Exact solutions and residual symmetries of the Ablowitz–Kaup–Newell–Segur system*

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The residual symmetries of the Ablowitz–Kaup–Newell–Segur (AKNS) equations are obtained by the truncated Painlevé analysis. The residual symmetries for the AKNS equations are proved to be nonlocal and the nonlocal residual symmetries are extended to the local Lie point symmetries of a prolonged AKNS system. The local Lie point symmetries of the prolonged AKNS equations are composed of the residual symmetries and the standard Lie point symmetries, which suggests that the residual symmetry method is a useful complement to the classical Lie group theory. The calculation on the symmetries shows that the enlarged equations are invariant under the scaling transformations, the space–time translations, and the shift translations. Three types of similarity solutions and the reduction equations are demonstrated. Furthermore, several types of exact solutions for the AKNS equations are obtained with the help of the symmetry method and the Bäcklund transformations between the AKNS equations and the Schwarzian AKNS equation.

Keywords: residual symmetries, Ablowitz–Kaup–Newell–Segur equation, exact solution, Bäcklund transformation

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1. Introduction

Partial differential equations (PDEs) are widely used to describe complex phenomena in various fields.[1–5] The study of the exact solutions of nonlinear evolution equations plays an important role in the soliton theory and the explicit formulas of PDEs.[6,7] The explicit formulas may also provide physical information and help us to understand the mechanism of related physical models. Many effective methods for obtaining the explicit solutions of PDEs have been presented, such as the classical and non-classical Lie group approaches, the truncated Painlevé expansion method, the Bäcklund transformation, the variational method, the multi-linear variable separation approach, the function expansion method, and the mapping method.[8–14]

One important solitary wave system is the Ablowitz–Kaup–Newell–Segur (AKNS) system

$$p_t + \frac{1}{2}\mathrm{i}p_{xx} - \mathrm{i}p^2 q = 0, \tag{1}$$

$$q_t - \frac{1}{2}\mathrm{i}q_{xx} + \mathrm{i}q^2 p = 0, \tag{2}$$

where subscripts x and t represent the partial differentiations. Several exact solutions of the AKNS equation were studied in Ref. [15]. Soliton scattering with changing amplitude was proposed in a negative order AKNS equation.[16] When $q = p^*$, equations (1) and (2) turn to the following nonlinear Schrödinger equation

$$\mathrm{i}p_t - \frac{1}{2}p_{xx} + |p|^2 p = 0. \tag{3}$$

The Painlevé analysis is considered as one of the most powerful and systematic methods to analyze the integrability of nonlinear systems.[17–22] Several years ago, Painlevé and his contemporaries identified a class of second-order nonlinear ordinary differential equations, in which the movable singularities exhibited by the general solution are only poles in the complex time plane.[17] Such systems are called the Painlevé type, possessing the Painlevé property. Ablowitz, Ramani, and Segur (ARS) pointed out the intimate connection between the singularity structure and the integrability of the system, particularly for the soliton equations.[21] Weiss, Tabor, and Carnevale (WTC) extended the Painlevé analysis from ordinary differential equations (ODEs) to partial differential equations (PDEs), and defined that a PDE has the Painlevé property when its solutions are single-valued on the movable singularity manifolds.[22] It means that the general solution to the model

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can be expanded locally in a Laurent-like series

$$v = v(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{\infty} v_i f^{i+a} \tag{4}$$

with sufficiently arbitrary functions of the equal orders of the differential equation system, where a is a negative integer, $v = v(x_1, x_2, \ldots, x_n)$ is a solution of the PDE, and $f = f(x_1, x_2, \ldots, x_n)$ is an analytic function. The formula $f(x_1, x_2, \ldots, x_n) = 0$ determines the singularity manifold.

The truncated Painlevé analysis and the function expansion method are usually applied to find traveling wave and non-traveling wave solutions for PDEs. For the coupled AKNS equations, the truncated Painlevé expansions can be written as[23]

$$p = p_0/f + p_1, \tag{5}$$

$$q = q_0/f + q_1, \tag{6}$$

where

$$p_0 = f_x \mathrm{e}^{\mathrm{i}u}, \quad p_1 = -\left[\mathrm{i}\frac{f_t}{f_x} + 2\mathrm{i}\lambda + \frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{\mathrm{i}u}, \tag{7}$$

$$q_0 = f_x \mathrm{e}^{-\mathrm{i}u}, \quad q_1 = \left[\mathrm{i}\frac{f_t}{f_x} + 2\mathrm{i}\lambda - \frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{-\mathrm{i}u}, \tag{8}$$

with λ being an arbitrary constant. From the above truncated Painlevé expansions, we can prove that the transformation

$$p = p_1 = -\left[\mathrm{i}\frac{f_t}{f_x} + 2\mathrm{i}\lambda + \frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{\mathrm{i}u}, \tag{9}$$

$$q = q_1 = \left[\mathrm{i}\frac{f_t}{f_x} + 2\mathrm{i}\lambda - \frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{-\mathrm{i}u} \tag{10}$$

is a Bäcklund transformation between the AKNS equations and the Schwarzian AKNS equation[23]

$$\left(\frac{f_t}{f_x}\right)_t = \left[\frac{3{f_t}^2}{2{f_x}^2} - \frac{1}{4}\left(\frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{{f_{xx}}^2}{{f_x}^2}\right) + 4\lambda\frac{f_t}{f_x}\right]_x. \tag{11}$$

