Cluster algebra structure on the finite dimensional representations of affine quantum group $U_q(\widehat{A_3})$*

Yang Yan-Min(杨彦敏), Ma Hai-Tao(马海涛), Lin Bing-Sheng(林冰生), and Zheng Zhu-Jun(郑驻军)†

Department of Mathematics, South China University of Technology, Guangzhou 510641, *China*

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In this paper, we prove one case of conjecture given by Hernandez and Leclerc. We give a cluster algebra structure on the Grothendieck ring of a full subcategory of the finite dimensional representations of affine quantum group $U_q(\widehat{A_3})$. As a conclusion, for every exchange relation of cluster algebra, there exists an exact sequence of the full subcategory corresponding to it.

Keywords: affine quantum group, cluster algebra, monoidal categorification

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1. Introduction

Affine quantum group (or quantum affine algebra) is a Hopf algebra that is a q-deformation of the universal enveloping algebra of an affine Lie algebra. They were introduced independently by Drinfeld and Jimbo as a special case of their general construction of a quantum group from a Cartan matrix.[1,2] One of their principal applications has been the study of the Yang–Baxter equation, which is a necessary condition for the solvability of statistical models.

Cluster algebras are a class of commutative rings introduced by Fomin and Zelevinsky.[3–5] A cluster algebra of rank n is an integral domain $\mathcal{A}$, together with some subsets of size n called clusters whose union generates the algebra $\mathcal{A}$. The clusters satisfy various conditions.

In Drinfeld's approach, quantum groups are universal enveloping algebras of certain Lie algebras, generally semisimple or affine. Given a Cartan matrix of a Lie algebra, a cluster algebra can be obtained. Naturally, affine quantum groups and cluster algebras are linked by Lie theory.

In recent years, there has been much interest in the studies of categorification of theories in fundamental mathematics and theoretical physics.[6–12] In general, categorification is a process of replacing set-theoretic theorems by category-theoretic analogues. It replaces sets by categories, functions by functors, and equations between functions by natural transformations of functors. Categorification can be thought of as the process of enhancing an algebraic object to a more sophisticated one, while "decategorification" is the process of reducing the categorified object back to the simpler original object. Thus, a useful categorification should possess a richer structure not seen in the underlying object. The categorification of physical theories may extend the mathematical structures of existing physical theories and help us solve the remaining problems in fundamental physics. This can also help us better understand the physical essence.

Khovanov and Lauda used the diagrammatic methods to study the categorification of quantum groups.[14] Khovanov has constructed a categorification of the Heisenberg algebra based on a graphical category that can act naturally on the category of representations of all symmetric groups,[13] and Cautis *et al.* have also carried out many significant works about the categorifications of the quantum Heisenberg algebras.[8]

In 2010, Hernandez and Leclerc defined the monoidal categorifications of cluster algebras and gave a model of monoidal category for certain cluster algebras.[15] An abelian monoidal category $\mathcal{M}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if the Grothendieck ring of $\mathcal{M}$ and $\mathcal{A}$ are isomorphic as rings. The cluster monomials of $\mathcal{A}$ are the classes of all the real simple objects of $\mathcal{M}$ and the cluster variables of $\mathcal{A}$ (including the frozen ones) are the classes of all the real prime simple objects of $\mathcal{M}$.

Hernandez and Leclerc conjectured that a series of certain full subcategories of finite dimensional representations of affine quantum group $U_q(\hat{g})$ for simple Lie algebra g are monoidal categorifications of certain cluster algebras.[15]

We give a proof of the above conjecture in the case of $g = sl_4$ for $\ell = 2$ in the present work.

The paper is organized as follows. In Section 2, we give some notations and our main results. Some discussion is given in Section 3. In the appendix, we prove assistant Proposition 1. We give the polynomial ring isomorphism between cluster algebra $\mathcal{A}$ and Grothendieck ring R_2 first. Then, we calculate the truncated q-characters of all real prime objects. Lastly, we check the cluster monomials are the classes of all the real simple objects.

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†Corresponding author. E-mail: zhengzj@scut.edu.cn

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2. Main result

Let g be a simple Lie algebra of type ADE. We denote the vertex set of its Dynkin diagram by I, the corresponding affine quantum group by $U_q(\hat{g})$ (parameter $q \in \mathbb{C}^*$ is not a root of unity), the category of finite-dimensional $U_q(\hat{g})$ representations of type 1 by $\mathcal{C}$. For $\ell \in \mathbb{Z}_{\geq 0}$, denote the such full subcategory of $\mathcal{C}$ by $\mathcal{C}_\ell$: for any V of $\mathcal{C}_\ell$, the roots of the Drinfeld polynomials of every composition factor of V belong to $\{q^{-2k-\xi_i} \mid 0 \leq k \leq \ell,\ i \in I\}$, where $\xi_i = \varepsilon$ for $i \in I_\varepsilon$, and $I = I_0 \sqcup I_1$ is a bipartition of I.

