Residual symmetry reductions and interaction solutions of the (2+1)-dimensional Burgers equation*

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In nonlinear physics, it is very difficult to study interactions among different types of nonlinear waves. In this paper, the nonlocal symmetry related to the truncated Painlevé expansion of the (2+1)-dimensional Burgers equation is localized after introducing multiple new variables to extend the original equation into a new system. Then the corresponding group invariant solutions are found, from which interaction solutions among different types of nonlinear waves can be found. Furthermore, the Burgers equation is also studied by using the generalized tanh expansion method and a new Bäcklund transformation (BT) is obtained. From this BT, novel interactive solutions among different nonlinear excitations are found.

Keywords: residual symmetry, Bäcklund transformation, symmetry reduction solution, generalized tanh expansion method

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1. Introduction

Due to the important role that the nonlinear theory plays in mathematics, physics, chemistry, biology, communications, astrophysics, and geophysics, much endeavor has been devoted to finding important and significant properties of different types of nonlinear equations, among which, the study on the integrable systems occupies the most important place. The integrable systems are introduced by scientists with some ideal conditions to describe complex nonlinear phenomena, such as the Korteweg–de Vries (KdV) equation,[1,2] the nonlinear Schrödinger equation (NLS),[3,4] the sine-Gordon (SG) equation,[5–8] the Davey–Stewartson (DS) equation,[9] and the Boussinesq equation.[10] Various effective methods including the inverse scattering transformation (IST), the Darboux transformation, the Bäcklund transformation,[11] Hirota's direct method,[12] the mapping and deformation approach,[13,14] the standard and extended truncated Painlevé analysis[15] have been well developed to find exact solutions for the integrable models.

Since Sophus Lie introduced the Lie group theory to study differential equations,[16] the symmetry theory has been developed to study nonlinear equations.[17–22] It is well known that by using Lie point symmetries, the dimensions of nonlinear equations can be reduced through classical or non-classical Lie group approaches and the exact solutions of these equations can be subsequently constructed. However, for the integrable systems, there exist infinitely many non-local symmetries which are linked with potential symmetries,[23,24] inverse recursion operators,[25] negative hierarchies, conformal invariance, Darboux transformations, Bäklund transformations, and so on. It would be very interesting to use these non-local symmetries to study the important properties of the corresponding equations. A direct idea is to localize these symmetries by extending the original system then use the traditional Lie point symmetry theory to reduce the extended system[26] and find new exact reduction solutions which are usually hard to find by the traditional Darboux transformation or Bäcklund transformation method.

The Painlevé analysis is usually used to study the integrability of nonlinear systems. The Painlevé integrable demands that there exist Laurent series on open sets of the complex time variable solutions and these solutions are consistent, or match on the overlapping pieces of the sets on which they are defined. In other words, this notion of integrability is the existence of a meromorphic solution. When the equations have the Painlevé property, the truncated Painlevé expansion method can be used to find the exact solutions of the equations. The recent studies on the Painlevé analysis reveals that the truncated Painlevé expansion is a Bäcklund transformation and the residue of the expansion is just the non-local symmetry related to this Bäcklund transformation, which is called the residual symmetry. In Ref. [27], the truncated Painlevé expansion method was further developed to the general hyperbolic tangent (tanh) function expansion approach by allowing the expansion coefficients to be arbitrary functions of space–time, and interaction solutions between solitons and any Schrödinger waves were obtained. Unlike the usual tanh expansion method, which loses much essential information of the original equation, the

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generalized tanh expansion method can be used to find much more general solutions, retrieving the missing essential properties.

In this paper, we concentrate on the (2+1)-dimensional Burgers equation with constants a, b in the form

$$u_t = uu_y + avu_x + bu_{yy} + abu_{xx}, \tag{1a}$$

$$u_x = v_y. \tag{1b}$$

We use the residual symmetry reduction and the generalized tanh function expansion approach to obtain new Bäcklund transformation transformations and interaction solutions between different solitons. An equivalent form of Eq. (1) has been derived from the generalized Painlevé integrability classification in Ref. [28].

The Burgers equation is a fundamental partial differential equation in the field of fluid mechanics. It also occurs in various areas of applied physics, such as the modeling of gas dynamics and traffic flow. The exact solutions of the Burgers equation have been studied by many authors. In Ref. [29], infinitely many symmetries and the exact solutions of the Burgers equation were obtained by the repeated symmetry reduction approach.

