Singular and non-topological soliton solutions for nonlinear fractional differential equations

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In this article, the fractional derivatives are described in the modified Riemann–Liouville sense. We propose a new approach, namely an ansatz method, for solving fractional differential equations (FDEs) based on a fractional complex transform and apply it to solve nonlinear space–time fractional equations. As a result, the non-topological as well as the singular soliton solutions are obtained. This method can be suitable and more powerful for solving other kinds of nonlinear fractional FDEs arising in mathematical physics.

Keywords: solitons, ansatz method, the space–time fractional Boussinesq equation, the space–time fractional (2+1)-dimensional breaking soliton equations

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1. Introduction

Fractional differential equations are the generalizations of classical differential equations with integer orders. In recent years, fractional differential equations have played an important role in different research areas such as mechanics, engineering, signal processing, stochastic dynamical system, plasma physics, electricity, electrochemistry, biology, control theory, systems identification, economics and finance.[1–4]

Finding approximate and exact solutions to the fractional differential equations is an important task. Many powerful and reliable methods have been proposed to obtain the exact solutions of fractional differential equations. For example, these methods include the (G'/G)-expansion method,[5–7] the first integral method,[8–10] the fractional sub-equation method,[11–13] the exp-function method,[14–16] the fractional functional variable method,[17,18] the fractional modified trial equation method,[19] Kudryashov method,[20,21] ansatz method,[22] and so on.

The theory of solitons is one of the very important areas of research in ocean dynamics, optics, plasma physics, fluid dynamics, semiconductors, and engineering. In these areas, the study of solitary waves attracts a lot of attention. Therefore, it is important to focus on solitary waves in a detailed manner from a mathematical point of view. Biswas *et al.* obtained optical solitons and soliton solutions with higher-order dispersion by using the ansatz method.[23–25] This method was used by many authors.[26–31] But, applications of this method are rather rare in the nonlinear fractional differential equations.

The rest of this paper is organized as follows: In Section 2, we describe the modified Riemann–Liouville derivative and explain how we can convert fractional differential equations into integer-order differential equations. In Sections 3 and 4, we apply this method to establish the exact solutions for the space–time fractional Boussinesq equation and the space–time fractional (2+1)-dimensional breaking soliton equations and to employ a variety of exact solutions to determine non-topological soliton solutions and singular soliton solutions. Conclusions are presented in Section 5.

2. Description of the modified Riemann–Liouville derivative

Fractional calculus theory is almost more than two decades old in the literature. There are several approaches to the generalization of the notion of differentiation to fractional orders, e.g., Riemann–Liouville, Grünwald–Letnikow, and Caputo.[32,33] But the first major contribution to give proper definition is due to Jumarie's modified Riemann–Liouville derivative[34,35] as shown below.

Definition 1 Assume that $f: R \to R$, $x \to f(x)$ denotes a continuous (but not necessarily differentiable) function. The modified Riemann–Liouville derivative of order α is defined by the expression[36]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]\,\mathrm{d}\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]\,\mathrm{d}\xi, & 0 < \alpha < 1 \\ (f^{(n)}(x))^{(\alpha-n)}, & n \le \alpha \le n+1, \quad n \ge 1. \end{cases} \tag{1}$$

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in which $\Gamma(\cdot)$ is the Gamma function defined by[37]

$$\Gamma(\alpha)=\lim_{n\to\infty}\frac{n!n^{\alpha}}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n)} \tag{2}$$

or

$$\Gamma(x)=\int_0^{\infty}\mathrm{e}^{-t}t^{x-1}\,\mathrm{d}x. \tag{3}$$

It imposes advantages of fast convergence, higher stability, and higher accuracy to derive different types of numerical algorithms.

Definition 2 The Mittag–Leffler function with two parameters is defined as[38]

$$E_{\alpha,\beta}(x)=\sum_{i=0}^{\infty}\frac{x^i}{\Gamma(\alpha\mathrm{i}+\beta)},\quad \Re e(\alpha)>0,\quad \beta,x\in\mathbb{C}, \tag{4}$$

this function is used to solve fractional differential equations as the exponential function in integer order systems.

We list some important properties for the modified Riemann–Liouville derivative as follows:

Property 1

$$D_x^{\alpha}x^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha},\quad \gamma>0; \tag{5}$$

Property 2

$$D_x^{\alpha}(cf(x))=cD_x^{\alpha}f(x),\quad c=\text{constant}; \tag{6}$$

Property 3

$$D_x^{\alpha}\{af(x)+bg(x)\}=aD_x^{\alpha}f(x)+bD_x^{\alpha}g(x), \tag{7}$$

where a and b are constants.

