Thermal vacuum state corresponding to squeezed chaotic light and its application*

Wan Zhi-Long(万志龙)[a)b)], Fan Hong-Yi(范洪义)[b)†], and Wang Zhen(王 震)[a)]

[a)] School of Mathematical and Chemical Industry, Changzhou Institute of Technology, Changzhou 213002, China
[b)] Department of Material Science and Engineering, University of Science and Technology of China, Heifei 230026, China

(Received 28 June 2015; revised manuscript received 13 August 2015; published online 20 October 2015)

For the density operator (mixed state) describing squeezed chaotic light (SCL) we search for its thermal vacuum state (a pure state) in the real-fictitious space. Using the method of integration within ordered product (IWOP) of operators we find that it is a kind of one- and two-mode combinatorial squeezed state. Its application in evaluating the quantum fluctuation of photon number reveals: the stronger the squeezing is, the larger a fluctuation appears. The second-order degree of coherence of SCL is also deduced which shows that SCL is classic. The new thermal vacuum state also helps to derive the Wigner function of SCL.

Keywords: thermal vacuum state, squeezed chaotic light, quantum fluctuation of photon number, second-order degree of coherence

PACS: 03.65.–w, 03.65.Yz, 05.30.–d, 42.50.–p **DOI:** 10.1088/1674-1056/24/12/120301

1. Introduction

It is common knowledge that a quantum state at finite temperature is usually expressed by a mixed state ρ, $\rho = \mathrm{e}^{-\beta H}/Z$, H is the system's Hamiltonian, and $Z = \mathrm{tr}\,\mathrm{e}^{-\beta H}$ is the partition function. In Ref. [1] for conveniently calculating the ensemble average of an observable A in a mixed state Umezawa–Takahashi replaced the ensemble average

$$\langle A\rangle = \mathrm{tr}\,(\rho A) \tag{1}$$

by an expectation-value of A in an appropriate pure state $|\psi\rangle$

$$\langle A\rangle = \langle\psi|A|\psi\rangle, \tag{2}$$

where $|\psi\rangle$ is named a thermal vacuum state, and for different density operator ρ it corresponds to different $|\psi\rangle$. For example, for a chaotic field with[2]

$$\rho_{\mathrm{c}} = \sum_{n=0}^{\infty}\gamma(1-\gamma)^n|n\rangle\langle n|, \quad \mathrm{tr}\,\rho_{\mathrm{c}} = 1,$$
$$|n\rangle = a^{\dagger n}|0\rangle/\sqrt{n!}, \quad \gamma = 1-\exp\left(-\frac{\hbar\omega}{\kappa T}\right). \tag{3}$$

Umezawa–Takahashi found the corresponding thermal vacuum state (a pure state)[3,4]

$$|0(\beta)\rangle = \mathrm{sech}\,\theta\exp\left[a^{\dagger}\tilde{a}^{\dagger}\tanh\theta\right]|0,\tilde{0}\rangle = S(\theta)|0,\tilde{0}\rangle, \tag{4}$$

where $|0,\tilde{0}\rangle$ is annihilated by $\tilde{a}$ and a, $[\tilde{a},\tilde{a}^{\dagger}] = 1$, and $S(\theta)$ is the thermal operator, $S(\theta) \equiv \exp\left[\theta\left(a^{\dagger}\tilde{a}^{\dagger} - a\tilde{a}\right)\right]$, which has the similar form to a two-mode squeezing operator except for the tilde mode in $S(\theta)$ being a fictitious one, and θ is a parameter related to the temperature by $\tanh\theta = \exp(-\hbar\omega/2\kappa T)$, κ is the Boltzmann constant.

In our opinion, such a theory not only simplifies the statistical average calculation, but also provides an explicit two-mode state exhibiting quantum entanglement between system and reservoir. Then we are challenged by the question: What is the thermal vacuum state[5–8] for the squeezed chaotic light (SCL)? We shall employ the method of integration within ordered product (IWOP)[9] of operators to find it, which can greatly simplify photon number average and quantum fluctuation calculation for SCL.

Our paper is arranged as follows. In Section 2, we briefly review the themal vacuum state theory and the partial trace method. In Section 3, we deduce the normal ordering form of the density operator for the squeezd chaotic light by using of the IWOP method, and obtain the thermal vacuum state corresponding to SCL, which turns out to be a one- and two-combinatorial mode squeezed state. As applications, in Sections 4–5 we employ this new thermal vacuum state to calculate the photon number distribution and fluctuation, and the second-order degree of coherence. In Section 6 we derive the Wigner function of SCL.[10–12] A brief summary is given in the last section.

