From fractional Fourier transformation to quantum mechanical fractional squeezing transformation*

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By converting the triangular functions in the integration kernel of the fractional Fourier transformation to the hyperbolic function, i.e., $\tan\alpha \to \tanh\alpha$, $\sin\alpha \to \sinh\alpha$, we find the quantum mechanical fractional squeezing transformation (FrST) which satisfies additivity. By virtue of the integration technique within the ordered product of operators (IWOP) we derive the unitary operator responsible for the FrST, which is composite and is made of $\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}$ and $\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]$. The FrST may be implemented in combinations of quadratic nonlinear crystals with different phase mismatches.

Keywords: fractional Fourier transformation, fractional squeezing transformation, unitary operator, the IWOP technique

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1. Introduction

In optical communication, image manipulation, and signal analysis, the fractional Fourier transformation (FrFT)[1] is a very useful tool. The concept of the FrFT was originally introduced for signal processing in 1980 by Namias as a Fourier transform of fractional order.[2,3] But FrFT did not have a great impact on optics until it was defined physically based on propagation in quadratic graded-index media (GRIN media). Mendlovic and Ozaktas[4–6] defined the α-th FrFT as follows. Let the original function be input from one side of the quadratic GRIN medium, at $z=0$. Then, the light distribution observed at the plane $z=z_0$ corresponds to the α equal to the (z_0/L)-th fractional Fourier transform of the input fraction, where $L \equiv (\pi/2)(n_1/n_2)^{1/2}$ is a characteristic distance, n_1, n_2 are the medium's physical parameters involved in the refractive index $n(r) = n_1 - n_2 r^2/2$, and r is the radial distance from the optical z axis. For real parameter α, the one-dimensional α-angle FrFT of a function f is denoted by $F_\alpha[f]$ and defined by

$$F_\alpha[f](p) = \frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}} \times \int_{-\infty}^{\infty} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{p^2+x^2}{\tan\alpha} - \frac{2px}{\sin\alpha}\right)\right] f(x)\,\mathrm{d}x. \quad (1)$$

The conventional Fourier transform is simply $F_{\pi/2}$. The composition $F_\alpha \circ F_\beta$ of two FrFT's with parameters α and β is defined by

$$\left(F_\alpha \circ F_\beta\right)[f] \equiv F_\alpha\left[F_\beta[f]\right], \quad (2)$$

the remarkable property of F_α is that it satisfies additivity, i.e.,

$$F_\alpha \circ F_\beta = F_{\alpha+\beta}. \quad (3)$$

In the context of quantum mechanics, function f turns to quantum state $|f\rangle$, the value $f(x)$ of f at given point x turns to the matrix element $\langle x|f\rangle$. The usual Fourier transform simply changes the basis from the coordinate basis $|x\rangle$ to the momentum basis $|p\rangle$

$$f(p) = \langle p|f\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|f\rangle\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}px}}{\sqrt{2\pi}} f(x)\,\mathrm{d}x, \quad (4)$$

where

$$|x\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{1}{2}x^2 + \sqrt{2}xa^{\dagger} - \frac{1}{2}a^{\dagger 2}\right)|0\rangle, \quad (5)$$

$$|p\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{1}{2}p^2 + \mathrm{i}\sqrt{2}pa^{\dagger} + \frac{1}{2}a^{\dagger 2}\right)|0\rangle, \quad (6)$$

here $a^{\dagger}$ and a are the creation and annihilation operators, respectively, obeying $\left[a, a^{\dagger}\right] = 1$, $|0\rangle$ is the vacuum state, and the α-angle fractional Fourier transform is[7,8]

$$F_\alpha[f](p) = \langle p|\,\mathrm{e}^{\mathrm{i}(\pi/2-\alpha)a^{\dagger}a}\,|f\rangle = \int_{-\infty}^{\infty} K_\alpha(p,x) f(x)\,\mathrm{d}x, \quad (7)$$

where the integration kernel $K_\alpha(p,x)$ is

$$K_\alpha(p,x) = \langle p|K_\alpha|x\rangle \equiv \langle p|\,\mathrm{e}^{\mathrm{i}(\pi/2-\alpha)a^{\dagger}a}\,|x\rangle = \frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{p^2+x^2}{\tan\alpha} - \frac{2px}{\sin\alpha}\right)\right], \quad (8)$$

