Conservation laws of the generalized short pulse equation*

Zhang Zhi-Yong(张智勇)[a)†] and Chen Yu-Fu(陈玉福)[b)]

[a)] College of Sciences, North China University of Technology, Beijing 100144, China
[b)] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

(Received 20 July 2014; revised manuscript received 2 September 2014; published online 10 December 2014)

We show that the generalized short pulse equation is nonlinearly self-adjoint with differential substitution. Moreover, any adjoint symmetry is a differential substitution of nonlinear self-adjointness, and vice versa. Consequently, the general conservation law formula is constructed and new conservation laws for some special cases are found.

Keywords: nonlinear self-adjointness with differential substitution, adjoint symmetry, conservation law

PACS: 02.20.Sv, 02.30.Jr, 11.30.–j **DOI:** 10.1088/1674-1056/24/2/020201

1. Introduction

The short pulse equation

$$u_{xt} = \frac{1}{6}\left(u^3\right)_{xx} + u, \tag{1}$$

was first introduced by Schäfer and Wayne as a model equation to describe the propagation of ultra-short light pulses in silica optical fibres.[1] Later, the properties of Eq. (1), such as integrability, bi-Hamiltonian structure, and its exact solutions, have been extensively studied. (See Refs. [2]–[4] and the references therein.)

This paper is concerned with the conservation laws of the generalized short pulse equation

$$u_{xt} = [g(u)]_{xx} + f(u), \tag{2}$$

from the point of view of nonlinear self-adjointness with differential substitution, where $f = f(u)$ and $g = g(u)$ are two nonconstant analytic functions of u. Obviously, equation (1) is a particular case of Eq. (2) with $g(u) = u^3/6$ and $f(u) = u$. Complete Lie symmetry classification and reductions of Eq. (2) have been performed in Ref. [5].

The conservation law has an important characteristic of studying partial differential equations (PDEs), such as integrability and linearization, constants of motion, analysis of solutions, and numerical solution methods.[6–13] Thus many researchers have developed new and effective methods to construct conservation laws of PDEs.[13–17] The method that we have adopted here is the nonlinear self-adjointness with differential substitution.[16,17] Approximate nonlinear self-adjointness and approximate conservation laws were studied in Refs. [17]–[19].

We will now briefly review the main idea of nonlinear self-adjointness with differential substitution. Consider a system of m PDEs with r-th order

$$E^\alpha(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{3}$$

where $x = (x^1, \ldots, x^n)$ is an independent variable set and $u = (u^1, \ldots, u^m)$ is a dependent variable set, $u_{(i)}$ denotes all i-th x derivatives of u. Note that the summation convention for repeated indices will be used unless otherwise stated.

Let $\mathcal{L}$ be the formal Lagrangian of system (3), which is written as

$$\mathcal{L} = v^\beta E^\beta(x, u, u_{(1)}, \ldots, u_{(r)}), \tag{4}$$

where v^β are the introduced new dependent variables, and then the adjoint equations of system (3) will be defined by

$$(E^\alpha)^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \tag{5}$$

where $\delta/\delta u^\alpha$ is the Euler operator written as

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}}.$$

The definition of nonlinear self-adjointness with differential substitution is then given as follows.[16,17]

Definition 1 The system (3) is said to be nonlinearly self-adjoint with differential substitution if the adjoint system (5) is satisfied for all solutions of system (3) upon a differential substitution $v = \varphi(x, u, u_{(1)}, \ldots, u_{(s)})$ such that $v \neq 0$.

Definition 1 means that the adjoint system (5), after having been inserted by the differential substitution $v = \varphi(x, u, u_{(1)}, \ldots, u_{(s)})$, holds identically in the solution space of the original system (3), thus Definition 1 is equivalent to the following equality:

$$(E^\alpha)^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)})\big|_{v=\varphi(x,u,u_{(1)},\ldots,u_{(s)})} = 0, \quad \text{when } E^\alpha = 0, \tag{6}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11301012 and 11271363), the Excellent Young Teachers Program of North China University of Technology (Grant No. 14058) and the Doctoral Fund of North China University of Technology (Grant No. 41).

