A note on local unitary equivalence of isotropic-like states*

Zhang Ting-Gui(张廷桂)[a)†], Hua Bo-Bo(华波波)[b)], Li Ming(李 明)[c)],
Zhao Ming-Jing(赵明镜)[d)‡], and Yang Hong(杨 红)[e)]

[a)] School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China
[b)] School of Mathematical Sciences, LMNS, Fudan University, Shanghai 200433, China
[c)] College of Science, China University of Petroleum, Qingdao 266580, China
[d)] Department of Mathematics, School of Science, Beijing Information Science and Technology University, Beijing 100192, China
[e)] College of Physics and Electronic Engineering, Hainan Normal University, Haikou 571158, China

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We consider the local unitary equivalence of a class of quantum states in a bipartite case and a multipartite case. The necessary and sufficient condition is presented. As special cases, the local unitary equivalent classes of isotropic state and Werner state are provided. Then we study the local unitary similar equivalence of this class of quantum states and analyze the necessary and sufficient condition.

Keywords: mixed state, local unitary equivalence, local unitary similar equivalence

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1. Introduction

Entanglement is one of the most extraordinary features of quantum physics. It plays a vital role in quantum information processing, including quantum teleportation, quantum cryptography, quantum computation, etc.[1] Two entangled states are said to be equivalent in implementing the same quantum information task if they can be obtained from each other via local operation and classical communication (LOCC). In particular, all the LOCC equivalent quantum pure states are interconvertible by local unitary operators (LU).[2] As many properties like quantum correlation,[3] quantum entanglement,[4] and quantum discord[5,6] keep invariant under local unitary transformations, it is significant to classify and characterize quantum states in terms of local unitary transformations.

There are a lot of researches to deal with the LU problem, one approach is to construct invariants of local unitary transformations.[7–14] Usually the invariants of mixed states are dependent on pure state decomposition. Recently, the invariants of bipartite states independent of the pure states decomposition are studied in Ref. [15]. The LU problem for multipartite pure qubits states has been solved in Refs. [16] and [17]. By exploiting the high order singular value decomposition technique and local symmetries of the states, Reference [18] presents a practical scheme of classification under local unitary transformations for general multipartite pure states with arbitrary dimensions, which extends the results of n-qubit pure states[16,17] to those of n-qudit pure states. For mixed states, Reference [19] solved the LU problem of arbitrary dimensional bipartite non-degenerated quantum systems by presenting a complete set of invariants, such that two density matrices are locally unitary equivalent if and only if all these invariants have equal values. In Ref. [20] the case of multipartite systems is studied and a complete set of invariants is presented for a special class of mixed states. Recently, we have studied the local unitary equivalence of multipartite mixed states by using the technology of matrix realignment and partial transpose[21] and solved the LU problem for multiqubit mixed states with Bloch representation.[22]

In this paper, we study the LU problem for a special class of quantum states. The necessary and sufficient condition is provided. Especially, the local unitary equivalence class of isotropic states[23] and Werner states[24] are obtained. Then we study the local unitary similar equivalence of this class of states and give the necessary and sufficient condition.

2. Local unitary equivalence

Two multipartite mixed states ρ and ρ' in $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ are said to be equivalent under local unitary transformations if there exist unitary operators U_i on the i-th Hilbert space H_i such that

$$\rho' = (U_1 \otimes U_2 \otimes \cdots \otimes U_n)\rho(U_1 \otimes U_2 \otimes \cdots \otimes U_n)^{\dagger}. \quad (1)$$

First, we consider the case of a bipartite system. Let H be an N-dimensional complex Hilbert space with $|i\rangle$, $i =$

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†Corresponding author. E-mail: tinggui333@163.com
‡Corresponding author. E-mail: zhaomingjingde@126.com

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$1,2,\ldots,N$ being an orthonormal basis. A general pure state on $H\otimes H$ is in the form of

$$|\phi\rangle = \sum_{i,j=1}^{N} a_{ij}|i\rangle\otimes|j\rangle,\ \ a_{ij}\in\mathbb{C}, \tag{2}$$

with the normalization $\sum_{i,j=1}^{N} a_{ij}a_{ij}^{*}=1$ (x^{*} denotes the complex conjugation of x). Let A denote the matrix given by $(A)_{ij}=a_{ij}$, we call A the matrix representation of pure state $|\phi\rangle$. The following quantities are associated with the state $|\phi\rangle$ and given by Eq. (2).

$$I_{\alpha} = \mathrm{Tr}(AA^{\dagger})^{\alpha},\ \ \alpha=1,2,\ldots,N, \tag{3}$$

where $A^{\dagger}$ denotes the adjoint of the matrix A. As is well known, two bipartite pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ in $H\otimes H$ are local unitary equivalent if and only if their matrix representations give the same values of quantities (3).