In Section 2 of this paper, the residual symmetry of the Burgers system is localized in a new extended system and then a new Bäcklund transformation is obtained via Lie's first theorem. In Section 3, the localized symmetry group is analyzed and categorized according to different transformation properties. The reduction solutions relating to space–time and the Möbious transformation symmetries are also obtained subsequently. In Section 4, the generalized tanh expansion approach is applied to the Burgers equation to obtain a new Bäcklund transformation. Using the Bäcklund transformation, some special explicit novel exact solutions are found and the interactions between them are also investigated. The last section is devoted to a short summary and discussion.

2. Localization of non-local residual symmetries of the Burgers equation

By means of the Painlevé analysis, various integrable properties, such as the Bäcklund transformation, the Lax pairs, infinitely many symmetries, and the bilinear forms, can be easily found if the studied model possesses the Painlevé property. For Eq. (1), the truncated Painlevé expansion takes the general form

$$u = \sum_{i=0}^{\alpha} u_i f^{i-\alpha}, \quad v = \sum_{j=0}^{\beta} v_j f^{j-\beta}, \tag{2}$$

where u_α and v_β are arbitrary solutions of the discussed system, while $u_{\alpha-1}, u_{\alpha-2}, \ldots, u_0$ and $v_{\beta-1}, v_{\beta-2}, \ldots, v_0$ are all related to the derivatives of f. In order to determine α and β, we first substitute $u \sim u_0 f^\alpha$ and $v \sim v_0 f^\beta$ into Eq. (1) and then balance the nonlinear and the dispersion terms to give $\alpha = 1$, $\beta = 1$. Hence the truncated Painlevé expansion reads

$$u = \frac{u_0}{f} + u_1, \quad v = \frac{v_0}{f} + v_1. \tag{3}$$

It is interesting that the residues u_0, v_0 are symmetries of Eq. (1) with solutions u_1 and v_1.

With the property of being Painlevé integrable, the Burgers system (1) is transformed by using Eq. (3) into its Schwarzian form

$$S(f) = 0, \tag{4}$$

which means that equation (4) is form invariant under the Möbious transformation

$$f \to \frac{a_1 f + b_1}{a_2 f + b_2}, \quad a_1 a_2 \neq b_1 b_2. \tag{5}$$

In other words, equation (4) possesses three symmetries $\sigma^f = d_1$, $\sigma^f = d_2 f$, and $\sigma^f = d_3 f^2$ with arbitrary constants d_1, d_2, and d_3.

In order to give the concrete expression of u_0, v_0, we substitute Eq. (3) into Eq. (1) and obtain

$$\begin{aligned} &-abu_{1,xx} - av_1 u_{1,x} - u_1 u_{1,y} + u_{1,t} - bu_{1,yy} + f^{-1}(-av_1 u_{0,x} \\ &- av_0 u_{1,x} - abu_{0,xx} - u_1 u_{0,y} + u_{0,t} - u_0 u_{1,y} - bu_{0,yy}) \\ &+ f^{-2}(av_1 u_0 f_x + 2abu_{0,x} f_x + abu_0 f_{xx} + bu_0 f_{yy} - av_0 u_{0,x} \\ &+ 2bu_{0,y} f_y - u_0 u_{0,y} + u_1 u_0 f_y - u_0 f_t) - f^{-3} u_0 (2abf_x^2 \\ &- av_0 f_x + 2bf_y^2 - u_0 f_y) = 0, \end{aligned} \tag{6}$$

$$u_{1,x} - v_{1,y} + f^{-1}(-v_{0,y} + u_{0,x}) + f^{-2}(v_0 f_y - u_0 f_x) = 0. \tag{7}$$

Vanishing the coefficient of f^{-3}, we have

$$u_0 = 2bf_y, \quad v_0 = 2bf_x. \tag{8}$$

Substituting Eq. (8) into Eq. (7) yields the Burgers equation (1b)

$$u_{1,x} - v_{1,y} = 0. \tag{9}$$

Vanishing the coefficient of f^{-2} in Eq. (6) by using Eq. (8), we have

$$av_1 f_x + abf_{xx} + bf_{yy} + u_1 f_y - f_t = 0, \tag{10}$$

while the coefficient of f^{-1} in Eq. (6) equals Eq. (9) by using Eqs. (8) and (10). Finally, vanishing the coefficient of f^0 in Eq. (6) yields

$$u_{1,t} - abu_{1,xx} - av_1 u_{1,x} - u_1 u_{1,y} - bu_{1,yy} = 0, \tag{11}$$

which is just the Burgers equation (1a) with solutions u_1 and v_1.

Now, using the standard truncated Painlevé expansion

$$u = \frac{2bf_y}{f} + u_1, \quad v = \frac{2bf_x}{f} + v_1,$$

we have the following Bäcklund transformation theorem.