We consider the following general nonlinear FDEs of the type

$$P(u,D_t^{\alpha}u,D_x^{\beta}u,D_t^{\alpha}D_t^{\alpha}u,D_t^{\alpha}D_x^{\beta}u,D_x^{\beta}D_x^{\beta}u,\ldots)=0,\quad 0<\alpha,\beta<1, \tag{8}$$

where $D_t^{\alpha}u$ and $D_x^{\alpha}u$ are the modified Riemann–Liouville derivatives and P is a polynomial in $u=u(x,t)$ and its fractional derivatives.

By using the fractional complex transform we can convert fractional differential equations into integer-order differential equations

$$u(x,t)=U(\xi),\quad \xi=\frac{kx^{\beta}}{\Gamma(1+\beta)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{9}$$

where k and c are nonzero arbitrary constants. Also, by using the chain rule

$$D_t^{\alpha}u=\sigma_t'\frac{\mathrm{d}U}{\mathrm{d}\xi}D_t^{\alpha}\xi,\quad D_x^{\alpha}u=\sigma_x'\frac{\mathrm{d}U}{\mathrm{d}\xi}D_x^{\alpha}\xi, \tag{10}$$

where σ_t' and σ_x' are called the sigma indexes see Refs. [39]–[41], without loss of generality we can take $\sigma_t'=\sigma_x'=l$, where l is a constant.

Substituting Eq. (9) with Eq. (5) and Eq. (10) into Eq. (8), we can rewrite Eq. (8) in the following nonlinear ordinary differential equation (ODE);

$$Q(U,U',U'',U''',\ldots)=0. \tag{11}$$

where Q is a polynomial in $U(\xi)$ and its total derivatives U', U'', U''',..., such that

$$U'=\mathrm{d}U/\mathrm{d}\xi,\quad U''=\mathrm{d}^2U/\mathrm{d}\xi^2,\ldots.$$

If possible, we should integrate Eq. (11) term-by-term one or more times.

3. The space–time fractional Boussinesq equation

The first equation is called the space–time fractional Boussinesq equation which has the form[42]

$$D_t^{2\alpha}u+bD_x^{2\alpha}u+\beta D_x^{2\alpha}(u^2)+\gamma D_x^{4\alpha}u=0,\quad t>0,\ 0<\alpha\le 1. \tag{12}$$

Here $\gamma=\text{const}>0$ is the dispersion parameter depending on the compression and rigidity characteristics of the material, $\beta=\text{const}$ is the coefficient controlling nonlinearity, $u(x,t)$ is the vertical deflection, and the quadratic nonlinearity $(u^2)_{xx}$ accounts for the curvature of the bending beam. α is a parameter describing the order of the fractional time and space derivative.

The bright and singular soliton solutions to Eq. (12) will be obtained with the help of the ansatz method.[43,44] In order to solve Eq. (12), we use the following transformations,

$$u(x,t)=U(\xi), \tag{13}$$

$$\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{14}$$

where k and c are non-zero constants.

Substituting Eq. (14) with Eqs. (5) and (10) into Eq. (12), this equation (Eq. (12)) reduced into an ODE

$$c^2U''+bk^2U''+\beta k^2\left(U^2\right)''+\gamma k^4U''''=0, \tag{15}$$

where "U'"$=\mathrm{d}U/\mathrm{d}\xi$. By integrating twice and setting the constants of integration to be zero, we obtain

$$\left(c^2+bk^2\right)U+\beta k^2U^2+\gamma k^4U''=0. \tag{16}$$

3.1. The bright (non-topological) soliton solution

The solitary wave ansatz for the bright (non-topological) 1-soliton solution, i.e., the hypothesis, is[45,46]

$$U(\xi)=A\mathrm{sech}^p\xi, \tag{17}$$

where

$$\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)}. \tag{18}$$

Here, A, c, and k are nonzero arbitrary constants. The unknown p will be determined during the course of derivation of

the solutions of Eq. (1).