The phase u should satisfy the consistent conditions

$$u_x = 2\frac{f_t}{f_x} + 2\lambda, \tag{12}$$

$$u_t = 3\frac{{f_t}^2}{{f_x}^2} + 8\lambda\frac{f_t}{f_x} + 6\lambda^2 - \frac{1}{2}\left(\frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{{f_{xx}}^2}{{f_x}^2}\right). \tag{13}$$

In addition to the truncated Painlevé analysis, the symmetry method is also a powerful method researching the PDEs. There are several methods proposed and proved efficient for finding the symmetries of the PDEs, including the famous classical and non-classical Lie group approaches, and the direct method.[9,10,24–28] The residual symmetry method was newly proposed in Ref. [29]. We will study the exact solutions of the AKNS equations by combining the residual symmetry method and the Bäcklund transformation. The paper is organized as follows. In Section 2, we propose the nonlocal residual symmetries of the AKNS equations and enlarge the AKNS equations to a prolonged AKNS system in order to research the local symmetry of the AKNS equations. Section 3 is devoted to similarity solutions and reduction equations of the prolonged AKNS system. In Section 4, the exact solutions of the AKNS equations are searched for by means of the symmetry method and the Bäcklund transformation. The final section presents our conclusion and discussion.

2. Nonlocal residual symmetries of the AKNS equations

The symmetry equations of Eqs. (1) and (2) are

$${\sigma^p}_t + \frac{1}{2}\mathrm{i}{\sigma^p}_{xx} - 2\mathrm{i}pq\sigma^p - \mathrm{i}p^2\sigma^q = 0, \tag{14}$$

$${\sigma^q}_t - \frac{1}{2}\mathrm{i}{\sigma^q}_{xx} + \mathrm{i}\sigma^p q^2 + 2\mathrm{i}pq\sigma^q = 0, \tag{15}$$

which means that equations (1) and (2) are form invariant under the transformations

$$p \longrightarrow p + \epsilon\sigma^p, \quad q \longrightarrow q + \epsilon\sigma^q \tag{16}$$

with infinitesimal parameter ϵ. By means of the standard classical Lie group theory, we can easily obtain the following theorem for the AKNS equations.

Theorem 1 The AKNS equations (1) and (2) possess the Lie point symmetries

$$\begin{aligned}\sigma^p &= (A_1x + 2A_1\lambda t + A_3)p_x + (2A_1t + A_4)p_t \\ &\quad - (A_2 - 2\mathrm{i}A_1\lambda x)p,\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma^q &= (A_1x + 2A_1\lambda t + A_3)q_x + (2A_1t + A_4)q_t \\ &\quad - (2\mathrm{i}A_1\lambda x - A_2 - 2A_1)q,\end{aligned} \tag{18}$$

where A_1, A_2, A_3, and A_4 are arbitrary integral constants.

Furthermore, from the AKNS equations (1) and (2) and the corresponding symmetry equations (14) and (15), we find another theorem.

Theorem 2 If f is a solution of the Schwarzian AKNS equation (11), then

$$\sigma = (\sigma_1, \sigma_2) = (p_0, q_0) = \left(f_x \mathrm{e}^{\mathrm{i}u}, f_x \mathrm{e}^{-\mathrm{i}u}\right) \tag{19}$$

is a nonlocal symmetry solution of the AKNS equations (1) and (2).

Proof Substituting formula $(\sigma_1, \sigma_2) = (p_0, q_0) = (f_x \mathrm{e}^{\mathrm{i}u}, f_x \mathrm{e}^{-\mathrm{i}u})$ and the Schwarzian AKNS equation (11) into the symmetry equations (14) and (15), both equations (14) and (15) turn to $0 = 0$, which means that $(\sigma_1, \sigma_2) = (p_0, q_0)$ is a good choice for the symmetry of the AKNS equations.

It is worthwhile to note that the symmetry components p_0 and q_0 are residues of the truncated Painlevé expansions expressed by Eqs. (5) and (6). Reference [29] defined the nonlocal symmetry as the residual symmetry. In order to research the localized symmetry, we have to prolong the original system so that the nonlocal symmetry of the original system can

be changed to a local Lie point symmetry of the prolonged system. For the AKNS equations, the prolonged system can be written as

$$p_t+\frac{1}{2}\mathrm{i}p_{xx}-\mathrm{i}p^2q=0, \tag{20}$$

$$q_t-\frac{1}{2}\mathrm{i}q_{xx}+\mathrm{i}q^2p=0, \tag{21}$$

$$p=-\left[\mathrm{i}\frac{f_t}{f_x}+2\mathrm{i}\lambda+\frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{\mathrm{i}u}, \tag{22}$$

$$q=\left[\mathrm{i}\frac{f_t}{f_x}+2\mathrm{i}\lambda-\frac{1}{2}\frac{f_{xx}}{f_x}\right]\mathrm{e}^{-\mathrm{i}u}, \tag{23}$$

