 0.0159559 |
| | 0.150 | 0.0039632 | 0.0040020 | 0.0044376 |
| 3d | 0.025 | 0.0424581 | 0.0424587 | 0.0424588 |
| | 0.050 | 0.0324634 | 0.0324700 | 0.0324736 |
| | 0.075 | 0.0228610 | 0.0228947 | 0.0229146 |
| | 0.100 | 0.0142465 | 0.0143561 | 0.0144257 |
| 4p | 0.025 | 0.0191776 | 0.0191762 | 0.0191787 |
| | 0.050 | 0.0108685 | 0.0108687 | 0.0108852 |
| | 0.075 | 0.0044723 | 0.0044750 | 0.0045636 |
| | 0.100 | 0.0003980 | 0.0004067 | 0.0007380 |
| 4d | 0.025 | 0.0193727 | 0.0193735 | 0.0193753 |
| | 0.050 | 0.0105205 | 0.0105312 | 0.0105633 |
| | 0.075 | 0.0035583 | 0.0036127 | 0.0038001 |
| 4f | 0.025 | 0.0193488 | 0.0193516 | 0.0193525 |
| | 0.050 | 0.0099252 | 0.0099719 | 0.0099876 |
| | 0.075 | 0.0021621 | 0.0024207 | 0.0025321 |
| 5p | 0.025 | 0.0090293 | 0.0090287 | 0.0090320 |
| 5d | 0.025 | 0.0090698 | 0.0090707 | 0.0090768 |
| 5f | 0.025 | 0.0089774 | 0.0089820 | 0.0089892 |
| 5g | 0.025 | 0.0088049 | 0.0088169 | 0.0088194 |

## 6. Concluding remarks

In this paper, path integral formalism was used to solve two exponential-type potentials. We have calculated the energy levels of the Feynman kernel with the Schiöberg and the Eckart potentials for angular momentum $\ell \neq 0$, approximating the centrifugal term. First, we have shown that the most appropriate approximation scheme of the centrifugal barrier for this type of potential is the one given by Eq. (4). Then, the Duru–Kleinert transformation allows us to convert a nonsolvable propagator to another solvable one, for which the energy spectrum and the wave functions are known. This, allows us to determine analytically the energy spectrum of the starting potential.

Our results show that the obtained eigenvalues are in very good agreement with those given by other approximate methods and turn out to be in good agreement with the numerical data. The quality of our results proves that we can provide analytical eigenenergies and corresponding eigensolutions of the spectrum of the potential because the treatment of the barrier allows a tractable calculation leading to analytical results.

When the barrier approximation is fixed, we note that the Feynman integral method gives results closer to a numerical method than solving the Schrödinger equation via the N.U. method.

To conclude, our method is efficient in solving this type of potential. We intend in the near future to expand the path integral method to a more general form of potentials and relativistic cases.

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