2. Brief review of the thermal vacuum state theory

For a given real optical field described by density operator ρ, how can one get its thermal vacuum state $|\psi(\beta)\rangle$? In other words, which condition must be required for the thermal vacuum state $|\psi(\beta)\rangle$ such that its partial trace can lead to ρ?

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11175113, 11447202, and 11574295).
†Corresponding author. E-mail: fhym@ustc.edu.cn
© 2015 Chinese Physical Society and IOP Publishing Ltd

Let the symbol $\mathrm{Tr} = \mathrm{tr}\,\widetilde{\mathrm{tr}}$ represent performing trace both for real and ficitious modes, then we have

$$\begin{aligned}\langle A\rangle &= \langle\psi(\beta)|A|\psi(\beta)\rangle = \mathrm{Tr}\left[A|\psi(\beta)\rangle\langle\psi(\beta)|\right]\\ &= \mathrm{tr}\left\{A\left[\widetilde{\mathrm{tr}}|\psi(\beta)\rangle\langle\psi(\beta)|\right]\right\}.\end{aligned} \tag{5}$$

Since $|\psi(\beta)\rangle$ involves both real and fictitious modes, we should note that

$$\widetilde{\mathrm{tr}}|\psi(\beta)\rangle\langle\psi(\beta)| \neq \langle\psi(\beta)|\,\psi(\beta)\rangle. \tag{6}$$

Comparing Eq. (5) with Eq. (1) we see

$$\widetilde{\mathrm{tr}}|\psi(\beta)\rangle\langle\psi(\beta)| = \rho, \tag{7}$$

which indicates that if we can find a pure state $|\psi(\beta)\rangle$ in the enlarged Hilbert space, then the partial trace over the tilde freedom of $|\psi(\beta)\rangle\langle\psi(\beta)|$ should lead to the density operator ρ of the system in a mixed state form.[13]

For example, let us check if $\widetilde{\mathrm{tr}}|0(\beta)\rangle\langle 0(\beta)| = \rho_{\mathrm{c}}$. By employing the completeness relation of coherent state in the fictitious space

$$\int\frac{\mathrm{d}^2z}{\pi}|\tilde{z}\rangle\langle\tilde{z}| = 1,\quad \tilde{a}|\tilde{z}\rangle = z|\tilde{z}\rangle \tag{8}$$

and the normally ordered expansion of $\mathrm{e}^{\lambda a^\dagger a}$

$$\mathrm{e}^{\lambda a^\dagger a} = :\,\exp\left[\left(\mathrm{e}^{\lambda}-1\right)a^\dagger a\right]:, \tag{9}$$

where $:\ :$ denotes normal ordering, as well as the IWOP method we start from Eq. (4) to derive

$$\begin{aligned}\widetilde{\mathrm{tr}}\left[|0(\beta)\rangle\langle 0(\beta)|\right] &= \widetilde{\mathrm{tr}}\left[\int\frac{\mathrm{d}^2z}{\pi}|\tilde{z}\rangle\langle\tilde{z}|\,0(\beta)\rangle\langle 0(\beta)|\right]\\ &= \mathrm{sech}^2\theta\int\frac{\mathrm{d}^2z}{\pi}\langle\tilde{z}|\,\mathrm{e}^{a^\dagger\tilde{a}^\dagger\tanh\theta}\left|0,\tilde{0}\right\rangle\left\langle 0,\tilde{0}\right|\mathrm{e}^{a\tilde{a}\tanh\theta}|\tilde{z}\rangle = \mathrm{sech}^2\theta\int\frac{\mathrm{d}^2z}{\pi}\langle\tilde{z}|\,\mathrm{e}^{a^\dagger z^*\tanh\theta}\left|0,\tilde{0}\right\rangle\left\langle 0,\tilde{0}\right|\mathrm{e}^{az\tanh\theta}|\tilde{z}\rangle\\ &= \mathrm{sech}^2\theta\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{a^\dagger z^*\tanh\theta}|0\rangle\langle 0|\,\mathrm{e}^{az\tanh\theta}\,\mathrm{e}^{-|z|^2} = \mathrm{sech}^2\theta\int\frac{\mathrm{d}^2z}{\pi}:\mathrm{e}^{-|z|^2+a^\dagger z^*\tanh\theta+az\tanh\theta-a^\dagger a}:\\ &= \mathrm{sech}^2\theta:\mathrm{e}^{a^\dagger a\left(\tanh^2\theta-1\right)}: \equiv \gamma:\exp\left[-\gamma a^\dagger a\right]:\\ &= \mathrm{sech}^2\theta\,\mathrm{e}^{a^\dagger a\ln\tanh^2\theta}\overset{\tanh\theta=\exp(-\hbar\omega/2\kappa T)}{=\!=\!=}\left(1-\mathrm{e}^{-\hbar\omega/kT}\right)\exp\left(-\frac{\hbar\omega}{kT}a^\dagger a\right) = \rho_{\mathrm{c}},\end{aligned} \tag{10}$$

which reveals that performing a trace over the fictitious mode of $|0(\beta)\rangle$ really leads to ρ_{c} of the chaotic optical field.