$$\left(\frac{f_t}{f_x}\right)_t=\left[\frac{3f_t^2}{2f_x^2}-\frac{1}{4}\left(\frac{f_{xxx}}{f_x}-\frac{3}{2}\frac{{f_{xx}}^2}{f_x^2}\right)+4\lambda\frac{f_t}{f_x}\right]_x, \tag{24}$$

$$u_x=2\frac{f_t}{f_x}+2\lambda, \tag{25}$$

$$u_t=3\frac{f_t^2}{f_x^2}+8\lambda\frac{f_t}{f_x}+6\lambda^2-\frac{1}{2}\left(\frac{f_{xxx}}{f_x}-\frac{3}{2}\frac{{f_{xx}}^2}{f_x^2}\right), \tag{26}$$

$$f_x=g. \tag{27}$$

Let us consider a one-parameter Lie group of infinitesimal transformation

$$\bar{x} = x+\epsilon X+\mathrm{O}(\epsilon^2), \tag{28}$$

$$\bar{t} = t+\epsilon T+\mathrm{O}(\epsilon^2), \tag{29}$$

$$\bar{p} = p+\epsilon P+\mathrm{O}(\epsilon^2), \tag{30}$$

$$\bar{q} = q+\epsilon Q+\mathrm{O}(\epsilon^2), \tag{31}$$

$$\bar{u} = u+\epsilon U+\mathrm{O}(\epsilon^2), \tag{32}$$

$$\bar{f} = f+\epsilon F+\mathrm{O}(\epsilon^2), \tag{33}$$

$$\bar{g} = g+\epsilon G+\mathrm{O}(\epsilon^2), \tag{34}$$

where $X\equiv X(x,t,p,t,u,f,g)$, $T\equiv T(x,t,p,t,u,f,g)$, $U\equiv U(x,t,p,t,u,f,g)$, $F\equiv F(x,t,p,t,u,f,g)$, $G\equiv G(x,t,p,t,u,f,g)$, which leave Eqs. (20)–(27) invariant. The corresponding five symmetry components $\{\sigma^p,\ \sigma^q,\ \sigma^u,\ \sigma^f,\ \sigma^g\}$ can be assumed to have the form

$$\sigma^p=Xp_x+Tp_t-P, \tag{35}$$

$$\sigma^q=Xq_x+Tq_t-Q, \tag{36}$$

$$\sigma^u=Xu_x+Tu_t-U, \tag{37}$$

$$\sigma^f=Xf_x+Tf_t-F, \tag{38}$$

$$\sigma^g=Xg_x+Tg_t-G. \tag{39}$$

According to the standard process of the classical Lie point symmetry method, the symmetry equations of the enlarged system (20)–(27) are

$${\sigma^p}_t+\frac{1}{2}\mathrm{i}{\sigma^p}_{xx}-2\mathrm{i}pq\sigma^p-\mathrm{i}p^2\sigma^q=0, \tag{40}$$

$${\sigma^q}_t-\frac{1}{2}\mathrm{i}{\sigma^q}_{xx}+\mathrm{i}{\sigma^p}q^2+2\mathrm{i}pq\sigma^q=0, \tag{41}$$

$$\begin{aligned}&2{\sigma^p}{f_x}^2+\mathrm{i}\,\mathrm{e}^{\mathrm{i}u}{f_x}^2{\sigma^u}_x+\mathrm{e}^{\mathrm{i}u}{\sigma^f}_xf_x-\mathrm{e}^{\mathrm{i}u}f_{xx}{\sigma^f}_x\\&-\sigma^u\mathrm{e}^{\mathrm{i}u}{f_x}^2u_x-2\sigma^u\mathrm{e}^{\mathrm{i}u}{f_x}^2\lambda+\mathrm{i}\sigma^u\mathrm{e}^{\mathrm{i}u}f_xf_{xx}=0,\end{aligned} \tag{42}$$

$$\begin{aligned}&2\sigma^q{f_x}^2-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}u}{f_x}^2{\sigma^u}_x+\mathrm{e}^{-\mathrm{i}u}{\sigma^f}_xf_x-\mathrm{e}^{-\mathrm{i}u}f_{xx}{\sigma^f}_x\\&-\sigma^u\mathrm{e}^{-\mathrm{i}u}{f_x}^2u_x-2\sigma^u\mathrm{e}^{-\mathrm{i}u}{f_x}^2\lambda\\&-\mathrm{i}\sigma^u\mathrm{e}^{-\mathrm{i}u}f_xf_{xx}=0,\end{aligned} \tag{43}$$

$$\begin{aligned}&3{f_t}^2{\sigma^f}_{xx}+4[(\lambda{\sigma^f}_{xx}-{\sigma^f}_{xt})f_x-4{\sigma^f}_xf_{xt}\\&+(6{\sigma^f}_t+4\lambda{\sigma^f}_x)f_{xx}]f_t+\frac{9}{4}{f_{xx}}^2{\sigma^f}_{xx}\\&+\left[2f_{tt}{\sigma^f}_x-4({\sigma^f}_t+2\lambda{\sigma^f}_x)f_{xt}+(4\lambda{\sigma^f}_t-{\sigma^f}_{xxx})f_{xx}\right.\\&\left.-f_{xxx}{\sigma^f}_{xx}+\frac{1}{2}f_{xxxx}{\sigma^f}_x\right]f_x-f_{xxx}f_{xx}{\sigma^f}_x+({\sigma^f}_{tt}-4\lambda{\sigma^f}_{xt}\\&+\frac{1}{4}{\sigma^f}_{xxxx}){f_x}^2=0,\end{aligned} \tag{44}$$