Therefore from Eqs. (17) and (18), it is possible to get

$$\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2} = Ap^2\mathrm{sech}^p\xi - Ap(p+1)\mathrm{sech}^{p+2}\xi \tag{19}$$

and

$$U^2(\xi) = A^2\mathrm{sech}^{2p}\xi. \tag{20}$$

Thus, substituting the ansatz Eqs. (17)–(20) into Eq. (16), yields

$$\begin{aligned}&\left(c^2+bk^2\right)A\mathrm{sech}^p\xi + \beta k^2A^2\mathrm{sech}^{2p}\xi\\ &\quad+\gamma k^4Ap^2\mathrm{sech}^p\xi - \gamma k^4Ap(p+1)\mathrm{sech}^{p+2}\xi = 0.\end{aligned} \tag{21}$$

Now, from Eq. (21), equating the exponents $p+2$ and $2p$ leads to

$$p+2=2p, \quad p=2. \tag{22}$$

From Eq. (21), setting the coefficients of $\mathrm{sech}^{p+2}\xi$ and $\mathrm{sech}^{2p}\xi$ terms to zero, we obtain

$$\beta k^2A^2 - \gamma k^4Ap(p+1) = 0, \tag{23}$$

by using Eq. (22), we have

$$A = \frac{6\gamma k^2}{\beta}. \tag{24}$$

We find, from setting the coefficients of $\mathrm{sech}^p\xi$ terms in Eq. (21) to zero

$$\left(c^2+bk^2\right)A + \gamma k^4A^2 = 0, \tag{25}$$

also we get

$$c = \pm k\sqrt{-(4\gamma k^2+b)}. \tag{26}$$

From Eq. (26) it is important to note that

$$4\gamma k^2 + b < 0. \tag{27}$$

Thus, the 1-soliton solution of Eq. (12) is given by

$$u_{1,2}(x,t) = A\mathrm{sech}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right), \tag{28}$$

where A is given by Eq. (24) and c is given by Eq. (26). Also, the constraint condition is shown in Eq. (27). The evolution of exact solution for Eq. (28) with $\alpha = 0.5$ and $\alpha = 1.0$ is shown in Fig. 1.

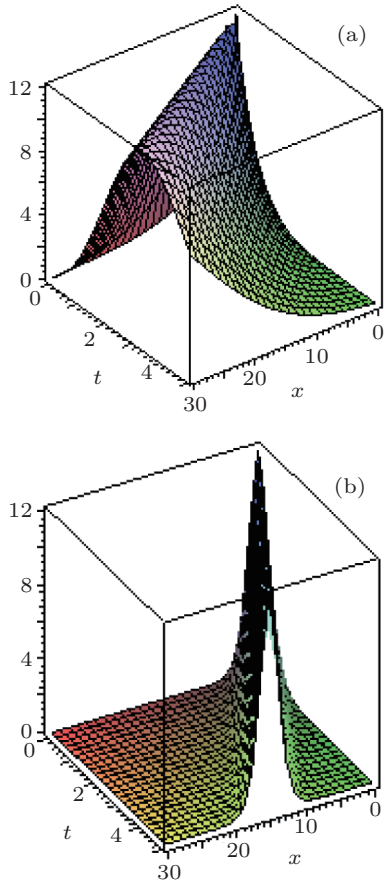


Fig. 1. (color online) The exact solution for Eq. (28) with $\alpha = 0.5$ (a) and $\alpha = 1$ (b) respectively, when $k = 1$, $\gamma = -2$, $\beta = -1$, $b = -1$.

3.2. Singular soliton solution

For singular solitons, the starting hypothesis is given by[47–49]

$$U(\xi) = A\mathrm{csch}^p\xi, \tag{29}$$

where

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{30}$$

where A, k, and c are nonzero arbitrary constants. The unknown p will be determined during the course of derivation of the solutions of Eq. (16). Equation (29) and its derivatives gives

$$\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2} = Ap^2\mathrm{csch}^p\xi + Ap(p+1)\mathrm{csch}^{p+2}\xi \tag{31}$$

and

$$U^2(\xi) = A^2\mathrm{csch}^{2p}\xi. \tag{32}$$

Substituting Eqs. (29)–(32) into Eq. (16), we have

$$\begin{aligned}&(c^2 + bk^2)A\mathrm{csch}^p\xi + \beta k^2A^2\mathrm{csch}^{2p}\xi\\ &\quad+ \gamma k^4\left\{Ap^2\mathrm{csch}^p\xi + Ap(p+1)\mathrm{csch}^{p+2}\xi\right\} = 0.\end{aligned} \tag{33}$$