Theorem 1 Transformation (12) is a Bäcklund transformation between solutions $\{u_1, v_1\}$ and $\{u, v\}$ if the former is related to f by Eq. (10).

Since u_0 and v_0 are non-local symmetries relating to Bäcklund transformation (3), one can naturally believe that they can be localized to a Lie point symmetry such that we can use Lie' s first theorem to recover the original Bäcklund transformation. To this end, we introduce new variables to eliminate the space derivatives of f

$$g \equiv f_x, \tag{12}$$

$$h \equiv f_y, \tag{13}$$

and investigate the symmetries between different variables. In other words, we have to solve the linearized form of the extended equations (1), (10), (12), and (13)

$$\sigma_{u_1,x} - \sigma_{v_1,y} = 0, \tag{14a}$$

$$-ab\sigma_{v_1,xy} + \sigma_{u_1,t} - b\sigma_{u_1,yy} - \sigma_{u_1}u_{1,y} - u_1\sigma_{u_1,y} - a\sigma_{v_1}v_{1,y} - av_1\sigma_{v_1,y} = 0, \tag{14b}$$

$$\sigma_{f,x} - \sigma_g = 0, \tag{14c}$$

$$\sigma_{f,y} - \sigma_h = 0, \tag{14d}$$

$$\sigma_{f,t} - b\sigma_{f,yy} - av_1\sigma_{f,x} - a\sigma_{v_1}f_x - ab\sigma_{f,xx} - f_y\sigma_{u_1} - u_1\sigma_{f,y} = 0. \tag{14e}$$

It can be easily verified that the solution of Eq. (14) has the form

$$\sigma_{u_1} = 2bh, \tag{15a}$$

$$\sigma_{v_1} = 2bg, \tag{15b}$$

$$\sigma_g = -\frac{fg}{b}, \tag{15c}$$

$$\sigma_h = -\frac{fh}{b}, \tag{15d}$$

$$\sigma_f = -\frac{f^2}{2b}, \tag{15e}$$

if d_3 is fixed as $-1/(2b)$.

The result (15) indicates that the residual symmetries (8) are localized in the properly extended system (1), (10), (12), and (13) with the Lie point symmetry vector

$$V = 2bh\partial_{u_1} + 2bg\partial_{v_1} - \frac{fg}{b}\partial_g - \frac{fh}{b}\partial_h - \frac{f^2}{2b}\partial_f. \tag{16}$$

In other words, the symmetries related to the truncated Painlevé expansion are just a special Lie point symmetry of the extended system.

Now we have obtained the localized residual symmetries, an interesting question is what kind of finite transformation would correspond to the Lie point symmetry (16). We have the following theorem.

Theorem 2 If $\{u_1, v_1, g, h, f\}$ is a solution of the extended system (1), (10), (12), and (13), then so is $\{\hat{u_1}, \hat{v_1}, \hat{g}, \hat{h}, \hat{f}\}$ given by

$$\hat{u_1} = u_1 + \frac{4hb^2\epsilon}{\epsilon f + 2b}, \tag{17a}$$

$$\hat{v_1} = v_1 + \frac{4gb^2\epsilon}{\epsilon f + 2b}, \tag{17b}$$

$$\hat{g} = \frac{4gb^2}{(\epsilon f + 2b)^2}, \tag{17c}$$

$$\hat{h} = \frac{4hb^2}{(\epsilon f + 2b)^2}, \tag{17d}$$

$$\hat{f} = \frac{2bf}{\epsilon f + 2b}, \tag{17e}$$

with arbitrary group parameter ϵ.

Proof Using Lie's first theorem on vector (16) with the corresponding initial condition

$$\frac{\mathrm{d}\hat{u_1}(\epsilon)}{\mathrm{d}\epsilon} = 2b\hat{h}(\epsilon), \quad \hat{u_1}(0) = u_1, \tag{18}$$

$$\frac{\mathrm{d}\hat{v_1}(\epsilon)}{\mathrm{d}\epsilon} = 2b\hat{g}(\epsilon), \quad \hat{v_1}(0) = v_1, \tag{19}$$

$$\frac{\mathrm{d}\hat{g}(\epsilon)}{\mathrm{d}\epsilon} = -\frac{\hat{f}(\epsilon)\hat{g}(\epsilon)}{b}, \quad \hat{g}(0) = g, \tag{20}$$

$$\frac{\mathrm{d}\hat{h}(\epsilon)}{\mathrm{d}\epsilon} = -\frac{\hat{f}(\epsilon)\hat{h}(\epsilon)}{b}, \quad \hat{h}(0) = h, \tag{21}$$

$$\frac{\mathrm{d}\hat{f}(\epsilon)}{\mathrm{d}\epsilon} = -\frac{\hat{f}^2(\epsilon)}{2b}, \quad \hat{f}(0) = f, \tag{22}$$

one can easily obtain the solutions of the above equations given in Theorem 2, thus the theorem is proved.