$${\sigma^f}_t-\frac{1}{2}{\sigma^u}_xf_x-\frac{1}{2}u_x{\sigma^f}_x+\lambda{\sigma^f}_x=0, \tag{45}$$

$$\begin{aligned}&2{\sigma^u}_t{f_x}^3-3u_x{f_x}^3{\sigma^u}_x-2{\sigma^u}_x{f_x}^3\lambda+{\sigma^f}_{xxx}{f_x}^2\\&-f_xf_{xxx}{\sigma^f}_x-3f_xf_{xx}{\sigma^f}_{xx}+3{\sigma^f}_x{f_{xx}}^2=0,\end{aligned} \tag{46}$$

$$\sigma^g-{\sigma^f}_x=0. \tag{47}$$

Putting Eqs. (35)–(39) into the prolonged system (20)–(27) and identifying all the coefficients of $\{p,q,u,f,g\}$, we obtain the over-determined equations for the unknown functions $X(x,t,p,q,u,f,g)$, $T(x,t,p,q,u,f,g)$, $P(x,t,p,q,u,f,g)$, $Q(x,t,p,q,u,f,g)$, $U(x,t,p,q,u,f,g)$, $F(x,t,p,q,u,f,g)$, $G(x,t,p,q,u,f,g)$. Solving the determinant equations, we obtain the following theorem.

Theorem 3 The prolonged AKNS equations (20)–(27) possess the Lie point symmetries

$$\begin{aligned}\sigma^p &= (C_1x+2C_1\lambda t+C_3)p_x+(2C_1t+C_4)p_t\\&\quad-(C_5-2\mathrm{i}C_1\lambda x)p-C_6\mathrm{e}^{\mathrm{i}u}g,\end{aligned} \tag{48}$$

$$\begin{aligned}\sigma^q &= (C_1x+2C_1\lambda t+C_3)q_x+(2C_1t+C_4)q_t\\&\quad-(2\mathrm{i}C_1\lambda x-C_5-2C_1)q-C_6\mathrm{e}^{-\mathrm{i}u}g,\end{aligned} \tag{49}$$

$$\begin{aligned}\sigma^u &= (C_1x+2C_1\lambda t+C_3)u_x+(2C_1t+C_4)u_t\\&\quad+(\mathrm{i}C_1+\mathrm{i}C_5+2C_1\lambda x),\end{aligned} \tag{50}$$

$$\begin{aligned}\sigma^f &= (C_1x+2C_1\lambda t+C_3)f_x+(2C_1t+C_4)f_t\\&\quad+(C_6f^2-C_2f-C_6),\end{aligned} \tag{51}$$

$$\begin{aligned}\sigma^g &= (C_1x+2C_1\lambda t+C_3)g_x+(2C_1t+C_4)g_t\\&\quad+(2C_6f-C_2+C_1)g,\end{aligned} \tag{52}$$

where C_1, C_2, C_3, C_4, C_5, and C_6 are arbitrary integral constants. Obviously, $C_6\mathrm{e}^{\mathrm{i}u}g$ in σ^p, $C_6\mathrm{e}^{-\mathrm{i}u}g$ in σ^q, C_6f^2 in σ^f, and C_6fg in σ^g are related to the residual symmetries.

The vector field associated with the above group of transformations can be written as

$$\begin{aligned}V &= (C_1x+2C_1\lambda t+C_3)\frac{\partial}{\partial x}+(2C_1t+C_4)\frac{\partial}{\partial t}\\&\quad-(\mathrm{i}C_1+\mathrm{i}C_5+2C_1\lambda x)\frac{\partial}{\partial u}\\&\quad+[(C_5-2\mathrm{i}C_1\lambda x)p+C_6\mathrm{e}^{\mathrm{i}u}g]\frac{\partial}{\partial p}\end{aligned}$$

$$+[(2\mathrm{i}C_1\lambda x-C_5-2C_1)q+C_6\mathrm{e}^{-\mathrm{i}u}g]\frac{\partial}{\partial q}+(C_2f+C_6-C_6f^2)\frac{\partial}{\partial f}+(C_2-C_1-2C_6f)g\frac{\partial}{\partial g}. \quad (53)$$

All invariant linearly independent infinitesimal generators can be written as

$$V=C_1V_1+C_2V_2+C_3V_3+C_4V_4+C_5V_5+C_6V_6, \quad (54)$$

where

$$\begin{aligned}
V_1 &= (x+2\lambda t)\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-2\mathrm{i}\lambda xp\frac{\partial}{\partial p}+(2\mathrm{i}\lambda x-2)q\frac{\partial}{\partial q}\\
&\quad-(\mathrm{i}+2\lambda x)\frac{\partial}{\partial u}-g\frac{\partial}{\partial g},\\
V_2 &= f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g},\ V_3=\frac{\partial}{\partial x},\ V_4=\frac{\partial}{\partial t},\\
V_5 &= p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q}-\mathrm{i}\frac{\partial}{\partial u},\\
V_6 &= \mathrm{e}^{\mathrm{i}u}g\frac{\partial}{\partial p}+\mathrm{e}^{-\mathrm{i}u}g\frac{\partial}{\partial q}+(1-f^2)\frac{\partial}{\partial f}-2fg\frac{\partial}{\partial g}. \quad (55)
\end{aligned}$$

It is obvious that V_1 ,V_2, and V_5 are scaling transformations, V_3 and V_4 indicate space and time translations, and V_6 is the Galilean translations.