†Corresponding author. E-mail: zhiyong-2008@163.com

© 2015 Chinese Physical Society and IOP Publishing Ltd

which is called the determining system of differential substitution.

Since the key point of studying nonlinear self-adjointness with differential substitution is to search for the differential substitution, in the next section we will show that the adjoint symmetries admitted by Eq. (2) are the required differential substitutions, and then give the general conservation law formula.

2. Nonlinear self-adjointness with differential substitution

Following the idea of nonlinear self-adjointness with differential substitution, we first introduce the formal Lagrangian of Eq. (2) in the symmetric form of mixed derivatives

$$\mathcal{L} = v\left[\frac{1}{2}u_{xt} + \frac{1}{2}u_{tx} - f(u) - g'(u)u_{xx} - g''(u)u_x^2\right], \tag{7}$$

where v is a new introduced dependent variable, and then the adjoint equation of Eq. (2) is

$$\frac{\delta \mathcal{L}}{\delta u} = D_t D_x v - f'(u)v - g'(u)D_x^2 v = 0.$$

Assume the differential substitution $v = \psi(x,t,u,u_x,u_t,\dots)$, which does not contain u_{xt} nor its differential results since they can be expressed through Eq. (2), then by the differential substitution determining equations (6), we will obtain

$$D_t D_x v - f'(u)v - g'(u)D_x^2 v = 0, \tag{8}$$

when equation (2) holds.

On the other hand, by means of the infinitesimal symmetry criterion for PDEs,[16] the determining equation of a generalized symmetry $X = \varphi(x,t,u,u_x,u_t,\dots)\partial_u$ is

$$\begin{aligned}\mathscr{D}_t\mathscr{D}_x\varphi - \left[f'(u) + g^{(3)}(u)u_x^2 + g''(u)u_{xx}\right]\varphi \\ - 2g''(u)u_x\mathscr{D}_x\varphi - g'(u)\mathscr{D}_x^2\varphi = 0,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathscr{D}_t &= \partial_t + u_t\partial_u + \left[f(u) + g'(u)u_{xx} + g''(u)u_x^2\right]\partial_{u_x} + u_{tt}\partial_{u_t}\dots \\ \mathscr{D}_x &= \partial_x + u_x\partial_u + \left[f(u) + g'(u)u_{xx} + g''(u)u_x^2\right]\partial_{u_t} + u_{xx}\partial_{u_x}\dots\end{aligned}$$

are total derivative operators which express u_{xt} and their derivatives through Eq. (2). The adjoint equation of Eq. (9) is

$$\mathscr{D}_t D_x\omega - f'(u)\omega - g'(u)D_x^2\omega = 0, \tag{10}$$

which is the determining equation of an adjoint symmetry $X = \omega(x,t,u,u_x,u_t,\dots,)\partial_u$. Obviously, solutions of Eq. (10) are not solutions of Eq. (9), thus in general an adjoint symmetry is not a symmetry of Eq. (2).

Obviously, in the solution space of Eq. (2), the determining equation of differential substitution (8) is identical to the adjoint symmetry determining equation (10), thus we have the following theorem:

Theorem 1 Any adjoint symmetry of Eq. (2) is a differential substitution of nonlinear self-adjointness, and vice versa.

Thus, in order to find the differential substitutions of nonlinear self-adjointness, we will perform adjoint symmetry classification of Eq. (2).