Similarly, from Eq. (33), equating the exponents $p+2$ and $2p$ leads to

$$p+2=2p, \tag{34}$$

and

$$p = 2. \tag{35}$$

From Eq. (33), setting the coefficients of $\mathrm{csch}^{p+2}\xi$ and $\mathrm{csch}^{2p}\xi$ terms to zero, we obtain

$$\beta k^2 A^2 + \gamma k^4 A p(p+1) = 0, \tag{36}$$

by using Eq. (35) and after some calculations, we have

$$A = -\frac{6\gamma k^2}{\beta}. \tag{37}$$

We find, from setting the coefficients of $\mathrm{csch}^{p}\xi$ terms in Eq. (33) to zero

$$\left(c^2 + bk^2\right)A + \gamma k^4 A p^2 = 0, \tag{38}$$

also we get

$$c = \pm k\sqrt{-(4\gamma k^2 + b)}. \tag{39}$$

From Eq. (39) it is important to note that

$$4\gamma k^2 + b < 0. \tag{40}$$

Thus, the singular soliton solution of Eq. (12) is given by

$$u_{1,2}(x,t) = A\mathrm{csch}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right), \tag{41}$$

where A is given by Eq. (37), c is given by Eq. (39) and the constraint condition is shown in Eq. (40). The evolution of the exact solution for Eq. (41) with $\alpha = 0.5$ and $\alpha = 1.0$ is shown in Fig. 2.

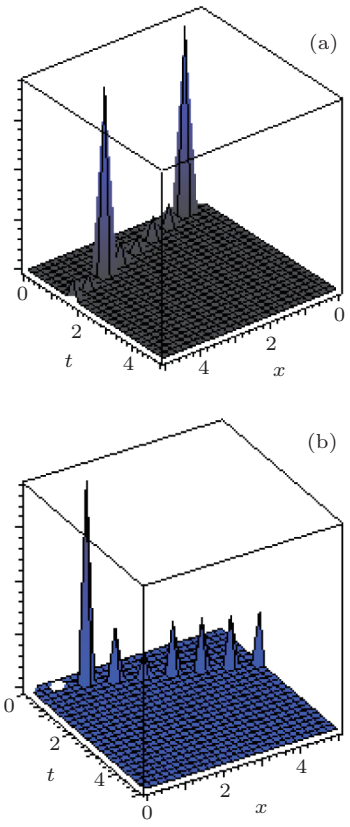


Fig. 2. (color online) The exact solution for (41) with $\alpha = 0.5$ (a) and $\alpha = 1$ (b) respectively, when $k = 1$, $\gamma = -2$, $\beta = -1$, $b = -1$.

4. The space–time fractional (2+1)-dimensional breaking soliton equations

We consider the space–time fractional (2+1)-dimensional breaking soliton equations[50]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a\frac{\partial^{3\alpha} u}{\partial x^{2\alpha} y^\alpha} + 4au\frac{\partial^\alpha v}{\partial x^\alpha} + 4a\frac{\partial^\alpha u}{\partial x^\alpha}v = 0, \tag{42a}$$

$$\frac{\partial^\alpha u}{\partial y^\alpha} - \frac{\partial^\alpha v}{\partial x^\alpha} = 0, \tag{42b}$$

where $0 < \alpha \leq 1$. In Ref. [50], Wen and Zheng applied the fractional sub-equation method to construct exact solutions of these equations. Equations (42) have been discussed in Ref. [51] using three different types of method, namely, the $(G\prime/G)$-expansion method, the functional variable method, and the exp-function method. When $\alpha = 1$, equations (42) are called the (2+1)-dimensional breaking soliton equations and some periodic wave solutions, non-traveling wave solutions, and Jacobi elliptic function solutions were found in Refs. [52]–[55].

Now, we introduce the following transformations

$$\begin{aligned} u(x,y,t) &= U(\xi), \\ v(x,y,t) &= V(\xi), \end{aligned} \tag{43}$$

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{wy^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{44}$$

where k, w, and c are non-zero constants.

Substituting Eq. (44) with Eqs. (5) and (10) into Eq. (42), equations (42) can be reduced into an ODE

$$\begin{aligned} &-cU' + ak^2 wU''' + 4akUV' + 4akU'V = 0, \\ &wU' - kV' = 0, \end{aligned} \tag{45}$$

where "U'"$= \mathrm{d}U/\mathrm{d}\xi$ and "V'"$= \mathrm{d}V/\mathrm{d}\xi$.