3. The thermal vacuum state for the squeezed chaotic light

We now turn to the density operator of squeezed chaotic light[14]

$$\begin{aligned}&\rho_{\mathrm{s}} = \left(1-\mathrm{e}^{\lambda}\right)S(r)\,\mathrm{e}^{\lambda a^\dagger a}S^{-1}(r) = S(r)\,\rho_{\mathrm{c}}S^{-1}(r),\\ &\lambda = -\frac{\hbar\omega}{kT},\end{aligned} \tag{11}$$

where

$$S(r) = \exp\left[\frac{r}{2}\left(a^{\dagger 2}-a^2\right)\right] = \exp\left[-\frac{\mathrm{i}}{2}(QP+PQ)r\right] \tag{12}$$

is the single-mode squeezing operator

$$SPS^{-1} = \mathrm{e}^{r}P,\quad SQS^{-1} = \mathrm{e}^{-r}Q \tag{13}$$

and

$$Q = \frac{a^\dagger+a}{\sqrt{2}},\quad P = \frac{a-a^\dagger}{\sqrt{2}\mathrm{i}}. \tag{14}$$

In order to get the thermal vacuum state of ρ_{s} we first derive the normal ordering form of ρ_{s} with the help of the IWOP method. Using the Weyl expansion formula of an operator A in the Fock space[15]

$$A = 2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\,\langle-\beta|A|\beta\rangle\exp\left[2\left(\beta^*a-\beta a^\dagger+a^\dagger a\right)\right]\vdots, \tag{15}$$

where $\vdots\ \vdots$ denotes the Weyl ordering symbol,

$$|\beta\rangle = \exp\left(-\frac{1}{2}|\beta|^2+\beta a^\dagger\right)|0\rangle$$

is a coherent state, we have

$$\begin{aligned}\mathrm{e}^{\lambda a^\dagger a} &= 2\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|:\exp\left[\left(\mathrm{e}^{\lambda}-1\right)a^\dagger a\right]:|\beta\rangle\vdots\exp\left[2\left(\beta^*a-\beta a^\dagger+a^\dagger a\right)\right]\vdots\\ &= 2\int\frac{\mathrm{d}^2\beta}{\pi}\vdots\exp\left[-\left(\mathrm{e}^{\lambda}+1\right)|\beta|^2+2\left(\beta^*a-\beta a^\dagger+a^\dagger a\right)\right]\vdots\\ &= \frac{2}{\mathrm{e}^{\lambda}-1}\vdots\exp\left[2\frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1}a^\dagger a\right]\vdots = \frac{2}{\mathrm{e}^{\lambda}-1}\vdots\exp\left[\frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1}\left(P^2+Q^2\right)\right]\vdots.\end{aligned} \tag{16}$$

According to the property that Weyl ordered operators are ordering-invariant under similar transformations,[15] thus using Eqs. (16) and (13) we have

$$\rho_{\rm s}=\left(1-{\rm e}^{\lambda}\right)S(r)\,{\rm e}^{\lambda a^{\dagger}a}S^{-1}(r)=\frac{2\left(1-{\rm e}^{\lambda}\right)}{{\rm e}^{\lambda}+1}\vdots\exp\left\{\frac{{\rm e}^{\lambda}-1}{{\rm e}^{\lambda}+1}\left[{\rm e}^{2r}P^{2}+{\rm e}^{-2r}Q^{2}\right]\right\}\vdots. \tag{17}$$