3. Symmetry reductions related to residual symmetries

Next let us consider the Lie point symmetry of the extended system in the general form

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial u_1} + V_1\frac{\partial}{\partial v_1} + G\frac{\partial}{\partial g} + H\frac{\partial}{\partial h} + F\frac{\partial}{\partial f}, \tag{23}$$

which means that the extended system (1), (10), (12), and (13) is invariant under the following transformation:

$$\{x, y, t, u, v, g, h, f\} \to \{x + \epsilon X, y + \epsilon Y, t + \epsilon T, u + \epsilon U, v + \epsilon V, g + \epsilon G, h + \epsilon G, h + \epsilon H, f + \epsilon F\} \tag{24}$$

with infinitesimal parameter ϵ. Equivalently, the symmetry in form (23) can be written in a function form

$$\sigma_{u_1} = Xu_{1x} + Yu_{1y} + Tu_{1t} - U_1, \tag{25a}$$

$$\sigma_{v_1} = Xv_{1x} + Yv_{1y} + Tv_{1t} - V_1, \tag{25b}$$

$$\sigma_g = Xg_x + Yg_y + Tg_t - G, \tag{25c}$$

$$\sigma_h = Xh_x + Yh_y + Th_t - H, \tag{25d}$$

$$\sigma_f = Xf_x + Yf_y + Tf_t - F. \tag{25e}$$

Substituting Eq. (25) into Eq. (14) and eliminating $u_{1,t}, v_{1,y}, g_t, h_x, h_t, f_x, f_y$, and f_t in terms of the extended system, we obtain more than 250 determining equations for functions X, Y, T, U_1, V_1, G, H, and F. By using a computer algebra, we finally obtain the desired result

$$\begin{aligned}
X &= \frac{c_1 xt}{2} + \frac{c_2 x}{2} + x_0,\\
Y &= \frac{c_1 ty}{2} + \frac{c_2 y}{2} + c_4 t + c_5,\\
T &= \frac{c_1 t^2}{2} + c_2 t + c_3,\\
U_1 &= -\frac{c_1 y}{2} - \frac{c_1 u_1 t}{2} + c_6 h - \frac{c_2 u_1}{2} - c_4,\\
V_1 &= -\frac{2x_{0,t} + av_1 c_1 t + av_1 c_2 - 2ac_6 g + c_1 x}{2a},\\
G &= -\frac{g(bc_1 t + bc_2 - 2bc_7 + 2c_6 f)}{2b},\\
H &= -\frac{h(bc_1 t + bc_2 - 2bc_7 + 2c_6 f)}{2b},\\
F &= -\frac{c_6 f^2 - 2bc_7 f - 2bc_8}{2b},
\end{aligned} \tag{26}$$

with arbitrary constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and arbitrary function x_0 of t.

The obtained symmetries can be categorized by different transformation invariances as follows.

(i) The residual symmetries and and the symmetries relating to the Möbious transformation can be obtained by setting $c_1 = c_2 = c_3 = c_4 = c_5 = x_0 = 0$ and $c_6 = c_7 = c_8 = -1$

$$\begin{aligned}
&\sigma_{u_1} = h, \quad \sigma_{v_1} = g, \quad \sigma_g = -\frac{(f-b)g}{b},\\
&\sigma_h = -\frac{(f-b)h}{b}, \sigma_f = -\frac{f^2 - 2bf - 2b}{2b}.
\end{aligned} \tag{27}$$

(ii) The symmetries relating to the time transformation can be found by setting $c_1 = c_2 = c_4 = c_5 = c_6 = c_7 = c_8 = x_0 = 0$ and $c_3 = 1$

$$\begin{aligned}
&\sigma_{u_1} = u_{1,t}, \sigma_{v_1} = v_{1,t},\\
&\sigma_g = g_t, \sigma_h = h_t, \sigma_f = f_t.
\end{aligned} \tag{28}$$

(iii) The symmetries relating to the y-space transformation can be found by setting $c_1 = c_2 = c_3 = c_4 = c_6 = c_7 = c_8 = x_0 = 0$ and $c_5 = 1$

$$\begin{aligned}
&\sigma_{u_1} = u_{1,y}, \quad \sigma_{v_1} = v_{1,y}, \quad \sigma_g = g_y,\\
&\sigma_h = h_y, \quad \sigma_f = f_y.
\end{aligned} \tag{29}$$