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i.e., from the point of view of quantum mechanics, the α-angle fractional Fourier transform is the composite transformation including both the basis changing and unitary transformation generated by the unitary operator $K_\alpha = \mathrm{e}^{\mathrm{i}(\pi/2-\alpha)a^\dagger a}$.

An interesting question arises: if we make change the integration kernel $K_\alpha(p,x)$ in Eq. (8) to

$$\Re_\alpha(p,x) = \frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right], \quad (9)$$

i.e., changing the triangular function to the hyperbolic function

$$\tan\alpha \to \tanh\alpha, \quad \sin\alpha \to \sinh\alpha, \quad (10)$$

then to what kind of transformations will $\Re_\alpha$ belong? In the following we shall demonstrate that $\Re(\alpha)$ is just the integration kernel of a new quantum-mechanical fractional squeezing transformation (FrST). As is well known, the squeezing transformation has been a major topic in quantum optics because it produces squeezed state which exhibits less quantum fluctuation than a coherent state in one quadrature at the expense of more fluctuation in another quadrature.[9–11] In this letter we shall firstly propose fractional squeezing transformation and then prove that it also satisfies additivity. The work is arranged as follows: in Section 2 we derive the unitary operator responsible for the FrST, we are able to do this because we have invented the integration technique within ordered product (IWOP) of operators[12,13] which can directly perform integration over Dirac's ket–bra operators.[14–16] In Section 3, we prove that FrST also satisfies additivity, while it is essential that the operator $\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$ is the core operator to realize the additivity of FrST. We expect that the FrST may be implemented in combinations of quadratic nonlinear crystals with different phase mismatches.

2. The unitary operator responsible for the fractional squeezing transformation

To find if the unitary operator is responsible for the fractional squeezing transformation, we multiply $\Re(\alpha)$ by $\left|x'\right|_{x'=p}\rangle$ from the left and $\langle x|$ from the right, i.e., we consider the following integration over the ket–bra

$$\begin{aligned} S_\alpha &\equiv \iint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\left|x'\right|_{x'=p}\rangle\Re_\alpha(p,x)\langle x| \\ &= \frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\iint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\left|x'\right|_{x'=p}\rangle \\ &\quad\times\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right]\langle x|, \end{aligned} \quad (11)$$

using the normally ordered form of the vacuum projector

$$|0\rangle\langle 0| =: \mathrm{e}^{-a^\dagger a}: \quad (12)$$

and Eqs. (5) and (6), as well as the IWOP technique we perform the integration within : : (note $a^\dagger$ and a are commutable within : : and can be taken as c-number parameters),

$$\begin{aligned} S_\alpha &= \frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\iint_{-\infty}^{\infty}\frac{\mathrm{d}x\mathrm{d}p}{\pi^{1/2}} \\ &\quad\times\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha}\right] \\ &\quad\times :\exp\left[-\frac{p^2}{2} + \sqrt{2}pa^\dagger - \frac{a^{\dagger 2}}{2} - \frac{x^2}{2}\right. \\ &\quad\left.+\sqrt{2}xa - \frac{a^2}{2} - a^\dagger a\right]:, \end{aligned} \quad (13)$$

where the integration over $\mathrm{d}x$ is (keep knowing $a^\dagger$ and a as c-number parameters within : :)