2.1. Classification of adjoint symmetry

To simplify our calculations, we use the following equivalence transformation for Eq. (2). In particular, by simple calculations, we have the following proposition:

Proposition 1 Any transformation of the form

$$\begin{aligned}&\tilde{x} = c_1x + c_2, \quad \tilde{t} = c_3t + c_4, \\ &\tilde{u} = c_5u + c_6, \quad \tilde{f}(\tilde{u}) = c_5(c_1c_3)^{-1}f(u), \\ &\tilde{g}(\tilde{u}) = c_1c_5c_3^{-1}g(u),\end{aligned} \tag{11}$$

where $c_1c_3c_5 \neq 0$, is an equivalence transformation of Eq. (2), i.e., transformation (11) maps Eq. (2) into

$$\tilde{u}_{\tilde{x}\tilde{t}} = [\tilde{g}(\tilde{u})]_{\tilde{x}\tilde{x}} + \tilde{f}(\tilde{u}).$$

In what follows, we consider the adjoint symmetry of Eq. (2) in the form of

$$\omega = A(x,t,u)u_x + B(x,t,u)u_t + C(x,t,u),$$

and then, by substituting it into Eq. (10) and separating with respect to x-, t-derivatives of u, we obtain $B = \alpha(t), A = -\alpha(t)g'(u) + \beta(x,t)$, and $C = \gamma(t)u + \mu(x,t)$, where α,β,γ, and μ satisfy

$$\begin{aligned}&\beta g'' = \beta_t - g'\beta_x = 0, \\ &g''(\beta_x + \gamma) = \beta_{xt} + \gamma' - g'\beta_{xx} = 0, \\ &f(\beta_x + \alpha' + \gamma) - \gamma u f' - \mu f' - g'\mu_{xx} + \mu_{xt} = 0.\end{aligned} \tag{12}$$

For arbitrary functions $f(u)$ and $g(u)$, the differential substitution is $\omega = u_t - g'(u)u_x$. For special $f(u)$ and $g(u)$, the differential substitutions of nonlinear self-adjointness are extended. Note that c_i $(i = 1,\dots,6)$ are arbitrary constants such that the differential substitution $\omega \neq 0$.

2.1.1. $g(u) = u$

In this case, equation (2) becomes

$$u_{xt} = u_{xx} + f(u), \tag{13}$$

which is obtained from the variational principle with the Lagrangian $L = \frac{1}{2}u_tu_x - \frac{1}{2}u_x^2 - \int f(u)\,\mathrm{d}u$, thus the adjoint symmetry and symmetry are identical.

Then, after routine computations, for arbitrary $f(u)$, the symmetry is

$$\omega = (c_1t + c_2)u_t - [c_1(x + 2t) + c_2 + c_3]u_x.$$

In particular, for $f(u)=u^p(p\neq 1)$, the symmetry is then

$$\omega = (c_1t+c_2)u_t-[c_1(x+2t)+c_2+c_3]\times u_x+c_4u+c_4(p-1)(x+t)u_x.$$

2.1.2. $g(u)\neq u$

In this case, one has $\beta=\gamma=0$. If $f(u)=u$, there is no additional new differential substitution to be found. Thus for $f(u)\neq u$, equation (12) becomes

$$\begin{aligned}&\mu_{xt}=0,\\&\alpha' f-\mu f'-g'\mu_{xx}=0.\end{aligned}\tag{14}$$

From the first equation in Eq. (14), we obtain $\mu=m(x)+n(t)$, where $m(x)$ and $n(t)$ are two arbitrary functions. The second equation then becomes

$$\alpha' f-nf'=mf'+m_{xx}g'.$$

Since the left side in the above equation is a function of t and u while the right side is a function about x and u, we thus have $f=f'=g'$, which means $f=g=\mathrm{e}^u$, and

$$\alpha'-n=m+m_{xx}=c,$$

which gives $n=\alpha'-c$, $m=c_1\sin x+c_2\cos x+c$, thus the adjoint symmetry is

$$\omega=\alpha u_t-\alpha\mathrm{e}^u u_x+\alpha'+c_1\sin x+c_2\cos x,$$

where $\alpha=\alpha(t)$ is an arbitrary function.