Denote the Grothendieck ring of $\mathcal{C}_\ell$ by R_ℓ, then $R_\ell = \mathbb{Z}\left[[W_{1,q^{2k+\xi_i}}^{(i)}]_{i\in I, 0\leq k\leq \ell}\right]$.

To study the finite-dimensional representations of $U_q(\hat{g})$, Frenkel and Reshetikhin introduced q characters.[16] The morphism of q characters is an injective ring homomorphism from the Grothendieck ring R of $\mathcal{C}$ to $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*}$. We denote the simple object with the highest weight monomial m by $L(m)$. Frenkel and Mukhin proved that $\chi_q(L(m)) = m(1+\sum_p M_p)$, where each M_p is a monomial in variables $A_{i,a}^{-1} = Y_{i,aq}^{-1} Y_{i,aq^{-1}}^{-1} \prod_{j\neq i} Y_{j,a}^{-a_{ij}}$, $i \in I, a \in \mathbb{C}^*$. Besides, for every dominant monomial m, they define a polynomial $\mathrm{FM}(m)$ which equals $\chi_q(L(m))$ under certain conditions.[17] We call this algorithm the FM algorithm.

Let V be an object of $\mathcal{C}$ and any subset J of I, $\chi_q(V)$ has a unique decomposition[15]

$$\chi_q(V) = \sum \lambda_m \varphi_J(m),$$

where m is J dominant, $\lambda_m \in \mathbb{Z}_{\geq 0}$, and $\varphi_J(m)$ is defined as follows. If $m = \prod_{j\in J}\prod_{k=1}^{n_j} Y_{j,a_k^{(j)}} \prod_{i\in I\setminus J}\prod_{k=1}^{n_i} Y_{i,a_k^{(i)}}$, set $\overline{m} = \prod_{j\in J}\prod_{k=1}^{n_j} Y_{j,a_k^{(j)}}$, and $L(\overline{m})$ becomes a simple object of $U_q(\hat{g}_J)$ which is a subalgebra of $U_q(\hat{g})$ generated by the Drinfeld generators attached to the vertices of J. Write $\chi_q(L(\overline{m})) = \overline{m}(1+\sum_p \overline{M_{J_p}})$, then set $\varphi_J(m) = m(1+\sum M_{J_p})$, where M_{J_p} are obtained from the corresponding $\overline{M_{J_p}}$ by replacing each $\overline{A_{j,a}^{-1}}$ by $A_{j,a}^{-1}$.

For $P, \widetilde{P} \in \mathbb{Z}[Y_{i,r}^{\pm}; i \in I, r \in \mathbb{Z}]$, $P \leq \widetilde{P}$ means that $\widetilde{P} - P \in \mathbb{N}[Y_{i,r}^{\pm}; i \in I, r \in \mathbb{Z}]$.

We define a useful polynomial $N(m) \leq \chi_q(L(m))$ for dominant monomial m. We will see later that $N(m) = \chi_q(L(m))$ for some simple objects $L(m)$.

(i) $\forall i \in I$, by definition, $\varphi_i(m)$ can be written as such expression $\varphi_i(m) = m(1+\sum M_{i_p})$. Denote all such mM_{i_p} by m_{i_p};

(ii) m_{i_p} is j-dominant for $j \in I\setminus\{i\}$, and $\varphi_j(m_{i_p}) = m_{i_p}(1+\sum M_{i_{p_k}})$. Denote the monomials in $\varphi_j(m_{i_p})$ by $m_{i_{p_k}}$;

(iii) Repeat this procedure until every monomial contains all $A_{i,a}^{-1}$ with $i \in I$. Put all such distinguished monomials together and denote it by $N(m)$. Clearly, $N(m) \leq \chi_q(L(m))$.

To simplify the notations, we write $Y_{i,r}$ and $A_{i,r}$ instead of Y_{i,q^r} and A_{i,q^r} respectively.