Integrating the second equation in the system and ignoring constants of integration we obtain

$$wU = kV. \tag{46}$$

Substituting Eq. (46) into the first equation of the system we find

$$-cU' + ak^2 wU''' + 8awUU' = 0. \tag{47}$$

Integrating Eq. (47) and ignoring constants of integration, we find

$$-cU + ak^2 wU'' + 4awU^2 = 0. \tag{48}$$

4.1. The bright (non-topological) soliton solution

Substituting the ansatz [Eqs. (17)–(20)] into Eq. (48), yields

$$\begin{aligned} &-cA\mathrm{sech}^p\xi + 4awA^2\mathrm{sech}^{2p}\xi \\ &\quad + ak^2 w\left\{Ap^2\mathrm{sech}^p\xi - Ap(p+1)\mathrm{sech}^{p+2}\xi\right\} = 0. \end{aligned} \tag{49}$$

Now, from Eq. (49), equating the exponents $p+2$ and $2p$

leads to

$$p+2=2p, \quad p=2. \tag{50}$$

From Eq. (49), setting the coefficients of $\text{sech}^{p+2}\xi$ and $\text{sech}^{2p}\xi$ terms to zero, we obtain

$$4awA^2 - ak^2wAp(p+1)=0, \tag{51}$$

by using Eq. (50) and after some calculations, we have

$$A=\frac{3k^2}{2}. \tag{52}$$

We find, from setting the coefficients of $\text{sech}^{p}\xi$ terms in Eq. (49) to zero

$$-cA+ak^2wAp^2=0, \tag{53}$$

also we get

$$c=4awk^2. \tag{54}$$

Thus, the non-topological soliton solution of Eq. (42) is given by

$$u(x,y,t) = \frac{3k^2}{2}\text{sech}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{wy^\alpha}{\Gamma(1+\alpha)}-\frac{4awk^2t^\alpha}{\Gamma(1+\alpha)}\right), \tag{55}$$

and from Eq. (46) we get

$$v(x,y,t) = \frac{3kw}{2}\text{sech}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{wy^\alpha}{\Gamma(1+\alpha)}-\frac{4awk^2t^\alpha}{\Gamma(1+\alpha)}\right), \tag{56}$$

where A is given by Eq. (52) and c is given by Eq. (54).

4.2. Singular soliton solution

Substituting Eqs. (30)–(33) into Eq. (48), we have

$$-cA\text{csch}^p\xi+4awA^2\text{csch}^{2p}\xi + ak^2w\left\{Ap^2\text{csch}^p\xi+Ap(p+1)\text{csch}^{p+2}\xi\right\}=0. \tag{57}$$

Similarly, from Eq. (57), equating the exponents $p+2$ and $2p$ we obtain $p=2$. From Eq. (57), setting the coefficients of $\text{csch}^{p+2}\xi$ and $\text{csch}^{2p}\xi$ terms to zero, we obtain

$$4awA^2+ak^2wAp(p+1)=0, \tag{58}$$

after some calculations, we have

$$A=-\frac{3k^2}{2}. \tag{59}$$

We find, from setting the coefficients of $\text{csch}^{p}\xi$ terms in Eq. (57) to zero

$$-cA+ak^2wAp^2=0, \tag{60}$$

also we get

$$c=4awk^2. \tag{61}$$

Thus, the 1-soliton solution of Eq. (42) is given by

$$u(x,y,t) = -\frac{3k^2}{2}\text{csch}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{wy^\alpha}{\Gamma(1+\alpha)}-\frac{4awk^2t^\alpha}{\Gamma(1+\alpha)}\right), \tag{62}$$

and

$$v(x,y,t) = -\frac{3wk}{2}\text{csch}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{wy^\alpha}{\Gamma(1+\alpha)}-\frac{4awk^2t^\alpha}{\Gamma(1+\alpha)}\right). \tag{63}$$

5. Conclusion

In this paper, we studied the non-topological and singular soliton solutions of nonlinear differential equations with fractional order. As a result, two types of soliton solutions for them have been successfully found. There are constraint conditions that must hold in order for the solitons to exist. The ansatz method was effectively used to achieve the goal set for this work. We have predicted that fractional complex transform with the help of the ansatz method can be extended to solve many systems of nonlinear fractional partial differential equations in mathematical and physical sciences.

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