Skew-gradient representation of generalized Birkhoffian system*

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The skew-gradient representation of a generalized Birkhoffian system is studied. A condition under which the generalized Birkhoffian system can be considered as a skew-gradient system is obtained. The properties of the skew-gradient system are used to study the properties, especially the stability, of the generalized Birkhoffian system. Some examples are given to illustrate the application of the result.

Keywords: generalized Birkhoffian system, skew-gradient system, Lyapunov function, stability

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1. Introduction

The authors in Ref. [1] pointed out that an ordinary differential equation can be written in a "linear-gradient system" when its one or more integrations or Lyapunov function is given. The skew-gradient system is a kind of linear-gradient systems which has important significance, and is different from the gradient system given by Ref. [2]. The former's matrix is antisymmetric; while the latter's matrix is symmetric, and is an identity matrix. Quispel *et al.*[3,4] proved that an autonomous ordinary differential equation has a first integral if and only if it can be written as a skew-gradient system, and presented a general method to rewrite such a system as a skew-gradient system, then from which constructed several new integral-preserving discrete gradient methods. Hong *et al.*[5] discussed the skew-gradient representation of autonomous stochastic differential equations with a conserved quantity, then from which constructed direct/indirect discrete gradient approaches. The Birkhoffian system is a natural generalization of the Hamiltonian system. Great progress on its dynamical research has been achieved.[6–14] The generalized Birkhoffian system refers to the Birkhoffian system with added terms.[14] We have studied the gradient representation for the generalized Birkhoffian system.[15] In this paper, we will give a condition under which a generalized Birkhoffian system can be rewritten in a skew-gradient system, and discuss the dynamic behaviors, especially integration and stability of the generalized Birkhoffian system by using the property of the skew-gradient system.

2. Skew-gradient system

The differential equations of the skew-gradient system have the form[1]

$$\dot{\boldsymbol{X}} = A(\boldsymbol{X})\nabla V, \tag{1}$$

where

$$\boldsymbol{X} = (x_1, x_2, \ldots, x_m)^{\mathrm{T}},$$
$$A(\boldsymbol{X}) = \begin{pmatrix} 0 & a_{12}(\boldsymbol{X}) & \ldots & a_{1m}(\boldsymbol{X}) \\ -a_{12}(\boldsymbol{X}) & 0 & \ldots & a_{2m}(\boldsymbol{X}) \\ \vdots & \vdots & & \vdots \\ -a_{1m}(\boldsymbol{X}) & -a_{2m}(\boldsymbol{X}) & \ldots & 0 \end{pmatrix},$$
$$\nabla V = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_m}\right)^{\mathrm{T}}, \tag{2}$$

where V is called "energy" in Ref. [1]. The gradient system given in Ref. [2] has the form

$$\dot{\boldsymbol{X}} = -\nabla V. \tag{3}$$

It is clear that equation (1) is different from Eq. (3).

The skew-gradient system has the following important properties. (i) An autonomous ordinary differential equation has a first integral if and only if it can be written as a skew-gradient system. (ii) The energy function V is an integral of the system (1). (iii) If V can be a Lyapunov function, then the solution $x_i = x_{i0}$ $(i = 1, 2, \ldots, m)$ of the system (1) is stable.

The skew-gradient form (1) has many uses. For example, if the system has a first integral V, then its motion stays on the level set $\Sigma_C := \{\boldsymbol{X} : V(\boldsymbol{X}) = C\}$. Preserving this property leads to good nonlinear stability, especially if Σ_C is compact.[1] Another example, systems with a Lyapunov function V can be "weak" or "strong". Preserving these properties is important because they are equivalent to the existence of a stable, respectively asymptotically stable, attracting set.[1]

The skew-gradient representation can be used not only to discuss the integration and stability, but also to construct the integral-preserving discrete gradient approaches for the system. In addition, Hamiltonian systems, generalized Hamiltonian systems, and autonomous Birkhoffian systems are skew-

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gradient systems. Naturally, the results and methods on the skew-gradient system can be applied to these systems.

3. Generalized Birkhoffian system and skew-gradient system

The generalized Birkhoffian system refers to the Birkhoffian system with added terms. Great progress has been made on its dynamical research.[16]

In general, a generalized Birkhoffian system cannot be rewritten in a skew-gradient system. In the following, we give a condition under which a generalized Birkhoffian system can become a skew-gradient system. In such a case, one can easily discuss the integration and stability or construct integral-preserving discrete gradient methods for the system by using the results of the skew-gradient system.