Thus replacing P,Q by p,q respectively, the classical Weyl correspondence of $\rho_{\rm s}$ is

$$\frac{2\left(1-{\rm e}^{\lambda}\right)}{{\rm e}^{\lambda}+1}\exp\left\{\frac{{\rm e}^{\lambda}-1}{{\rm e}^{\lambda}+1}\left[{\rm e}^{2r}p^{2}+{\rm e}^{-2r}q^{2}\right]\right\}\equiv h(q,p), \tag{18}$$

which means that the classical correspondence of a Weyl ordered operator $\vdots h(Q,P)\vdots$ can be directly obtained by simply replacing $Q\longrightarrow q$, $P\longrightarrow p$. Then according to the Weyl correspondence rule, we have

$$\rho_{\rm s}=\iint_{-\infty}^{\infty}{\rm d}p{\rm d}q h(q,p)\Delta(q,p), \tag{19}$$

where $\Delta(q,p)$ is the Wigner operator whose normal ordering form[16] is

$$\Delta(q,p)=\frac{1}{\pi}:{\rm e}^{-(q-Q)^{2}-(p-P)^{2}}:. \tag{20}$$

Substituting Eqs. (18) and (20) into Eq. (19), we can get the normal ordering form of density operator for the squeezed chaotic light

$$\begin{aligned}\rho_{\rm s}&=\frac{2\left(1-{\rm e}^{\lambda}\right)}{\pi\left({\rm e}^{\lambda}+1\right)}\iint_{-\infty}^{\infty}{\rm d}p{\rm d}q:{\rm e}^{-(q-Q)^{2}-(p-P)^{2}}:\exp\left\{\frac{{\rm e}^{\lambda}-1}{{\rm e}^{\lambda}+1}\left[{\rm e}^{2r}p^{2}+{\rm e}^{-2r}q^{2}\right]\right\}\\&=\frac{1}{\tau_1\tau_2}:\exp\left[-\frac{Q^{2}}{2\tau_1^{2}}-\frac{P^{2}}{2\tau_2^{2}}\right]:,\end{aligned} \tag{21}$$

where

$$2\tau_1^{2}=(2\bar{n}+1)\,{\rm e}^{2r}+1,\quad 2\tau_2^{2}=(2\bar{n}+1)\,{\rm e}^{-2r}+1,\quad \bar{n}=\frac{1}{{\rm e}^{-\lambda}-1}. \tag{22}$$

Using Eq. (15) we can re-express Eq. (21) as

$$\begin{aligned}\rho_{\rm s}&=\frac{1}{\tau_1\tau_2}\exp\left[-\frac{a^{\dagger2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]:\exp\left[a^{\dagger}a\left(-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}\right)\right]:\exp\left[-\frac{a^{2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]\\&=\frac{1}{\tau_1\tau_2}\exp\left[-\frac{a^{\dagger2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]\exp\left[a^{\dagger}a\ln\left(1-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}\right)\right]\exp\left[-\frac{a^{2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right].\end{aligned} \tag{23}$$

Then we introduce the real mode coherent state

$$|z\rangle=\exp\left\{-\frac{1}{2}|z|^{2}+za^{\dagger}\right\}|0\rangle=\exp\left\{-\frac{1}{2}|z|^{2}\right\}\|z\rangle,\quad \|z\rangle=\exp\{za^{\dagger}\}|0\rangle \tag{24}$$

with the help of the IWOP method and using the integration formula

$$\int\frac{{\rm d}^{2}z}{\pi}\exp\left(-h|z|^{2}+\xi z+\eta z^{*}\right)=\frac{1}{h}\exp\left(\frac{\xi\eta}{h}\right),\quad {\rm Re}\,h>0, \tag{25}$$

we can rewrite Eq. (23) as

$$\rho_{\rm s}=\frac{1}{\tau_1\tau_2}\exp\left[-\frac{a^{\dagger2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]\int\frac{{\rm d}^{2}z}{\pi}:\exp\left[-|z|^{2}+\left(z^{*}a^{\dagger}+za\right)\sqrt{1-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}}-a^{\dagger}a\right]:\exp\left[-\frac{a^{2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]. \tag{26}$$

Noticing $:{\rm e}^{-a^{\dagger}a}:=|0\rangle\langle 0|$ and Eq. (24), we can rewrite Eq. (26) as

$$\rho_{\rm s}=\frac{1}{\tau_1\tau_2}\int\frac{{\rm d}^{2}z}{\pi}{\rm e}^{-|z|^{2}}\exp\left[-\frac{a^{\dagger2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]\left\|z^{*}\sqrt{1-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}}\right\rangle\left\langle z\sqrt{1-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}}\right\|\exp\left[-\frac{a^{2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]. \tag{27}$$