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\exp\left(\frac{\mathrm{i}x^2}{2\tanh\alpha} - \frac{\mathrm{i}xp}{\sinh\alpha} - \frac{x^2}{2} + \sqrt{2}xa\right) \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}\exp\left[-\frac{x^2}{2}(1-\mathrm{i}\coth\alpha) + x\left(\sqrt{2}a - \frac{\mathrm{i}p}{\sinh\alpha}\right)\right] \\ &= \sqrt{\frac{2}{1-\mathrm{i}\coth\alpha}}\exp\left[\frac{\left(\sqrt{2}a - \mathrm{i}p/\sinh\alpha\right)^2}{2(1-\mathrm{i}\coth\alpha)}\right], \end{aligned} \quad (14)$$

and the next integration over $\mathrm{d}p$ is

$$\begin{aligned} &\int_{-\infty}^{\infty}\mathrm{d}p\exp\left[-\frac{p^2}{2}\left(1-\mathrm{i}\coth\alpha + \frac{1}{\sinh}{}^2\alpha(1-\mathrm{i}\coth\alpha)\right)\right. \\ &\left.+\sqrt{2}p\left(a^\dagger - \frac{\mathrm{i}a}{\sinh\alpha(1-\mathrm{i}\coth\alpha)}\right)\right] \\ &= \int_{-\infty}^{\infty}\mathrm{d}p\exp\left[-\frac{p^2}{2}\left(\frac{(1-\mathrm{i}\coth\alpha)^2\sinh^2\alpha+1}{\sinh^2\alpha(1-\mathrm{i}\coth\alpha)}\right)\right. \\ &\left.+\sqrt{2}p\left(a^\dagger - \frac{\mathrm{i}a}{\sinh\alpha(1-\mathrm{i}\coth\alpha)}\right)\right] \\ &= \int_{-\infty}^{\infty}\mathrm{d}p\exp\left[\frac{\mathrm{i}p^2}{\tanh\alpha-\mathrm{i}}\right. \\ &\left.+\sqrt{2}p\left(a^\dagger - \frac{\mathrm{i}a}{\sinh\alpha(1-\mathrm{i}\coth\alpha)}\right)\right] \\ &= \sqrt{\frac{\pi(\tanh\alpha-\mathrm{i})}{-\mathrm{i}}} \\ &\quad\times\exp\left[-\frac{\tanh\alpha-\mathrm{i}}{2\mathrm{i}}\left(a^\dagger - \frac{\mathrm{i}a}{\sinh\alpha(1-\mathrm{i}\coth\alpha)}\right)^2\right], \end{aligned} \quad (15)$$

so by combining Eqs. (14) and (15) we obtain

$$\begin{aligned} S_\alpha &= \frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\sqrt{\frac{2}{1-\mathrm{i}\coth\alpha}}\sqrt{\frac{\pi(\tanh\alpha-\mathrm{i})}{-\mathrm{i}}} \\ &\quad\times :\exp\left[\mathrm{i}\frac{a^{\dagger 2}}{2}\tanh\alpha + a^\dagger a(\operatorname{sech}\alpha - 1) + \mathrm{i}\frac{a^2}{2}\tanh\alpha\right]: \\ &= \sqrt{\operatorname{sech}\alpha}:\exp\left[\mathrm{i}\frac{a^{\dagger 2}}{2}\tanh\alpha\right. \\ &\quad\left.+a^\dagger a(\operatorname{sech}\alpha-1) + \mathrm{i}\frac{a^2}{2}\tanh\alpha\right]:. \end{aligned} \quad (16)$$

Using

$$:\mathrm{e}^{a^\dagger a(\operatorname{sech}\alpha-1)}: = \mathrm{e}^{a^\dagger a\ln\operatorname{sech}\alpha}, \quad (17)$$

we see

$$S_\alpha = \exp\left(\mathrm{i}\frac{a^{\dagger 2}}{2}\tanh\alpha\right)\exp\left[\left(a^\dagger a+\frac{1}{2}\right)\ln\operatorname{sech}\alpha\right] \times \exp\left(\mathrm{i}\frac{a^2}{2}\tanh\alpha\right) = \exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]. \quad (18)$$

S_α is Abelian with respect to the parameter α and is really a squeezing operator (for a review of the squeezed state we refer to Ref. [7]). Further, due to