(iv) The symmetries relating to the scaling transformation can be found by setting $c_1 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = x_0 = 0$ and $c_2 = 2$

$$\begin{aligned}
\sigma_{u_1} &= xu_{1,x} + yu_{1,y} + 2tu_{1,t} + u_1,\\
\sigma_{v_1} &= xv_{1,x} + yv_{1,y} + 2tv_{1,t} + v_1,\\
\sigma_g &= xg_x + yg_y + 2tg_t + g,\\
\sigma_h &= xh_x + yh_y + 2th_t + h, \sigma_f = xf_x + yf_y + 2tf_t.
\end{aligned} \tag{30}$$

(v) The symmetries relating to time relevant x-space transformation can be found by setting $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$

$$\begin{aligned}
&\sigma_{u_1} = x_0 u_{1,x}, \quad \sigma_{v_1} = \frac{ax_0 v_{1,x} + x_{0,t}}{a},\\
&\sigma_g = x_0 g_x, \quad \sigma_h = x_0 h_x, \quad \sigma_f = x_0 f_x.
\end{aligned} \tag{31}$$

(vi) The symmetries relating to the Galileo transformation can be found by setting $c_1 = c_2 = c_3 = c_5 = c_6 = c_7 = c_8 = x_0 = 0$ and $c_4 = 1$

$$\begin{aligned}
&\sigma_{u_1} = 1 + tu_{1,y}, \quad \sigma_{v_1} = tv_{1,y}, \quad \sigma_g = tg_y,\\
&\sigma_h = th_y, \quad \sigma_f = tf_y.
\end{aligned} \tag{32}$$

(vii) The symmetries relating to the time-Möbious transformation can be found by setting $c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = x_0 = 0$ and $c_1 = 2$

$$\begin{aligned}
\sigma_{u_1} &= xtu_{1,x} + ytu_{1,y} + t^2 u_{1,t} + tu_1 + y,\\
\sigma_{v_1} &= xtv_{1,x} + ytv_{1,y} + t^2 v_{1,t} + tv_1 + \frac{x}{a},\\
\sigma_g &= xtg_x + ytg_y + t^2 g_t + gt,\\
\sigma_h &= xth_x + yth_y + t^2 h_t + ht,\\
\sigma_f &= xtf_x + ytf_y + t^2 f_t.
\end{aligned} \tag{33}$$

Now let us proceed to reduce the extended system by the standard Lie point symmetry group approach. Without loss of generality, we consider the symmetry reductions relating to the space–time invariant and the residual symmetries by setting $c_1 = c_2 = c_4 = c_7 = 0$ and $c_3 = c_5 = c_6 = c_8 = x_0 = 1$. Substituting Eq. (26) into Eq. (25) with Eqs. (12) and (13), we obtain

$$\begin{aligned}
\sigma_{u_1} &= u_{1,x} + u_{1,y} + u_{1,t} - f_y,\\
\sigma_{v_1} &= v_{1,x} + v_{1,y} + v_{1,t} - f_x,\\
\sigma_f &= \frac{1}{2b}(2bf_x + 2bf_y + 2bf_t + f^2 - 2b).
\end{aligned} \tag{34}$$

The corresponding group invariant solutions, which can be obtained by solving Eqs. (9)–(11) and (34) with $\sigma_{u_1} = \sigma_{v_1} = \sigma_f = 0$, have the general form

$$\begin{aligned}
f &= \sqrt{2b}\tanh\left(\frac{x+F}{\sqrt{2b}}\right), \quad u_1 = \sqrt{2b}F_Y \tanh\left(\frac{x+F}{\sqrt{2b}}\right) + U_1,\\
v_1 &= -\sqrt{2b}(F_Y + F_T - 1)\tanh\left(\frac{x+F}{\sqrt{2b}}\right) + V_1,
\end{aligned} \tag{35}$$

where the group invariant variables are $Y = -x + y$ and $T = -x + t$, while F, U_1, and V_1 are all group invariant functions of Y and T.