The nonsingular generalized Birkhoffian equations have the form[14]

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t} - \Lambda_{\nu}\right), \quad (\mu,\nu = 1,2,\ldots,2n), \tag{4}$$

where and in the following, the same indices in a term indicate to take the sum with it. In Eqs. (4), $B = B(t,\boldsymbol{a})$ is the Birkhoffian, $R_{\mu} = R_{\mu}(t,\boldsymbol{a})$ are Birkhoff's functions, and $\Lambda_{\mu} = \Lambda_{\mu}(t,\boldsymbol{a})$ are the added terms, and

$$\Omega_{\mu\nu} = \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \tag{5}$$

is called Birkhoff's tensor, satisfying

$$\Omega^{\mu\nu}\Omega_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{6}$$

For Eqs. (4), if

$$\frac{\partial R_{\nu}}{\partial t} = \frac{\partial B}{\partial t} = \frac{\partial \Lambda_{\nu}}{\partial t} = 0, \quad (\nu = 1,2,\ldots,2n), \tag{7}$$

then we have

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\nu}} - \Lambda_{\nu}\right). \tag{8}$$

In general, equations (8) still cannot become a skew-gradient system. If there is a function $V = V(\boldsymbol{a})$, satisfying

$$\frac{\partial B}{\partial a^{\nu}} - \Lambda_{\nu} = \frac{\partial V}{\partial a^{\nu}} \quad (\nu = 1,2,\ldots,2n), \tag{9}$$

then equations (8) become

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\frac{\partial V}{\partial a^{\nu}}. \tag{10}$$

Obviously, it is a skew-gradient system. So, under conditions (7) and (9), the generalized Birkhoffian system (1) becomes the skew-gradient system (10).

Taking the derivative of V with respect to time according to Eqs. (10), we have

$$\dot{V} = \frac{\partial V}{\partial a^{\mu}}\dot{a}^{\mu} = \frac{\partial V}{\partial a^{\mu}}\Omega^{\mu\nu}\frac{\partial V}{\partial a^{\nu}} = 0. \tag{11}$$

So, V is an integral of system (10). If V can further become a Lyapunov function, then one can discuss the stability of generalized Birkhoffian system (1) by using the property of the skew-gradient system.

For a Birkhoffian system, there are $\Lambda_{\mu} = 0$ $(\mu = 1,2,\ldots,2n)$, then an autonomous Birkhoffian system is a skew-gradient system, whose Birkhoffian is an integral. Similarly, if B can further become a Lyapunov function, then one can discuss the stability of the Birkhoffian system.

4. Illustrative examples

Example 1 A generalized Birkhoffian system of order 4 is

$$R_1 = a^2(a^1+1), \quad R_2 = 0, \quad R_3 = a^4(a^3+1),$$
$$R_4 = 0, \quad B = \frac{1}{2}\left[(a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2\right],$$
$$\Lambda_1 = -(a^1)^2, \quad \Lambda_2 = -(a^2)^3, \quad \Lambda_3 = \Lambda_4 = 0.$$

Try to rewrite it in a skew-gradient system, and discuss the stability of zero solution.

Clearly, condition (7) is satisfied. Condition (9) gives

$$a^1 + (a^1)^2 = \frac{\partial V}{\partial a^1}, \quad a^2 + (a^2)^3 = \frac{\partial V}{\partial a^2},$$
$$a^3 = \frac{\partial V}{\partial a^3}, \quad a^4 = \frac{\partial V}{\partial a^4}.$$

From which one can take

$$V = B + \frac{1}{3}(a^1)^3 + \frac{1}{4}(a^2)^4.$$

Then equations (10) give

$$\begin{pmatrix}\dot{a}^1\\ \dot{a}^2\\ \dot{a}^3\\ \dot{a}^4\end{pmatrix} = \begin{pmatrix} 0 & 1/(a^1+1) & 0 & 0\\ -1/(a^1+1) & 0 & 0 & 0\\ 0 & 0 & 0 & 1/(a^3+1)\\ 0 & 0 & -1/(a^3+1) & 0\end{pmatrix} \times \begin{pmatrix} a^1+(a^1)^2\\ a^2+(a^2)^3\\ a^3\\ a^4\end{pmatrix}.$$

Take function V as a Lyapunov function, which is positive-definite in the neighborhood of $a^1 = a^2 = a^3 = a^4 = 0$. Taking $\dot{V}$ according to the above equation obtains

$$\dot{V} = 0.$$

According to the Lyapunov theorem, zero solution $a^1 = a^2 = a^3 = a^4 = 0$ is stable.

Example 2 The Hojman–Urrutia equation has a Birkhoffian representation[6]

$$R_1 = a^2 + a^3, \quad R_2 = 0, \quad R_3 = a^4, \quad R_4 = 0,$$

$$B = \frac{1}{2}\left[(a^3)^2 + 2a^2a^3 - (a^4)^2\right].$$

Try to rewrite it in a skew-gradient system.

Equations (10) give

$$\begin{pmatrix} \dot{a}^1 \\ \dot{a}^2 \\ \dot{a}^3 \\ \dot{a}^4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a^3 \\ a^2 + a^3 \\ -a^4 \end{pmatrix},$$

Clearly, it is a skew-gradient system, and function B is an integral of the system. But B cannot become a Lyapunov function.

5. Conclusions

Generally speaking, a generalized Birkhoffian system is not a skew-gradient system. In this paper, we obtained a condition under which the generalized Birkhoffian system can become a skew-gradient system, namely expressions (7) and (9). When a generalized Birkhoffian system is rewritten in a skew-gradient system, its dynamic behaviors can be discussed by using the properties of the skew-gradient system. Especially, if function V can become a Lyapunov function, then one can discuss the stability of the system.

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