Further, using $\langle\tilde{0}\,|\tilde{z}\rangle={\rm e}^{-|z|^{2}/2}$, $|\tilde{z}\rangle$ is the coherent state[17,18] in the fictitious mode, equation (27) can be put into the form

$$\rho_{\rm s}=\frac{1}{\tau_1\tau_2}\int\frac{{\rm d}^{2}z}{\pi}\langle\tilde{z}|\exp\left[\tilde{a}^{\dagger}a^{\dagger}\sqrt{1-\frac{1}{2\tau_1^{2}}-\frac{1}{2\tau_2^{2}}}\right]\exp\left[-\frac{a^{\dagger2}}{4}\left(\frac{1}{\tau_1^{2}}-\frac{1}{\tau_2^{2}}\right)\right]|0\tilde{0}\rangle\langle 0\tilde{0}|$$

$$\times \exp\left[-\frac{a^2}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\exp\left[\tilde{a}a\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right]|\tilde{z}\rangle$$
$$=\frac{1}{\tau_1\tau_2}\widetilde{\mathrm{tr}}\left\{\exp\left[\tilde{a}^\dagger a^\dagger\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right]\exp\left[-\frac{a^{\dagger 2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]|0\tilde{0}\rangle\langle 0\tilde{0}|\exp\left[-\frac{a^2}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\right.$$
$$\left.\times\exp\left[\tilde{a}a\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right]\right\}\equiv\widetilde{\mathrm{tr}}\left[|\psi(\beta)\rangle_{ss}\langle\psi(\beta)|\right], \tag{28}$$

thus according to Eq. (7) we conclude

$$|\psi(\beta)\rangle_{\mathrm{s}}=\sqrt{\frac{1}{\tau_1\tau_2}}\exp\left[\tilde{a}^\dagger a^\dagger\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}-\frac{a^{\dagger 2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]|0\tilde{0}\rangle, \tag{29}$$

this is the thermal vacuum state corresponding to SCL. From another point of view, $|\psi(\beta)\rangle_{\mathrm{s}}$ in Eq. (29) is a kind of one- and two-mode combinatorial squeezed state in real–fictitious mode Fock space.

In particular, when $r=0$, there is no squeezing, from Eq. (22) we see $\tau_1^2=\tau_2^2=\bar{n}+1$, then $|\psi(\beta)\rangle_{\mathrm{s}}$ reduces to $|0(\beta)\rangle$

$$\frac{1}{\sqrt{\bar{n}+1}}\exp\left[\tilde{a}^\dagger a^\dagger\sqrt{\frac{\bar{n}}{\bar{n}+1}}\right]|0\tilde{0}\rangle=\sqrt{1-\mathrm{e}^{\lambda}}\exp\left(\mathrm{e}^{\lambda/2}\tilde{a}^\dagger a^\dagger\right)|0\tilde{0}\rangle=|0(\beta)\rangle, \tag{30}$$

which means that ρ_{s} reduces to ρ_{c}, as expected.

The pure state $|\psi(\beta)\rangle_{\mathrm{s}}$ is normalized and can be shown as follows:

$$\begin{aligned}{}_{\mathrm{s}}\langle\psi(\beta)|\psi(\beta)\rangle_{\mathrm{s}}&=\frac{1}{\tau_1\tau_2}\langle 0\tilde{0}|\exp\left[\tilde{a}a\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}-\frac{a^2}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\\&\quad\times\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}|z_1\tilde{z}_2\rangle\langle z_1\tilde{z}_2|\exp\left[\tilde{a}^\dagger a^\dagger\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}-\frac{a^{\dagger 2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]|0\tilde{0}\rangle\\&=\frac{1}{\tau_1\tau_2}\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}\exp\left[-|z_1|^2-|z_2|^2\right.\\&\quad\left.+(z_1^*z_2^*+z_1z_2)\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}-\left(\frac{z_1^2}{4}+\frac{z_1^{*2}}{4}\right)\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\\&=\frac{1}{\tau_1\tau_2}\int\frac{\mathrm{d}^2z_1}{\pi}\exp\left[-\left(\frac{1}{\tau_1^2}+\frac{1}{\tau_2^2}\right)\frac{|z_1|^2}{2}-\left(\frac{z_1^2}{4}+\frac{z_1^{*2}}{4}\right)\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]=1,\end{aligned} \tag{31}$$

this is equivalent to

$${}_{\mathrm{s}}\langle\psi(\beta)|\,\psi(\beta)\rangle_{\mathrm{s}}=\mathrm{tr}\left[\tilde{\mathrm{tr}}\,|\psi(\beta)\rangle_{\mathrm{ss}}\langle\psi(\beta)|\right]=\mathrm{Tr}\,|\psi(\beta)\rangle_{\mathrm{ss}}\langle\psi(\beta)|=1. \tag{32}$$