Here we mainly consider the local unitary equivalence of quantum states

$$\rho_1 = \frac{p_0}{N^2}I_N\otimes I_N + \sum_{i=1}^{K} p_i|\phi_i\rangle\langle\phi_i|, \tag{4}$$

and

$$\rho_2 = \frac{p_0}{N^2}I_N\otimes I_N + \sum_{i=1}^{K} p_i|\varphi_i\rangle\langle\varphi_i| \tag{5}$$

with $p_i\geq 0$ for $i=0,\ldots,K$, $\sum_{i=0}^{K}p_i=1$, and $p_i\neq p_j$ for $1\leq i<j\leq K$, $1\leq K\leq N^2$.

Lemma 1 Two arbitrary dimensional bipartite non-degenerate density matrices are equivalent under local unitary transformations if and only if there exist eigenstate decompositions $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$ such that the following invariants have the same values for both density matrices:

$$J^{s} = \mathrm{Tr}_2(Tr_1\rho^{s}),\ \ s=1,\ldots,N^2, \tag{6}$$

$$\mathrm{Tr}[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})]. \tag{7}$$

Proposition 1 For two bipartite mixed states in Eqs. (4) and (5), they are local unitary equivalent if and only if the corresponding matrix representations of $|\phi_i\rangle$ and $|\varphi_i\rangle$ yield the same values of the invariants (7).

Proof If ρ_1 and ρ_2 are local unitary equivalent, then $|\phi_i\rangle$ and $|\varphi_i\rangle$ are local unitary equivalent under the same local unitary operators. Therefore, $|\phi_i\rangle$ and $|\varphi_i\rangle$ give rise to the same values of the invariants (7).

On the other hand, if $|\phi_i\rangle$ and $|\varphi_i\rangle$ give rise to the same values of invariants (7) by Lemma 1, $|\phi_i\rangle$ and $|\varphi_i\rangle$ are local unitary equivalent under the same local unitary operators, hence ρ_1 and ρ_2 are local unitary equivalent.

Remark 1 In fact, if the eigenvalues are not all positive in Proposition 1, then the conclusion still holds true. Proposition 1 can be used to solve the local unitary equivalence of mixed state with only one degenerate eigenvalue. Because if one state has only one degenerate eigenvalue, then it can be transformed into the form like Proposition 1. That is,

$$\rho = \lambda_1|v_1\rangle\langle v_1| + \cdots + \lambda_s|v_s\rangle\langle v_s| + \sum_{i=s+1}^{N^2}\lambda_0|v_i\rangle\langle v_i|,$$

equivalently,

$$\rho = \lambda_0 I_{N^2} + (\lambda_1-\lambda_0)|v_1\rangle\langle v_1| + \cdots + (\lambda_s-\lambda_0)|v_s\rangle\langle v_s|,$$

where $\lambda_i\neq\lambda_j,\ \ i\neq j,\ i,j=0,1,\ldots,s$.

Now we can analyze the LU problem in a two-qubit system. First, when the quantum state has non-degenerate eigenvalues, then Lemma 1 is sufficient to determine the local unitary equivalence. Second, when the quantum state has eigenvalues with multiplicity not larger than 2, then one can solve the local unitary equivalence by the method proposed in Ref. [21]. Finally, if there is only one degenerate eigenvalue, then Proposition 1 can be used to deal with the LU problem of quantum states. Therefore, the LU problem of two-qubit quantum states can be solved in this way.

This proposition can also be used to judge which states are equivalent to isotropic states[23] under local unitary transformations, which are invariant under the transformation of the form $(U\otimes U^{*})\rho(U\otimes U^{*})^{\dagger}$. The isotropic state can be written as the mixture of the maximally mixed state and the maximally entangled state $|\psi^{+}\rangle=\frac{1}{\sqrt{d}}\sum_{a=0}^{d-1}|aa\rangle$,

$$\rho_{\mathrm{isot}} = \frac{p}{d^2}I_d\otimes I_d + (1-p)|\psi^{+}\rangle\langle\psi^{+}|,$$

where $0\leq p\leq 1$. Following Proposition 1, the state that is local unitary equivalent to the isotropic states is in the form of

$$\rho = \frac{p}{d^2}I_d\otimes I_d + (1-p)|\psi'\rangle\langle\psi'|,$$

where $|\psi'\rangle$ is a maximally entangled state.