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\left|x\right\rangle\left\langle p'\right|_{p'=x}\\ &=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\pi^{1/2}}\exp\left(-\frac{x^2}{2}+\sqrt{2}xa^\dagger-\frac{a^{\dagger 2}}{2}\right)\left|0\right\rangle\left\langle 0\right| \times\exp\left(-\frac{x^2}{2}-\sqrt{2}\mathrm{i}xa+\frac{a^2}{2}\right)\\ &=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}:\exp\left[-x^2+\sqrt{2}x\left(a^\dagger-\mathrm{i}a\right)-\frac{a^{\dagger 2}-a^2}{2}-a^\dagger a\right]:\\ &=:\exp\left[-\left(\mathrm{i}+1\right)a^\dagger a\right]:=\exp\left[-\frac{\mathrm{i}\pi}{2}a^\dagger a\right] \end{aligned} \quad (19)$$

or

$$\left|x'\right|_{x'=p}\rangle=\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}\left|p\right\rangle,\ \left\langle p\right|\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}=\left\langle x'\right|_{x'=p}\left|,\right. \quad (20)$$

we can rewrite Eq. (11) as

$$S_\alpha = \frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2} \times\iint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\left|p\right\rangle\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha}-\frac{\mathrm{i}xp}{\sinh\alpha}\right]\left\langle x\right|, \quad (21)$$

thus

$$\begin{aligned}&\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\iint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\left|p\right\rangle\\ &\times\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha}-\frac{\mathrm{i}xp}{\sinh\alpha}\right]\left\langle x\right|\\ &=\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}S_\alpha=\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]. \end{aligned} \quad (22)$$

Due to

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}a\,\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}&=a\,\mathrm{e}^{-\mathrm{i}\pi/2}=-\mathrm{i}a,\\ \mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}a^\dagger\,\mathrm{e}^{-\mathrm{i}\pi a^\dagger a/2}&=a^\dagger\,\mathrm{e}^{\mathrm{i}\pi/2}=\mathrm{i}a^\dagger, \end{aligned} \quad (23)$$

we see

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\\ &=\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}. \end{aligned} \quad (24)$$

Comparing Eq. (24) with Eq. (22) we name

$$\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$$

the fractional squeezing operator, which is a composite operator, whose matrix element is

$$\begin{aligned}&\left\langle p\right|\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\left|x\right\rangle\\ &=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}}\exp\left[\frac{\mathrm{i}(x^2+p^2)}{2\tanh\alpha}-\frac{\mathrm{i}xp}{\sinh\alpha}\right]=\aleph_\alpha\left(p,x\right). \end{aligned} \quad (25)$$

3. The additivity of fractional squeezing transformation

Let $\left\langle x\right|f\rangle=f\left(x\right)$, using Eqs. (20), (24), and (25) the FrST can be expressed as

$$\begin{aligned}\mathfrak{F}_\alpha\left[f\right]\left(p\right)&=\left\langle p\right|\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\left|f\right\rangle\\ &=\iint_{-\infty}^{\infty}\mathrm{d}p'\mathrm{d}x\left\langle p\right|\left.p'\right\rangle\left\langle p'\right|\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}\\ &\quad\times\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\left|x\right\rangle\left\langle x\right|\left.f\right\rangle\\ &=\int_{-\infty}^{\infty}\mathrm{d}p'\int\mathrm{d}x\delta\left(p-p'\right)\left\langle x'\right|_{x'=p'}\left|\right.\\ &\quad\times\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\left|x\right\rangle\left\langle x\right|\left.f\right\rangle\\ &=\int_{-\infty}^{\infty}\mathrm{d}x\left\langle x'\right|_{x'=p}\left|\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\right|x\rangle\left\langle x\right|\left.f\right\rangle\\ &=\int_{-\infty}^{\infty}\aleph_\alpha\left(p,x\right)f\left(x\right)\mathrm{d}x. \end{aligned} \quad (26)$$