Remark 1 Comparing Theorem 1, Theorem 2, and Theorem 3, we can find that the symmetries for p and q of the enlarged AKNS equations are richer than those of the AKNS equations. The standard Lie point symmetries and the residual symmetries together constitute the Lie point symmetries of the enlarged system. So, the residual symmetries are a useful supplement for the Lie group theory.

3. Similarity solutions related to the nonlocal residual symmetries

After determining the Lie point symmetries of the prolonged system (20)–(27), we then find the group invariant solutions and the group invariants which satisfy

$$\sigma^p=\sigma^q=\sigma^u=\sigma^f=\sigma^g=0. \quad (56)$$

The fact that the residual symmetries include exponents leads to such complicated calculations that we cannot obtain any similarity solutions when all the parameters are included. In order to obtain some similarity solutions and reduction equations, we have to simplify some parameters. In the following, C_6 is always treated as 0. The combination of Eqs. (48)–(52) and (56) leads to the following three types of non-trivial similarity solutions.

Case 1 $C_1\neq 0$. In this case, we have the first type of similarity solutions

$$\begin{aligned}
p &= P(\xi)(2C_1t+C_4)^{[(-2\mathrm{i}\lambda^2C_4+2\mathrm{i}\lambda C_3+C_5)/(2C_1)]}\\
&\quad\times\mathrm{e}^{[2\mathrm{i}\lambda(-x+\lambda t)+(2\mathrm{i}(2C_4\lambda-C_3)\lambda)/C_1]}, \quad (57)\\
q &= Q(\xi)(2C_1t+C_4)^{[(-2\mathrm{i}\lambda C_3-2\mathrm{i}\lambda^2C_4+C_5+2C_1)/(2C_1)]}\\
&\quad\times\mathrm{e}^{[2\mathrm{i}\lambda(x-\lambda t)+2\mathrm{i}(C_3-2C_4\lambda)\lambda/C_1]}, \quad (58)\\
u &= U(\xi)-\frac{\mathrm{i}(C_1+2\mathrm{i}\lambda C_3+C_5-2\mathrm{i}\lambda^2C_4)\ln(2C_1t+C_4)}{2C_1}\\
&\quad-2\lambda(x-\lambda t)-\frac{2(C_3-2C_4\lambda)\lambda}{C_1}, \quad (59)\\
f &= F(\xi)(2C_1t+C_4)^{\frac{C_2}{2C_1}}, \quad (60)\\
g &= G(\xi)(2C_1t+C_4)^{\frac{C_2-C_1}{2C_1}}, \quad (61)
\end{aligned}$$

where the group invariant is

$$\xi=-\frac{-C_1x+2C_4\lambda-C_3+2C_1\lambda t}{\sqrt{2C_1t+C_4}C_1}, \quad (62)$$

and the reduction functions $P(\xi)$, $Q(\xi)$, $U(\xi)$, $F(\xi)$, and $G(\xi)$ are governed by the following reduction equations:

$$\begin{aligned}
&P_{\xi\xi}+2\mathrm{i}C_1\xi P_\xi-2P^2Q\\
&+2(2\lambda C_3-\mathrm{i}C_5-2C_4\lambda^2)P=0, \quad (63)\\
&Q_{\xi\xi}-2\mathrm{i}C_1\xi Q_\xi-2PQ^2\\
&-2Q(2\lambda^2C_4-2\lambda C_3+\mathrm{i}C_5+2\mathrm{i}C_1)=0, \quad (64)\\
&2PG+\mathrm{i}GU_\xi\mathrm{e}^{\mathrm{i}U}+G_\xi\mathrm{e}^{\mathrm{i}U}=0, \quad (65)\\
&2QG\mathrm{e}^{\mathrm{i}U}-\mathrm{i}GU_\xi+G_\xi=0, \quad (66)\\
&3{G_\xi}^3-[4GG_{\xi\xi}+4FC_2(2G\xi C_1-3FC_2)]G_\xi\\
&+G^2G_{\xi\xi\xi}-4G^2(G{C_1}^2\xi-2GC_1\xi C_2\\
&-2C_1C_2F+3F{C_2}^2)=0, \quad (67)\\
&-2FC_2+GU_\xi+2G\xi C_1=0, \quad (68)\\
&-3G^2{U_\xi}^2-4C_1G^2U_\xi\xi-4(\mathrm{i}C_5-2\lambda C_3\\
&+2\lambda^2C_4+\mathrm{i}C_1)G^2+2GG_{\xi\xi}-3{G_\xi}^2=0, \quad (69)\\
&G=F_\xi. \quad (70)
\end{aligned}$$