2.2. Conservation law

We will use the following theorem to construct conservation laws.[20]

Theorem 2 Any infinitesimal symmetry (local and nonlocal)

$$X=\xi^i(x,u,u_{(1)},\dots)\frac{\partial}{\partial x^i}+\zeta^\sigma(x,u,u_{(1)},\dots)\frac{\partial}{\partial u^\sigma}$$

of system (3) leads to a conservation law $D_i(C^i)=0$ that is constructed by the formula

$$\begin{aligned}C^i=\ &\xi^i\mathcal{L}+W^\sigma\left[\frac{\partial\mathcal{L}}{\partial u_i^\sigma}-D_j\Big(\frac{\partial\mathcal{L}}{\partial u_{ij}^\sigma}\Big)+D_jD_k\Big(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\sigma}\Big)-\cdots\right]\\&+D_j(W^\sigma)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\sigma}-D_k\Big(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\sigma}\Big)+\cdots\right]\\&+D_jD_k(W^\sigma)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\sigma}-\cdots\right]+\cdots,\end{aligned}\tag{15}$$

where $W^\sigma=\zeta^\sigma-\xi^j u_j^\sigma$ and $\mathcal{L}$ is the formal Lagrangian, which is written in the symmetric form in the mixed derivatives.

By Theorem 2, after substituting formal Lagrangian (7) into Eq. (15) and then performing proper arrangements, we obtain the following conservation law formula of Eq. (2).

Theorem 3 A conservation law $D_tC^t_{(2)}+D_xC^x_{(2)}=0$ of Eq. (2) is given by

$$\begin{aligned}C^t_{(2)}&=\frac{1}{2}vD_x\eta-\frac{1}{2}\eta D_xv,\\C^x_{(2)}&=\Big(g'D_xv-g''u_xv-\frac{1}{2}D_tv\Big)\eta-g'vD_x\eta+\frac{1}{2}vD_t\eta,\end{aligned}$$

where the differential substitution $v=\psi$ determined by Eq. (8) and $X=\eta(x,t,u,u_x,u_t,\dots)\partial_u$ is a generalized symmetry of Eq. (2).

In what follows, we consider three special examples to illustrate the effectiveness of nonlinear self-adjointness with differential substitution in the construction of conservation laws. Note that for the following examples $X=\eta(x,t,u,u_x,u_t,\dots)\partial_u$ is a generalized symmetry of the corresponding PDEs.

Example 1 The first example is to consider Eq. (2). For arbitrary functions $f(u)$ and $g(u)$, the differential substitution is $v=u_t-g'u_x$, thus by Theorem 3 and considering $D_xv=u_{xt}-(g'u_x)_x=f$, we have

$$\begin{aligned}C^t_{(2)}&=-f\eta,\\C^x_{(2)}&=fg'\eta+(u_t-g'u_x)(D_t\eta-g'D_x\eta-g''u_x\eta).\end{aligned}$$

When $g(u)=u^3/6, f(u)=u$, equation (2) becomes the short pulse equation (1), which is admitted by the infinitesimal operators $X_1=u_x\partial_u, X_2=u_t\partial_u$, and $X_3=(u-xu_x+tu_t)\partial_u$. Then by using $\eta=u-xu_x+tu_t$ and after proper arrangements, we obtain

$$\begin{aligned}C^t_{(2)}&=u^2,\\C^x_{(2)}&=u^2u_xu_t-u_t^2-\frac{1}{4}u^4(1+u_x^2),\end{aligned}$$

which is identical to the results in Ref. [17].