4. Calculating photon number distribution in terms of $|\psi(\beta)\rangle_{\mathrm{s}}$

The introduction of $|\psi(\beta)\rangle_{\mathrm{s}}$ can greatly simplify the calculation of the statistical average. For instance, we calculate photon-number distribution in SCL, using Eq. (29) and the coherent state representation $\int(\mathrm{d}^2z/\pi)\,|z\rangle\langle z|=1$ we derive

$$\begin{aligned}{}_{\mathrm{s}}\left\langle\psi(\beta)\left|aa^\dagger\right|\psi(\beta)\right\rangle_{\mathrm{s}}&=\frac{1}{\tau_1\tau_2}\int\frac{\mathrm{d}^2z_1}{\pi}|z_1|^2\exp\left[-\left(\frac{1}{\tau_1^2}+\frac{1}{\tau_2^2}\right)\frac{|z_1|^2}{2}-\frac{z_1^2+z_1^{*2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\\&=\frac{1}{\tau_1\tau_2}\frac{\partial^2}{\partial f\partial g}\int\frac{\mathrm{d}^2z_1}{\pi}\exp\left[-\left(\frac{1}{\tau_1^2}+\frac{1}{\tau_2^2}\right)\frac{|z_1|^2}{2}+fz_1+gz_1^*-\frac{z_1^2+z_1^{*2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\Bigg|_{f=g=0}\\&=\frac{\partial^2}{\partial f\partial g}\exp\left[\frac{fg}{2}\left(\tau_1^2+\tau_2^2\right)+\frac{1}{4}\left(\tau_1^2-\tau_2^2\right)\left(f^2+g^2\right)\right]\Bigg|_{f=g=0}=\frac{1}{2}\left(\tau_1^2+\tau_2^2\right)\\&=\left(\bar{n}+\frac{1}{2}\right)\cosh 2r+\frac{1}{2}.\end{aligned} \tag{33}$$

Thus the average photon number is

$${}_{\mathrm{s}}\left\langle\psi(\beta)\left|a^\dagger a\right|\psi(\beta)\right\rangle_{\mathrm{s}}=\left(\bar{n}+\frac{1}{2}\right)\cosh 2r-\frac{1}{2}, \tag{34}$$

which indicates that the average photon number increases with the squeezing parameter r.

5. Quantum fluctuation of photon number

Also using $|\psi(\beta)\rangle_{\rm s}$ we calculate

$$
\begin{aligned}
{}_{\rm s}\langle\psi(\beta)|a^2a^{\dagger 2}|\psi(\beta)\rangle_{\rm s} &= \frac{1}{\tau_1\tau_2}\int\frac{{\rm d}^2z_1\,{\rm d}^2z_2}{\pi^2}|z_1|^4\exp\left[-|z_1|^2-|z_2|^2+(z_1^*z_2^*+z_1z_2)\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right.\\
&\left.-\left(\frac{z_1^2}{4}+\frac{z_1^{*2}}{4}\right)\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\\
&= \frac{1}{\tau_1\tau_2}\int\frac{{\rm d}^2z_1}{\pi}|z_1|^4\exp\left[-\left(\frac{1}{\tau_1^2}+\frac{1}{\tau_2^2}\right)\frac{|z_1|^2}{2}-\frac{z_1^2+z_1^{*2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\\
&= \frac{1}{\tau_1\tau_2}\left[\frac{\partial^2}{\partial f\partial g}\right]^2\left\{\int\frac{{\rm d}^2z_1}{\pi}\exp\left[-\left(\frac{1}{\tau_1^2}+\frac{1}{\tau_2^2}\right)\frac{|z_1|^2}{2}+fz_1+gz_1^*-\frac{z_1^2+z_1^{*2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\right\}\Bigg|_{f=g=0}\\
&= \left[\frac{\partial^2}{\partial f\partial g}\right]^2\exp\left[\frac{fg}{2}\left(\tau_1^2+\tau_2^2\right)+\frac{1}{4}\left(\tau_1^2-\tau_2^2\right)\left(f^2+g^2\right)\right]\Bigg|_{f=g=0}\\
&= \frac{1}{4}\left(\tau_2^2-\tau_1^2\right)^2+\frac{1}{2}\left(\tau_1^2+\tau_2^2\right)^2=\frac{1}{4}\left(2\bar{n}+1\right)^2\sinh^2 2r+\frac{1}{2}\left[\left(2\bar{n}+1\right)\cosh 2r+1\right]^2. \qquad (35)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
{}_{\rm s}\left\langle\psi(\beta)\left|\left(a^\dagger a\right)^2\right|\psi(\beta)\right\rangle_{\rm s} &= {}_{\rm s}\langle\psi(\beta)|\left(a^2a^{\dagger 2}-3aa^\dagger+1\right)|\psi(\beta)\rangle_{\rm s}\\
&= \frac{1}{4}\left(\tau_2^2-\tau_1^2\right)^2+\frac{1}{2}\left(\tau_1^2+\tau_2^2\right)^2-\frac{3}{2}\left(\tau_1^2+\tau_2^2\right)+1\\
&= \frac{1}{4}\left(2\bar{n}+1\right)^2\sinh^2 2r+\frac{1}{2}\left[\left(2\bar{n}+1\right)\cosh 2r+1\right]^2-\frac{3}{2}\left(2\bar{n}+1\right)\cosh 2r-\frac{1}{2}, \qquad (36)
\end{aligned}
$$