Subsequently, we consider the states that are local unitary equivalent to Werner states.[24] We need the technique of partial transpose of states. For a density matrix ρ in $H_1\otimes H_2$ with elements $\rho_{m\mu,nv}=\langle e_m\otimes f_\mu|\rho|e_n\otimes f_v\rangle$, the partial transposition of ρ is defined by[25]

$$\rho^{\mathrm{T}_2} = (I\otimes T)\rho = \sum_{mn,\mu v}\rho_{mv,n\mu}|e_m\otimes f_\mu\rangle\langle e_n\otimes f_v|,$$

where ρ^{T_2} denotes the transposition of ρ with respect to the second system, $|e_n\rangle$ and $|f_v\rangle$ are the bases associated with spaces H_1 and H_2 respectively. The LU problem of the original states can be transformed into that of theirs partial transposed states,[21] since two mixed states ρ_1 and ρ_2 in $H_1\otimes H_2$ are local unitary equivalent if and only if $\rho_1^{\mathrm{T}_2}$ and $\rho_2^{\mathrm{T}_2}$ are local unitary equivalent.

The arbitrary dimensional Werner states[24] are invariant under the transformations $(U\otimes U)\rho(U\otimes U)^{\dagger}$ for any unitary U. They can be written as

$$\rho_w = \frac{1}{d^3-d}[(d-f)I_d\otimes I_d + (df-1)\sum_{ij}|ij\rangle\langle ji|],$$

where $-1 \leq f \leq 1$. The partial transpose of ρ_w is

$$\rho_w^{\mathrm{T}_2} = \frac{1}{d^3-d}(d-f)I_d \otimes I_d + \frac{(df-1)}{d^2-1}|\psi^+\rangle\langle\psi^+|.$$

Therefore, the state that is local unitary equivalent to the Werner states is in the form of

$$\rho = \frac{1}{d^3-d}(d-f)I_d \otimes I_d + \frac{(df-1)}{d^2-1}(|\psi'\rangle\langle\psi'|)^{\mathrm{T}_2},$$

where $|\psi'\rangle$ is a maximally entangled state.

Now we consider the multipartite case. Before showing the equivalence of multipartite quantum states under local unitary transformations, we give a short review of the high order singular value decomposition developed in Ref. [26]. For any tensor $\mathscr{A}$ with order $d_1 \times d_2 \times \cdots \times d_N$, there exists a core tensor Σ such that

$$\mathscr{A} = (U_1 \otimes U_2 \otimes \cdots \otimes U_N)\Sigma, \tag{8}$$

where Σ forms the same order tensor as $\mathscr{A}$. Any $(N-1)$-order tensor $\Sigma_{i_n=i}$ obtained by fixing the n-th index to i, has the following properties $\langle \Sigma_{i_n=i}, \Sigma_{i_n=j}\rangle = \delta_{ij}\sigma_i^{(n)^2}$, with $\sigma_i^n \geq \sigma_j^n$ and $\forall i \leq j$ for all possible values of n. Here, the singular value σ_i^n symbolizes the Frobenius norm $\sigma_i^n = \|\Sigma_{i_n=i}\| \equiv \sqrt{\langle \Sigma_{i_n=i}, \Sigma_{i_n=i}\rangle}$, where the inner product $\langle A, B\rangle \equiv \Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_N} b_{i_1 i_2 \cdots i_N} a^*_{i_1 i_2 \cdots i_N}$.

To calculate the core tensor Σ, one first expresses $\mathscr{A}$ in matrix unfolding form $\mathscr{A}_n$. Then one derives the singular value decomposition of the matrix $\mathscr{A}_n = U_n \Lambda_n V_n$. The core tensor is then given by

$$\Sigma = \otimes_{n=1}^N U_n^\dagger \mathscr{A}. \tag{9}$$

Lemma 2 Two multipartite pure states are local unitary equivalent if and only if they have the same core tensor up to the local symmetry $\otimes_{n=1}^N P^{(n)}$, where $P^{(n)}$ is a block-diagonal matrix consisting of unitary blocks with the same partitions as those of the identical singular values of $\mathscr{A}_n$.

By Proposition 1 and Lemma 2, one can obtain the following result easily.

Proposition 2 Two multipartite mixed states of the form

$$\rho_1 = \frac{p}{NM\cdots T} I_N \otimes I_M \otimes \cdots \otimes I_T + (1-p)|\phi\rangle\langle\phi|$$

and

$$\rho_2 = \frac{p}{NM\cdots T} I_N \otimes I_M \otimes \cdots \otimes I_T + (1-p)|\varphi\rangle\langle\varphi|$$

are local unitary equivalent if and only if $|\phi\rangle$ and $|\varphi\rangle$ have the same core tensor up to the local symmetry $\otimes_{n=1}^N P^{(n)}$.