Substituting Eq. (35) into Eq. (12), we obtain the reduction solution for Burgers system (1)

$$u = \sqrt{2b}F_Y \tanh^{-1}\left(\frac{x+F}{\sqrt{2b}}\right) + U_1,$$

$$v = -\sqrt{2b}(F_Y + F_T - 1)\tanh^{-1}\left(\frac{x+F}{\sqrt{2b}}\right) + V_1. \quad (36)$$

In order to obtain the symmetry reduction equations for the group invariant functions F, U_1, and V_1, we first substitute Eq. (35) into Eq. (9) and find

$$U_{1,Y} + U_{1,T} + V_{1,Y} = 0. \quad (37)$$

Next, substituting Eq. (35) into Eq. (10) and vanishing different powers of $\tanh((x+F)/\sqrt{2b})$, we obtain

$$\begin{aligned}&(U_1 - aV_1)F_Y + 2baF_{YT} - (1+aV_1)F_T\\&+ b(a+1)F_{YY} + abF_{TT} + aV_1 = 0.\end{aligned} \quad (38)$$

Finally, we obtain three symmetry reduction equations by substituting Eq. (35) into Eq. (11) and vanishing different powers of $\tanh((x+F)/\sqrt{2b})$, two of them are trivial, leaving only the third one

$$\begin{aligned}&(aV_1 - U_1)U_{1,Y} + (aV_1 - U_1)F_Y^2 + [(1+aV_1)F_T\\&- baF_{TT} - 4baF_{YT} - 3b(a+1)F_{YY} - aV_1]F_Y\\&+ (1+aV_1)U_{1,T} - 2ba(F_{YY} + F_{YT})F_T + b[2aF_{YT}\\&- (a+1)U_{1,YY} - aU_{1,TT} + 2aF_{YY} - 2aU_{1,YT}] = 0.\end{aligned} \quad (39)$$

The entrance of the tanh part in Eq. (35) indicates the intrusion of an additional soliton to the background wave. In other words, the group invariant solution (35) is an interaction solution of one soliton and the background wave.

4. Bäcklund transformations and interaction solutions of the Burgers equation

The tanh function is the simplest automorphic function which possesses the property that its derivatives can be explicitly expressed by itself. Enlightened by the form of the reduction solutions of the Burgers equation in Eq. (35), we now use the general tanh function expansion method to study the interaction structure between different solitons.

The general truncated tanh expansion for (2+1)-dimensional Burgers system (1) has the form

$$u = w_1\tanh(\lambda t + \phi) + w_0, \quad (40a)$$

$$v = p_1\tanh(\lambda t + \phi) + p_0, \quad (40b)$$

where w_1, w_0, p_1, p_0, and ϕ are arbitrary functions of x, y, and t. The λt in Eq. (40) has been added for later convenience.

Now, by substituting the tanh expansion (40) into the Burgers system (1) and vanishing different powers of $\tanh(\phi)$ in both expansion equations, we can prove the following Bäcklund transformation theorem.

Theorem 3 If f_1 and f_2 are the solution of the Burgers system (1), then so are u and v given by

$$u = 2b(\tanh(\phi + \lambda t) - 1)\phi_y + f_1, \quad (41a)$$

$$v = 2b(\tanh(\phi + \lambda t) - 1)\phi_x + f_2, \quad (41b)$$

with

$$\phi_t = b\phi_{yy} - 2b\phi_y^2 + f_1\phi_y + ab\phi_{xx} - 2ab\phi_x^2 + af_2\phi_x - \lambda. \quad (42)$$

Proof Substituting Eq. (40) into Burgers system (1) with $\lambda = 0$ temporarily, we obtain

$$\begin{aligned}&-w_1(2b\phi_y^2 + 2ab\phi_x^2 - w_1\phi_y - ap_1\phi_x)\tanh^3(\phi) + (2bw_{1,y}\phi_y\\&+ w_0w_1\phi_y + abw_1\phi_{xx} - w_1w_{1,y} + 2abw_{1,x}\phi_x + ap_0w_1\phi_x\\&- ap_1w_{1,x} + bw_1\phi_{yy} - w_1\phi_t)\tanh^2(\phi) + (-abw_{1,xx}\\&+ 2bw_1\phi_y^2 - w_1w_{0,y} - bw_{1,yy} - w_0w_{1,y} - ap_1w_1\phi_x\\&- ap_1w_{0,x} - ap_0w_{1,x} - w_1^2\phi_y + 2abw_1\phi_x^2 + w_{1,t})\tanh(\phi)\\&- 2abw_{1,x}\phi_x - abw_1\phi_{xx} + w_1\phi_t - 2bw_{1,y}\phi_y + w_{0,t}\\&- abw_{0,xx} - w_0w_{0,y} - ap_0w_1\phi_x - bw_{0,yy} - w_0w_1\phi_y\\&- ap_0w_{0,x} - bw_1\phi_{yy} = 0,\end{aligned} \quad (43)$$

$$\begin{aligned}&(-w_1\phi_x + p_1\phi_y)\tanh^2(\phi) + (w_{1,x} - p_{1,y})\tanh(\phi) + w_1\phi_x\\&+ w_{0,x} - p_1\phi_y - p_{0,y} = 0.\end{aligned} \quad (44)$$