Example 2 The second example to be considered is

$$u_{xt}=u_{xx}+f(u).\tag{16}$$

For arbitrary function $f(u)$, the new added differential substitutions are $v_1=u_x$ and $v_2=tu_t-xu_x-2tu_x$. Then for the differential substitution $v_2+q_1v_1$ with an arbitrary constant q_1, the conservation laws $D_tC^t_{(16)}+D_xC^x_{(16)}=0$ of Eq. (16) are given by

$$\begin{aligned}C^t_{(16)}&=(xu_x+2tu_x-tu_t-q_1u_x)D_x\eta,\\C^x_{(16)}&=(q_1+1)f\eta+(tu_t-xu_x-2tu_x+q_1u_x)D_x\eta.\end{aligned}$$

In particular, for $f(u)=u^p(p\neq 1)$, the extended differential substitution is $v_3=u+(p-1)(x+t)u_x$, we then have

$$\begin{aligned}C^t_{(16)}&=\big(u+(p-1)(x+t)u_x\big)D_x\eta,\\C^x_{(16)}&=-u^p\eta-\big(u+(p-1)(x+t)u_x\big)D_x\eta.\end{aligned}$$

Example 3 The last example is

$$u_{xt}=(\mathrm{e}^uu_x)_x+\mathrm{e}^u,\tag{17}$$

whose conservation laws $D_tC^t_{(17)}+D_xC^x_{(17)}=0$ of Eq. (17) are

$$C^t_{(17)}=vD_x\eta,$$
$$C^x_{(17)}=(e^uD_xv-e^uu_xv-D_tv)\,\eta-e^uvD_x\eta,$$

where the differential substitutions $v=\alpha u_t-\alpha e^u u_x+\alpha'+c_1\sin x+c_2\cos x$ with an arbitrary function $\alpha=\alpha(t)$.

3. Conclusion

We show that the generalized short pulse equation is nonlinearly self-adjoint with differential substitution and its adjoint symmetries are the required differential substitutions. The general conservation law formula is then presented and new conservation laws are constructed.

References

[1] Schäfer T and Wayne C E 2004 *Physica D* **196** 90
[2] Brunelli J C 2006 *Phys. Lett. A* **353** 475
[3] Fu Z T, Chen Z, Zhang L N, Mao J Y and Liu S K 2010 *Appl. Math. Comput.* **215** 3899
[4] Sakovich A and Sakovich S 2005 *J. Phys. Soc. Jpn.* **74** 239
[5] Liu H, Li J and Liu L 2012 *Stud. Appl. Math.* **129** 103
[6] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer-Verlag)
[7] Wang X, Chen Y and Dong Z Z 2014 *Chin. Phys. B* **23** 010201
[8] Lou S Y, Hu X R and Chen Y 2012 *J. Phys. A: Math. Theor.* **45** 155209
[9] Song J F, Qu C Z and Qiao Z J 2011 *J. Math. Phys.* **52** 013503
[10] Su J R, Zhang S L and Li J N 2010 *Commun. Theor. Phys.* **53** 37
[11] Zhao G L, Chen L Q, Fu J L and Hong F Y 2013 *Chin. Phys. B* **22** 030201
[12] Sun X T, Han Y L, Wang X X, Zhang M L and Jia L Q 2012 *Acta Phys. Sin.* **61** 200204 (in Chinese)
[13] Bluman G W, Cheviakov A F and Anco S C 2010 *Applications of Symmetry Methods to Partial Differential Equations* (New York: Springer-Verlag)
[14] Kara A H and Mahomed F M 2006 *Nonlinear Dyn.* **45** 367
[15] Naz R, Mahomed F M and Mason D P 2008 *Appl. Math. Comput.* **205** 212
[16] Gandarias M L 2014 *Commun. Nonlinear Sci. Numer. Simul.* **19** 3523
[17] Ibragimov N H 2011 *Archives of ALGA* **7/8** 1
[18] Zhang Z Y 2013 *J. Phys. A: Math. Theor.* **46** 155203
[19] Z Y and Chen Y F 2014 *IMA J. Appl. Math. first published online May 8, 2014*
[20] Ibragimov N H 2007 *J. Math. Anal. Appl.* **333** 311