Remark 2 Two multipartite mixed states

$$\rho_1 = \frac{p}{NM\cdots T} I_N \otimes I_M \otimes \cdots \otimes I_T + (1-p)\rho$$

and

$$\rho_2 = \frac{p}{NM\cdots T} I_N \otimes I_M \otimes \cdots \otimes I_T + (1-p)\rho'$$

are local unitary equivalent if and only if the corresponding density matrices ρ and ρ' are local unitary equivalent. Therefore, the local unitary equivalence of two quantum states does not change under the disturbance of the white noise. For example,

$$\rho_1 = \frac{p}{8} I_2 \otimes I_2 \otimes I_2 + (1-p)\rho$$

and

$$\rho_2 = \frac{p}{8} I_2 \otimes I_2 \otimes I_2 + (1-p)\rho'$$

with

$$\rho = \frac{q}{2}(|000\rangle + |111\rangle)(\langle 000| + \langle 111|) + (1-q)|111\rangle\langle 111|$$

and

$$\begin{aligned}\rho' = {} & \frac{q}{2}(|001\rangle + |010\rangle + |100\rangle)(\langle 001| + \langle 010| + \langle 100|) \\ & + (1-q)|111\rangle\langle 111|,\end{aligned}$$

are not local unitary equivalent because ρ and ρ' are not local unitary equivalent.[20]

3. Local unitary similar equivalence

Definition 1 If there exists a unitary matrix U such that $(U \otimes U^*)\rho_1(U \otimes U^*)^\dagger = \rho_2$, we call states ρ_1 and ρ_2 being local unitary similar equivalent.

In Ref. [27] the author studies the unitary invariants and unitary similar equivalence, and the following Specht's theorem[28] has been presented. Next we use them to deal with the local unitary similar equivalent problem for bipartite mixed states.

Lemma 3 Let A and B be both $n \times n$ complex matrices. Then A and B are unitary similar, i.e, there is a unitary matrix U, such that $UAU^\dagger = B$, if and only if $\mathrm{tr}(\omega(A, A^\dagger)) = \mathrm{tr}(\omega(B, B^\dagger))$ holds for each word ω, where $\omega(A, A^\dagger)$ is the result of taking any monomial $\omega(x, y)$ in noncommuting variables x and y and replacing x with A and y with $A^\dagger$.

The proof of Specht's theorem can also be applied to two finite sets $\{A_i\}_{i=1}^t$ and $\{B_i\}_{i=1}^t$ of $n \times n$ matrices.[27,29]

Lemma 4 Let $\{A_i\}_{i=1}^t$ and $\{B_i\}_{i=1}^t$ be both $n \times n$ complex matrices. There is a unitary matrix U such that $U^\dagger A_i U = B_i$ for $i = 1, 2, \ldots, t$ if and only if for every word $\omega(x_1, y_1, x_2, y_2, \ldots, x_t, y_t)$ in the noncommuting variables x_i and y_i we have

$$\begin{aligned}&\mathrm{tr}(\omega(A_1, A_1^\dagger, A_2, A_2^\dagger, \ldots, A_t, A_t^\dagger)) \\ = {} & \mathrm{tr}(\omega(B_1, B_1^\dagger, B_2, B_2^\dagger, \ldots, B_t, B_t^\dagger)).\end{aligned}$$

For pure states $|\psi\rangle$ and $|\phi\rangle$ with coefficient matrices A and B respectively, if $U\otimes U^*|\psi\rangle = |\phi\rangle$, then $UAU^\dagger = B$. Utilizing this relation, we can obtain the necessary and sufficient condition for local unitary similar equivalence problem.

Proposition 3 For two bipartite mixed states in Eqs. (4) and (5), they are local unitary similar equivalent if and only if

$$\begin{aligned}&\mathrm{tr}\left(\omega(A_1,A_1^\dagger,A_2,A_2^\dagger,\ldots,A_K,A_K^\dagger)\right)\\&\quad=\mathrm{tr}(\omega(B_1,B_1^\dagger,B_2,B_2^\dagger,\ldots,B_K,B_K^\dagger))\end{aligned}$$

holds true for each word $\omega(x_1,y_1,x_2,y_2,\ldots,x_t,y_t)$ in the non-commuting variables x_i and y_i, with A_i and B_i being both the coefficient matrices of $|\phi_i\rangle$ and $|\varphi_i\rangle$ respectively.

4. Conclusions

In this work, we study the LU problem for a special class of states. The necessary and sufficient condition is provided. Consequently, the local unitary equivalent classes of isotropic state and Werner state are obtained. Then we investigate the local unitary similar equivalence for this class of state and obtain the necessary and sufficient condition.

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