then the photon number fluctuation is

$${}_{\rm s}\left\langle\psi(\beta)\left|\left(a^\dagger a\right)^2\right|\psi(\beta)\right\rangle_{\rm s}-{}_{\rm s}\langle\psi(\beta)|a^\dagger a|\psi(\beta)\rangle_{\rm s}^2=\frac{1}{2}\left(\tau_2^4-\tau_1^4\right)-\frac{1}{2}\left(\tau_1^2+\tau_2^2\right)=\left(\bar{n}+\frac{1}{2}\right)^2\cosh 4r, \qquad (37)$$

this result seems to be newly reported, thus we see that for the chaotic state, the stronger the squeezing is, the larger the fluctuation is. This is a new property of chaotic light. We have derived it by virtue of the thermal vacuum state $|\psi(\beta)\rangle_{\rm s}$.

Using Eqs. (35) and (36) we further examine the second-order coherence degree[19,20]

$$g_\rho^{(2)}=\frac{{}_{\rm s}\langle\psi(\beta)|a^{\dagger 2}a^2|\psi(\beta)\rangle_{\rm s}}{{}_{\rm s}\langle\psi(\beta)|a^\dagger a|\psi(\beta)\rangle_{\rm s}^2}=2+\frac{(\bar{n}+1/2)^2\sinh^2 2r}{\left[(\bar{n}+1/2)\cosh 2r-1/2\right]^2}=2+\frac{(\bar{n}+1/2)^2\sinh^2 2r}{\left[\bar{n}\cosh 2r+\sinh^2 r\right]^2}>2, \qquad (38)$$

which indicates that the squeezed chaotic light is bunching, and when $r\to 0$, $g_\rho^{(2)}=2$, it reduces to chaotic light, exhibiting the bunching effect, thus the squeezed chaotic light is still classic.

6. Wigner function of the squeezed chaotic field

The Wigner function of $|\psi(\beta)\rangle_{\rm s}$ is

$$W(\alpha)={}_{\rm s}\langle\psi(\beta)|\Delta(\alpha)|\psi(\beta)\rangle_{\rm s},\quad \alpha=(q+{\rm i}p)/\sqrt{2}, \qquad (39)$$

where $\Delta(\alpha)$ is the Wigner operator, its coherent state representation is

$$\Delta(\alpha)={\rm e}^{2|\alpha|^2}\int\frac{{\rm d}^2z}{\pi^2}|z\rangle\langle -z|\,{\rm e}^{-2(z\alpha^*-z^*\alpha)}. \qquad (40)$$

Substituting Eq. (41) into Eq. (40), using the completeness of the coherent state and the integration formula

$$\int\frac{{\rm d}^2z}{\pi}z^nz^{*m}\exp\left(\zeta|z|^2+\xi z+\eta z^*\right)={\rm e}^{-\xi\eta/\zeta}\sum_{l=0}^{\min(m,n)}\frac{m!n!\xi^{m-1}\eta^{n-1}}{l!(m-l)!(n-l)!(-\zeta)^{m+n-l+1}},\quad {\rm Re}\,\zeta<0, \qquad (41)$$