Case 2 $C_1=0$ and $C_3\neq 0$. The second type of reduction equations are

$$\begin{aligned}
&P_{\xi\xi}{C_4}^2-2(C_4C_5+\mathrm{i}{C_3}^2)P_\xi-2{C_3}^2P^2Q+P{C_5}^2=0, \quad (71)\\
&Q_{\xi\xi}{C_4}^2+2(C_4C_5+\mathrm{i}{C_3}^2)Q_\xi-2PQ^2{C_3}^2+Q{C_5}^2=0, \quad (72)\\
&\mathrm{i}C_4U_\xi G+G_\xi C_4-2GPC_3\mathrm{e}^{-\mathrm{i}U}\\
&-G(C_2+C_5+2\mathrm{i}C_3\lambda)=0, \quad (73)\\
&-\mathrm{i}C_4U_\xi G+G_\xi C_4-2\mathrm{e}^{\mathrm{i}U}C_3QG\\
&+(C_5-C_2+2\mathrm{i}C_3\lambda)G=0, \quad (74)\\
&C_2{C_4}^4{G_\xi}^2G-3{C_4}^5{G_\xi}^3+4C_4[{C_4}^4G_{\xi\xi}G+{C_3}^2C_2F(2GC_3\\
&-4\lambda GC_4-3C_2F)]G_\xi-C_2{C_4}^4G_{\xi\xi}G^2-{C_4}^5G_{\xi\xi\xi}G^2\\
&+4{C_3}^2C_2G(GC_3-C_2F)\\
&\times(2GC_3-4\lambda GC_4-3C_2F)=0, \quad (75)\\
&{C_4}^2U_\xi G+(2C_3C_4\lambda+C_4C_5\mathrm{i}-2{C_3}^2)G+2C_3C_2F=0, \quad (76)\\
&2{C_4}^2G_{\xi\xi}-3{C_4}^2G{U_\xi}^2-2(3\mathrm{i}C_4C_5-2C_3C_4\lambda-2{C_3}^2)GU_\xi\\
&-3{C_4}^2{G_\xi}^2/G+2C_2C_4G_\xi+(3{C_5}^2-{C_2}^2
\end{aligned}$$

$$+4\mathrm{i}C_5\lambda C_3-4\lambda^2{C_3}^2)G=0, \tag{77}$$

$$G=\frac{C_2F}{C_3}-\frac{F_\xi C_4}{C_3}, \tag{78}$$

where the group variable is $\xi=t-C_4x/C_3$, and the similarity solutions for the enlarged AKNS system (20)–(27) are

$$p = P\mathrm{e}^{C_5x/C_3}, \tag{79}$$

$$q = Q\mathrm{e}^{-C_5x/C_3}, \tag{80}$$

$$u = -\mathrm{i}C_5x/C_3+U, \tag{81}$$

$$f = \mathrm{e}^{C_2x/C_3}F, \tag{82}$$

$$g = \mathrm{e}^{C_2x/C_3}G. \tag{83}$$

Case 3 $C_1=0$ and $C_3=0$. The final type of the group variable can be written as

$$\xi=x. \tag{84}$$

The corresponding similarity solution is

$$p = P\mathrm{e}^{C_5t/C_4}, \tag{85}$$

$$q = Q\mathrm{e}^{-C_5t/C_4}, \tag{86}$$

$$u = -\mathrm{i}C_5t/C_4+U, \tag{87}$$

$$f = F\mathrm{e}^{C_2t/C_4}, \tag{88}$$

$$g = G\mathrm{e}^{C_2/C_4}, \tag{89}$$

where the group functions $P\equiv P(x)$, $Q\equiv Q(x)$, $U\equiv U(x)$, $F\equiv F(x)$, and $G\equiv G(x)$ are determined by the reduction equations

$$-2\mathrm{i}PC_5+P_{xx}C_4-2P^2QC_4=0, \tag{90}$$

$$Q_{xx}C_4-2\mathrm{i}QC_5-2PQ^2C_4=0, \tag{91}$$

$$2PG\mathrm{e}^{-\mathrm{i}U}+\mathrm{i}GU_x+2\mathrm{i}\lambda G+G_x=0, \tag{92}$$

$$2QG\mathrm{e}^{\mathrm{i}U}-\mathrm{i}GU_x-2\mathrm{i}\lambda G+G_x=0, \tag{93}$$

$$3{G_x}^3{C_4}^2+4G_x[FC_2(3FC_2+\lambda GC_4)-4GG_{xx}{C_4}^2]$$
$$+G^2G_{xxx}{C_4}^2-4C_2G^2(3FC_2+4\lambda GC_4)=0, \tag{94}$$

$$2FC_2-U_xGC_4+2\lambda GC_4=0, \tag{95}$$

$$-3G^2C_4{U_x}^2-4G^2C_4U_x\lambda-3{G_x}^2C_4+2G_{xx}GC_4$$
$$-4(C_4\lambda^2+\mathrm{i}C_5)G^2=0, \tag{96}$$

$$G=F_x. \tag{97}$$

4. Exact solutions of the AKNS equations

To obtain some exact solutions of the AKNS equations by means of the symmetry method, we need to obtain the solutions of the reduction equations. Different reduction equations lead to different results. We will consider the most difficult condition in this section, i.e., Eqs. (63)–(70). After calculation, we obtain the following three types of nontrivial exact solutions.