Vanishing the coefficients of $\tanh^3(\phi)$ in Eq. (43) and $\tanh^2(\phi)$ in Eq. (44), we obtain

$$w_1 = 2b\phi_y, \quad p_1 = 2b\phi_x. \quad (45)$$

The coefficients of $\tanh^2(\phi)$ and $\tanh(\phi)$ in Eq. (43) are equivalent after using Eq. (45), which reads

$$\phi_t = b\phi_{yy} + w_0\phi_y + ab\phi_{xx} + ap_0\phi_x. \quad (46)$$

After using Eq. (45), equation (44) is degenerated into the Burgers equation (1b)

$$w_{0,x} - p_{0,y} = 0. \quad (47)$$

Vanishing the coefficient of $\tanh^0(\phi)$ in Eq. (43), we obtain

$$f_{1,t} - f_1f_{1,y} - af_2f_{1,x} - bf_{1,yy} - abf_{1,xx} = 0, \quad (48)$$

with

$$f_1 = w_0 + 2b\phi_y, \quad (49)$$

$$f_2 = p_0 + 2b\phi_x. \quad (50)$$

It can be easily verified that f_1 and f_2 also satisfy

$$f_{1,x} - f_{2,y} = 0. \quad (51)$$

It is obvious that equations (48) and (51) are just the (2+1)-dimensional Burgers system. Substituting w_0 and p_0 by putting Eqs. (49) and (50) into Eq. (46) meanwhile replacing ϕ by $\lambda t + \phi$, we then obtain Eq. (42). Now substituting w_0, p_0, w_1, and p_1 into Eq. (40) by using Eqs. (45), (49), and (50), we finally obtain the Bäcklund transformation (41). The theorem is proved.

If the seed functions are taken as constants, i.e., $f_1 = c_1$ and $f_2 = c_2$, then equation (42) becomes a potential Burgers equation with source λ. Even for the seed solutions taken as constants, we can still obtain rich nontrivial solutions from the Bäcklund transformation (41). For instance, the straight line solution $\phi = k_1x + k_2y + \omega t$ of Eq. (42) leads to the single kink soliton solution

$$\begin{aligned} u &= -2bk_2\{\tanh[k_2(2bk_2 - c_1)t + ak_1(2bk_1 - c_2)t \\ &\quad - k_1x - k_2y] + 1\} + c_1, \qquad (52a) \\ v &= -2bk_1\{\tanh[k_2(2bk_2 - c_1)t + ak_1(2bk_1 - c_2)t \\ &\quad - k_1x - k_2y] + 1\} + c_2. \qquad (52b) \end{aligned}$$

The traveling wave solution of Eq. (42) with $f_1 = c_1$ and $f_2 = c_2$ leads to a kink soliton solution

$$\begin{aligned} \phi &= -\frac{(K - k_2c_1 - ak_1c_2 + \omega)(k_1x + k_2y + \omega t - e_0)}{4b(k_2^2 + ak_1^2)} \\ &\quad - \frac{1}{2}\ln\left[1 + \exp\left(\frac{-K(k_1x + k_2y + \omega t - e_1)}{b(k_2^2 + ak_1^2)}\right)\right], \qquad (53) \end{aligned}$$

where k_1, k_2, e_0, e_1, and ω are arbitrary constants and K is related to other constants by

$$\begin{aligned} K &= \{k_2^2c_1^2 - 2k_2c_1\omega + 2ak_1c_1k_2c_2 + \omega^2 - 2ac_2k_1\omega \\ &\quad + a^2c_2^2k_1^2 - 8k_2^2\lambda b - 8\lambda abk_1^2\}^{1/2}. \qquad (54) \end{aligned}$$

Substituting the kink soliton solution (53) into Eq. (41) with $f_1 = c_1$ and $f_2 = c_2$, we obtain the two-soliton fusion solution for u and v.

We draw the corresponding figures of Eqs. (53) and (41a) with Eq. (53) and the interaction structure between them by plotting projectively at the $y = 1$ plane, while the parameters are fixed as

$$\begin{aligned} &e_0 = 0, \ \ e_1 = 0, \ \ k_1 = 1, \ \ k_2 = 1, \ \ \omega = -1, \\ &a = 1, \ \ b = 1, \ \ c_1 = 3, \ \ c_2 = 3, \ \ \lambda = 1. \qquad (55) \end{aligned}$$

Figure 1 reveals that solution (53) of Eq. (42) is a kink soliton. From Fig. 2, which displays the soliton structure of Eq. (41a), we see that two kink solitons fused to one at some time. Figure 3 displays three solitons interaction structure between u of Eq. (41a) with $f_1 = c_1$ and ϕ of Eq. (53). It is interesting to find from this picture that three kink solitons fused to one after the interaction. Because the field v of Eq. (41b) resembles u in form, the interaction structure between field v with ϕ of Eq. (53) has the similar form as that in Fig. 3.