we obtain

$$
\begin{aligned}
W(\alpha) &= \mathrm{e}^{2|\alpha|^2}\int\frac{\mathrm{d}^2z}{\pi^2}{}_{\mathrm{s}}\langle\psi(\beta)|z\rangle\langle -z|\,\mathrm{e}^{-2(z\alpha^*-z^*\alpha)}|\psi(\beta)\rangle_{\mathrm{s}}\\
&= \frac{\mathrm{e}^{2|\alpha|^2}}{\tau_1\tau_2}\int\frac{\mathrm{d}^2z}{\pi^2}\langle 0\tilde{0}|\exp\left[-\frac{a^2}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]\exp\left[\tilde{a}a\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right]|z\rangle\\
&\quad\times\langle -z|\,\mathrm{e}^{-2(z\alpha^*-z^*\alpha)}\int\frac{\mathrm{d}^2z'}{\pi}\left|z'\right\rangle\left\langle z'\right|\exp\left[\tilde{a}^\dagger a^\dagger\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}\right]\exp\left[-\frac{a^{\dagger 2}}{4}\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]|\tilde{0}0\rangle\\
&= \frac{\mathrm{e}^{2|\alpha|^2}}{\pi\tau_1\tau_2}\int\frac{\mathrm{d}^2z}{\pi}\int\frac{\mathrm{d}^2z'}{\pi}:\exp\left[-\left|z'\right|^2-|z|^2-2\left(z\alpha^*-z^*\alpha\right)\right.\\
&\quad\left.+\left(z'z-z'^*z^*\right)\sqrt{1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}}-\frac{1}{4}\left(z^2+z^{*2}\right)\left(\frac{1}{\tau_1^2}-\frac{1}{\tau_2^2}\right)\right]:\\
&= \frac{1}{\pi\sqrt{1+4\tau_1^2\tau_2^2\left(1-\frac{1}{2\tau_1^2}-\frac{1}{2\tau_2^2}\right)}}\exp\left\{\frac{|\alpha|^2\left(1-\tau_1^2-\tau_2^2\right)+\left(\alpha^2+\alpha^{*2}\right)\left(\tau_1^2-\tau_2^2\right)}{1+4\tau_1^2\tau_2^2\left(1-1/2\tau_1^2-1/2\tau_2^2\right)}\right\}.
\end{aligned}
\tag{42}
$$

Due to Eq. (22), equation (43) can be re-written as

$$
W(q,p)=\frac{1-\mathrm{e}^{\lambda}}{\pi\left(\mathrm{e}^{\lambda}+1\right)}\exp\left[\frac{\mathrm{e}^{\lambda}-1}{\mathrm{e}^{\lambda}+1}\left(q^2\,\mathrm{e}^{-2r}+p^2\,\mathrm{e}^{2r}\right)\right]. \tag{43}
$$

7. Summary

In summary, by virtue of the partial trace method and the IWOP technique, we obtained the thermal vacuum state for the squeezed chaotic light. It helps us to successfully calculate the photon number disturibution, quantum fluctuation, and the second-order coherence degree, which shows that the squeezed chaotic light is still of a classic nature. For the Radon transformation property of squeezed chaotic light we can refer to Ref. [21]. The thermal vacuum state corresponding to squeezed chaotic light we found in this work may bring convenience for studying quantum decoherence of a squeezed state in the amplitude damping channel and quantum controlling of squeezing.

References

[1] Fan H Y, Jiang N Q 2008 *Phys. Scr.* **78** 045402
[2] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
[3] Fan H Y and Xu X F 2004 *Opt. Lett.* **29** 1048
[4] Takahashi Y and Umezawa H 1975 *Collective Phenomena* **2** 55
[5] Fan H Y and Liang X T 2000 *Chin. Phys. Lett.* **17** 174
[6] Hu L Y, Xu X X and Fan H Y 2010 *J. Opt. Soc. Am. B* **27** 286
[7] Xu Y J, Wang C C, Wang X C and Fan H Y 2012 *Int. J. Theor. Phys.* **51** 1062
[8] Yuan H C, Xu X X, Xu X F and Fan H Y 2011 *Mod. Phys. Lett. B* **25** 2353
[9] Fan H Y and Fan Y 2006 *Ann. Phys.* **321** 480
[10] Hu L Y and Fan H Y 2009 *Mod. Phys. Lett. A* **24** 2263
[11] Hu L Y and Fan H Y 2009 *Chin. Phys. B* **18** 1674
[12] Xu X X, Hu L Y and Fan H Y 2011 *Mod. Phys. Lett. B* **25** 31
[13] Fan H Y, Tang X B and Hu L Y 2010 *Commun. Theor. Phys.* **53** 45
[14] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
[15] Fan H Y 2008 *Ann. Phys.* **323** 1502
[16] Fan H Y 2000 *Mod. Phys. Lett. A* **15** 2297
[17] Fan H Y and Ruan T N 1983 *Commun. Theor. Phys.* **2** 1563
[18] Glauber R J 1963 *Phy. Rev.* **130** 2529
[19] Glauber R J 1963 *Phy. Rev.* **131** 2766
[20] Fan H Y and Wu Z 2015 *Acta Phys. Sin.* **64** 080303 (in Chinese)
[21] Meng X G and Xu Y J 2014 *Int. J. Theor. Phys.* **53** 1239