4.1. The first type solution

The first type solution for Eqs. (67)–(70) is

$$U=a_1,\ F=\frac{c_1}{\xi},\ G=-\frac{c_1}{\xi^2}, \tag{98}$$

where a_1 and c_1 are arbitrary constants and the other parameters should satisfy the condition constraint

$$C_1=-C_2=2\mathrm{i}\lambda^2C_4-2\mathrm{i}\lambda C_3-C_5. \tag{99}$$

From the similarity solutions (59)–(61), the exact solutions of the reduction equation given by Eqs. (98) and (99), and the variable ξ governed by Eq. (62), we obtain

$$u = a_1-2\lambda(x-\lambda t)-2\lambda(C_3-C_4\lambda)/C_1$$
$$-[(\mathrm{i}C_1-2\lambda C_3+\mathrm{i}C_5+2\lambda^2C_4)$$
$$\times\ln(2C_1t+C_4)]/(2C_1), \tag{100}$$

$$f = -\frac{c_1\sqrt{2C_1t+C_4}C_1(2C_1t+C_4)^{C_2/(2C_1)}}{-C_1x+2C_4\lambda-C_3+2\lambda C_1t}, \tag{101}$$

$$g = -\frac{c_1\sqrt{2C_1t+C_4}{C_1}^2(2C_1t+C_4)^{C_2/(2C_1)}}{(-C_1x+2C_4\lambda-C_3+2\lambda C_1t)^2}. \tag{102}$$

The substitution of Eqs. (100)–(102) into Bäcklund transformations (22) and (23) leads to the exact solution of the AKNS equations

$$p = -\frac{C_1\exp\left[\mathrm{i}a_1+2\mathrm{i}\lambda^2t-2\mathrm{i}\lambda x+\frac{2\lambda(-2C_4\lambda+C_3)}{-2\lambda^2C_4-\mathrm{i}C_5+2\lambda C_3}\right]}{-C_1x+2C_4\lambda-C_3+2\lambda C_1t}, \tag{103}$$

$$q = -\frac{C_1\exp\left[-\mathrm{i}a_1-2\mathrm{i}\lambda^2t+2\mathrm{i}\lambda x-\frac{2\lambda(-2C_4\lambda+C_3)}{-2\lambda^2C_4-\mathrm{i}C_5+2\lambda C_3}\right]}{-C_1x+2C_4\lambda-C_3+2\lambda C_1t}. \tag{104}$$

4.2. The second type solution

The second type exact solution for Eqs. (67)–(70) is

$$U=a_2+2C_2\xi^2,\ F=\frac{c_2}{\xi},\ G=-\frac{c_2}{\xi^2}, \tag{105}$$

with the parameters constrained by

$$C_1=-3C_2,\ C_5=-2\mathrm{i}\lambda C_3+2\mathrm{i}\lambda^2C_4+3C_2. \tag{106}$$

The combination of Eqs. (59)–(62) and (105)–(113) leads to the exact solution for u, f, and g

$$u = \frac{2C_2(-C_1x+2C_4\lambda-C_3+2\lambda C_1t)^2}{(2C_1t+C_4){C_1}^2}+a_2$$
$$-\frac{2\lambda(-2\lambda C_4+C_3)}{C_1}-2\lambda(x-\lambda t)-[\mathrm{i}(C_1+2\mathrm{i}\lambda C_3$$
$$+C_5-2\mathrm{i}\lambda^2C_4)\ln(2C_1t+C_4)]/(2\,C_1), \tag{107}$$

$$f = -\frac{c_2\sqrt{2C_1t+C_4}C_1(2C_1t+C_4)^{C_2/(2C_1)}}{-C_1x+2C_4\lambda-C_3+2\lambda C_1t}, \tag{108}$$

$$g = -\frac{c_2\sqrt{2C_1t+C_4}{C_1}^2(2C_1t+C_4)^{C_2/(2C_1)}}{(-C_1x+2C_4\lambda-C_3+2\lambda C_1t)^2}. \tag{109}$$

Putting the solutions of u, f, and g into formulas (22) and (23), we obtain

$$\begin{aligned} p &= -(2C_1t+C_4)^{[(2i\lambda C_3+C_5-2i\lambda^2C_4)/(2C_1)]} \\ &\quad\times \exp\left\{i\left[(a_2+2C_2\xi^2)+\frac{2\lambda(2\lambda C_4-C_3)}{C_1}\right.\right. \\ &\quad\left.\left.-2\lambda(x-\lambda t)\right]\right\}\left(2iC_2\xi-\frac{1}{\xi}\right), \qquad (110) \\ q &= (2C_1t+C_4)^{[-1-(C_5-2i\lambda^2C_4+2i\lambda C_3)/(2C_1)]} \\ &\quad\times \exp\left\{-i\left[(a_2+2C_2\xi^2)+\frac{2\lambda(2\lambda C_4-C_3)}{C_1}\right.\right. \\ &\quad\left.\left.-2\lambda(x-\lambda t)\right]\right\}\left(2i\xi C_2+\frac{1}{\xi}\right), \qquad (111) \end{aligned}$$

with the group variable ξ determined by

$$\xi=\frac{C_1x-2C_4\lambda+C_3-2C_1\lambda t}{\sqrt{2C_1t+C_4}C_1}.$$

4.3. The third type solution

The final type exact solution for Eqs. (67)–(70) is

$$U=a_3-2C_2\xi^2,\quad F=c_3\xi,\quad G=c_3, \qquad (112)$$

where a_3 and c_3 are arbitrary constants and the other parameters should satisfy the condition constraints

$$C_1=-3C_2,\quad C_5=-2i\lambda C_3+2i\lambda^2C_4+3C_2. \qquad (113)$$

The substitution of U, F, and G into formulas (65) and (66) leads to the exact solution for the reduction functions P and Q

$$P=2iC_2\xi\ \mathrm{e}^{i(a_3-2C_2\xi^2)},\quad Q=-2iC_2\xi\ \mathrm{e}^{-i(a_3-2C_2\xi^2)}. \qquad (114)$$