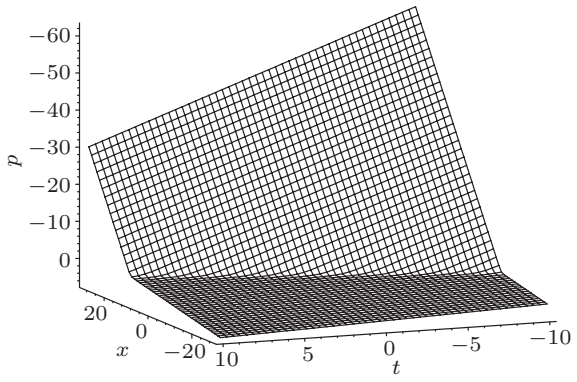


Fig. 1. The kink soliton solution (53).

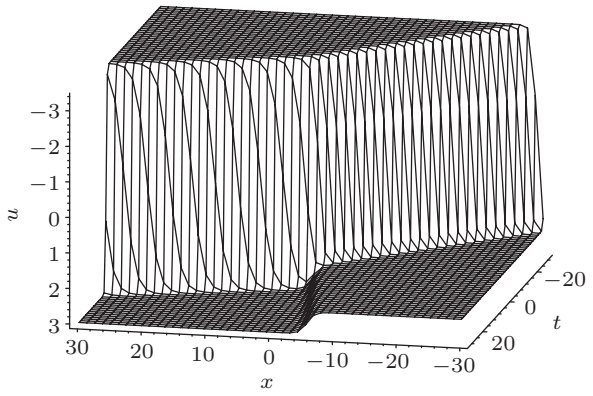


Fig. 2. The two kink solitons fusion structure of u in Eq. (41a) with Eq. (53).

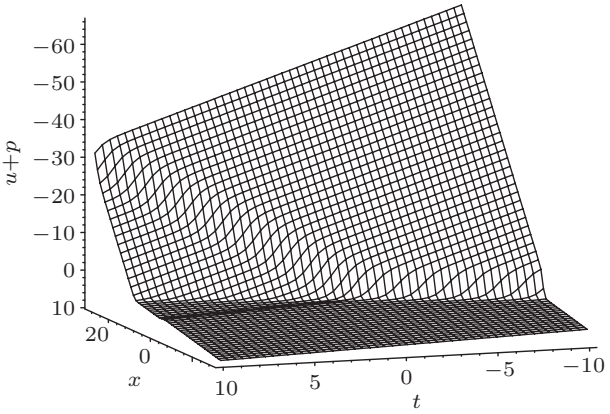


Fig. 3. Evolution of three kink solitons fusion interaction.

5. Discussion and conclusion

In summary, the (2+1)-dimensional Burgers equation is investigated by multiple ways to find the interactions among different nonlinear excitations. The nonlocal residual symmetries which correspond to the generators of the truncated Painlevé expansions are localized in a new prolonged system. The new Bäcklund transformation (Theorem 2) and the symmetry reduction solutions thus are obtained by using the classical Lie point symmetry approach. The reduction solutions

reveal that a new soliton is introduced to interact with the background waves. Enlightened by the form of the reduction solutions of the Burgers equation, we further study this system by a new generalized tanh expansion method. With this method, we give a new Bäcklund transformation (Theorem 3) for the Burgers system. From this Bäcklund transformation, we can obtain rich soliton structures even for the constant seed. Especially, two kink solitons fusion structure and the evolution of three kink solitons fusing to one are displayed by plotting projectively. This fact implies that starting from any seed solution, the Bäcklund transformation (Theorem 3) will introduce an additional soliton to the original seed.

In this paper, we have discussed the localization procedure only for one particular nonlocal symmetry, i.e., the residual symmetries. It is obvious that the the same procedure can be executed for other nonlocal symmetries, such as those obtained from the Bäcklund transformation, the bilinear forms, the negative hierarchies, the nonlinearizations,[30,31] and the self-consistent sources.[32] So it would be meaningful to apply this method to various nonlocal symmetries to find rich nonlinear interactive solutions among different excitations.

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