Using the properties of the hyperbolic function

$$\tanh\left(\alpha+\beta\right)=\frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta}, \quad (27)$$

$$\sinh\left(\alpha+\beta\right)=\sinh\alpha\cosh\beta+\sinh\beta\cosh\alpha, \quad (28)$$

we can directly calculate two successive integration transformations

$$\begin{aligned}&\left(\mathfrak{F}_\alpha\circ\mathfrak{F}_\beta\right)\left[f\right]\left(p\right)\\ &=\int_{-\infty}^{\infty}\aleph_\alpha\left(p,p'\right)\mathrm{d}p'\int_{-\infty}^{\infty}\aleph_\beta\left(p',x\right)f\left(x\right)\mathrm{d}x\\ &=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\alpha}\sqrt{2\pi\mathrm{i}\sinh\beta}}\\ &\quad\times\int_{-\infty}^{\infty}\mathrm{d}p'\exp\left[\frac{\mathrm{i}}{2}\left(\frac{p'^2+p^2}{\tanh\alpha}-\frac{2p'p}{\sinh\alpha}\right)\right]\\ &\quad\times\int_{-\infty}^{\infty}\mathrm{d}x\exp\left[\frac{\mathrm{i}}{2}\left(\frac{x^2+p'^2}{\tanh\beta}-\frac{2xp'}{\sinh\beta}\right)\right]f\left(x\right)\\ &=\frac{1}{\sqrt{2\pi\mathrm{i}\sinh\left(\alpha+\beta\right)}}\int_{-\infty}^{\infty}\mathrm{d}x\\ &\quad\times\exp\left[\frac{\mathrm{i}}{2}\left(\frac{x^2+p^2}{\tanh\left(\alpha+\beta\right)}-\frac{2xp}{\sinh\left(\alpha+\beta\right)}\right)\right]f\left(x\right)\\ &=\mathfrak{F}_{\alpha+\beta}\left[f\right]\left(p\right), \end{aligned} \quad (29)$$

so the FrST is additive. We can either discuss the additivity of FrST in terms of the Dirac symbol, which is expressed as

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x'\left\langle p\right|\mathrm{e}^{\mathrm{i}\pi a^\dagger a/2}$$

$$
\begin{aligned}
&\times \exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]|x'\rangle_{\ p'=x'}\langle p'|\,\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}\\
&\times \exp\left[\frac{\mathrm{i}\beta}{2}\left(a^2+a^{\dagger 2}\right)\right]|x\rangle\langle x|\,f\rangle\\
=&\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}x'\,\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}\exp\left[\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]|x'\rangle\langle x'|\\
&\times \exp\left[\frac{\mathrm{i}\beta}{2}\left(a^2+a^{\dagger 2}\right)\right]|x\rangle\langle x|\,f\rangle\\
=&\langle p|\,\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}\exp\left[\frac{\mathrm{i}\,(\beta+\alpha)}{2}\left(a^2+a^{\dagger 2}\right)\right]|f\rangle\,, \qquad (30)
\end{aligned}
$$

so the operator $\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}$ is the core operator to realize the additivity of FrST.

In Ref. [17] it is reported that combinations of quadratic nonlinear crystals with different phase mismatches can be studied numerically in terms of amplitude squeezing in the second-harmonic generation. Thus we predict that an appropriate choice of phase mismatches can yield fractional squeezing in the harmonic field with some stability. We expect that the FrST could be implemented by experimentalists with quadratic nonlinear crystals.

In summary, for the first time we propose a quantum mechanical FrST which satisfies additivity, which is found by converting the triangular functions in the integration kernel of the fractional Fourier transformation to the hyperbolic function, i.e., $\tan\alpha \to \tanh\alpha$, $\sin\alpha \to \sinh\alpha$. The key point to make up FrST is the composite operator $\exp\left[-\frac{\mathrm{i}\alpha}{2}\left(a^2+a^{\dagger 2}\right)\right]\mathrm{e}^{\mathrm{i}\pi a^{\dagger}a/2}$.

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