Putting the above formulas for P and Q into the similarity solutions for Eqs. (57) and (58), we obtain the exact solution for p and q

$$\begin{aligned} p &= 2iC_2\,\xi(2C_1t+C_4)^{[(-2i\lambda^2C_4+2i\lambda C_3+C_5)/(2C_1)]} \\ &\quad\times \exp\left\{i\left[(a_3-2C_2\xi^2)-2\lambda(x-\lambda t)\right.\right. \\ &\quad\left.\left.-\frac{2\lambda(-2\lambda C_4+C_3)}{C_1}\right]\right\}, \qquad (115) \\ q &= -2iC_2\xi(2C_1t+C_4)^{[-1-(C_5-2i\lambda^2C_4+2i\lambda C_3)/(2C_1)]} \\ &\quad\times \exp\left\{i\left[2\lambda(x-\lambda t)+\frac{2\lambda(-2\lambda C_4+C_3)}{C_1}\right.\right. \\ &\quad\left.\left.-(a_3-2C_2\xi^2)\right]\right\}, \qquad (116) \end{aligned}$$

with

$$\xi=-\frac{-C_1x+2C_4\lambda-C_3+2C_1\lambda t}{\sqrt{2C_1t+C_4}C_1}.$$

Equation (1.1) in Ref. [15] can be written in the form

$$Q_T+Q_{xx}-2Q^2R=0, \qquad (117)$$

$$R_T-R_{xx}+2QR^2=0. \qquad (118)$$

Soliton solutions, rational solutions, Matveev solutions, complexitons, and interaction solutions of Eqs. (117) and (118) were derived through a matrix method for constructing double Wronskian entries in Ref. [15]. When $t=2iT$, $q(x,t)=Q(x,T)$, and $p(x,t)=R(x,T)$, the AKNS equations expressed by Eqs. (1) and (2) are transformed to system (117) and (118). With the same transformations, the exact solutions of system (1) and (2) will turn to the exact solutions of system (117) and (118). When $t=2iT$, the exact solutions (103), (104); (110), (111); and (115), (116) are complex solutions. The complex solutions are not given in Ref. [15].

5. Discussion and summary

We study the AKNS equations by means of the residual symmetry method. The residual symmetry is nonlocal, while it can be localized by enlarging the AKNS equation to the prolonged AKNS system (20)–(27) including eight equations. The Lie point symmetries are researched for the prolonged AKNS system. The calculation shows that the enlarged AKNS system is invariant under some scaling transformations, space and time translations and shift translations. The symmetry reduction equations are classified to three types according to different symmetry parameter constraints.

We obtain several exact solutions for the original AKNS equations by means of combining the symmetry method and the Bäcklund transformations. When $q=p^*$, equations (1) and (2) turn to the nonlinear Schrödinger equation (3). From Eqs. (22) and (23), we know $q=p^*$ on condition that u is a real function. The exact solutions (103), (104); (110), (111); and (115), (116) show that u is a real function and $q=p^*$ if $C_3=C_4\lambda$. Correspondingly, the solutions for p in Eqs. (103), (104); (110), (111); and (115), (116) are the solutions of the nonlinear Schrödinger equation (3) if $C_3=C_4\lambda$. In fact, the exact solutions that can be obtained by the method presented in this paper are very rich and one can obtain other exact solutions for the AKNS equations by other reduction equations. For example, when $C_1=C_2=0$ and $C_3\neq0$, with the help of the second type reduction equation and similarity solution, we can obtain an exact solution for p and q in the form

$$\begin{aligned} p &= i\left(\frac{C_3}{C_4}-2\lambda\right)\exp\left\{i\left[\frac{t(3{C_3}^2-8C_3\lambda C_4)}{{C_4}^2}\right.\right. \\ &\quad\left.\left.-\frac{2xC_3}{C_4}+a_4+2x\lambda+6\lambda^2t\right]\right\}, \qquad (119) \\ q &= i\left(2\lambda-\frac{C_3}{C_4}\right)\exp\left\{-i\left[\frac{t(3{C_3}^2-8C_3\lambda C_4)}{{C_4}^2}\right.\right. \\ &\quad\left.\left.-\frac{2xC_3}{C_4}+a_4+2x\lambda+6\lambda^2t\right]\right\}. \qquad (120) \end{aligned}$$

It is worthwhile to note that the Lie point symmetries of the prolonged AKNS system constitute the residual symme-

tries and the traditional Lie point symmetries, which shows that the residual symmetry method is very useful for the symmetry theory.

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