Let V be an object of $\mathcal{C}_2$, $\chi_q(V) = m(1+\sum_p M_p)$, where m is a dominant monomial in the variables $Y_{i,\xi_i}, Y_{i,\xi_i+2}, Y_{i,\xi_i+4}$ $(i \in I)$, and the M_p are certain monomials in the $A_{i,r}^{-1} = Y_{i,r-1}Y_{i,r+1}\prod_{j\neq i} Y_{j,r}^{a_{ij}}$. Thus, it can be checked that for $r \geq 5$, there cannot exist a monomial mM_p such that mM_p is dominant.

For this reason, we can define the truncated q-character $\chi_q(V)_{\leq 4}$ of V as the polynomial obtained from $\chi_q(V)$ by keeping the monomials M_p which do not contain any $A_{i,r}^{-1}$ with $r \geq 5$.

Using the mathematical method of Proposition 6.1 in Ref. [15], it can be proved that, the map $V \mapsto \chi_q(V)_{\leq 4}$ is an injective homomorphism from the Grothendieck ring R_2 of $\mathcal{C}_2$ to $\mathcal{Y}$.

Similarly, we attach a polynomial $\mathrm{FM}(m)_{\leq 4}$ which is obtained from $\mathrm{FM}(m)$ by keeping only the monomials which do not contain any $A_{i,r}^{-1}$ with $r \geq 5$.

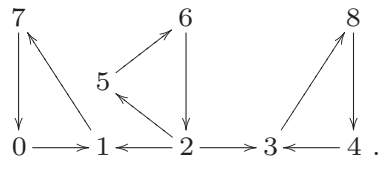
If $L(m)$ is minuscule, then $\chi_q(L(m)) = \mathrm{FM}(m)$. That is, $\chi_q(L(m))$ and $\mathrm{FM}(m)$ have the same dominant monomials with the same multiplicities. Hence, we have

$$\chi_q(L(m))_{\leq 4} = \mathrm{FM}(m)_{\leq 4}.$$

Lastly, we define $N(m)_{\leq 4}$ as the polynomial which is obtained from $N(m)$ by keeping only the monomials which do not contain any $A_{i,r}^{-1}$ with $r \geq 5$.

We have the following result of the cluster algebra of rank 6.

Proposition 1 If $\mathcal{A}$ is the cluster algebra of rank 6 with initial seed $(\boldsymbol{x}, \Gamma)$, where $\boldsymbol{x} = (x_0, \ldots, x_5, x_6, x_7, x_8)$, x_6, x_7, x_8 are frozen variables, Γ is



Then the map $\iota\colon \mathcal{A} \to R_2$,

$$x_0 \mapsto [L(Y_{1,0})], \quad x_1 \mapsto [L(Y_{1,4}Y_{2,1})], \quad x_2 \mapsto [L(Y_{1,4}Y_{2,1}Y_{3,4})],$$
$$x_4 \mapsto [L(Y_{3,0})], \quad x_3 \mapsto [L(Y_{2,1}Y_{3,4})], \quad x_5 \mapsto [L(Y_{2,5})],$$
$$x_6 \mapsto [L(Y_{2,1}Y_{2,3}Y_{2,5})], \ x_7 \mapsto [L(Y_{1,0}Y_{1,2}Y_{1,4})], \ x_8 \mapsto [L(Y_{3,0}Y_{3,2}Y_{3,4})]$$

can be extended to a ring isomorphism. Furthermore, $\mathcal{C}_2$ becomes a monoidal categorification of $\mathcal{A}$.

The proof of this proposition will be given in the appendix.

A partition of $I = I_0 \sqcup I_1$ is given by choosing $I_0 = \{1,3\}$. In order to distinguish the label of the vertices of Dynkin diagram E_6 from A_3, we denote the vertex set of Dynkin diagram E_6 by $\widetilde{I} = [0,5]$, and choose $\widetilde{I}_0 = \{1,3,5\}$.

For a finite mutation sequence $\mu = \mu_2\mu_1\mu_3\mu_2\mu_5\mu_0\mu_4$, denote $\Gamma_2 = \mu(\Gamma)$. If we ignore the label of vertices, Γ_2 is defined by

$$\begin{array}{ccccc} (1,1) & \leftarrow & (1,2) & \rightarrow & (1,3) \\ \downarrow & & \uparrow & & \\ (2,1) & \rightarrow & (2,2) & \leftarrow & (2,3) \\ \uparrow & & \downarrow & & \\ (3,1) & \leftarrow & (3,2) & \rightarrow & (3,3). \end{array}$$

Let $\mathcal{A}_2$ be the cluster algebra attached to the initial seed $(\boldsymbol{x}, \Gamma_2)$, where $\boldsymbol{x} = (x_{(i,k)} \mid i \in I, 1 \le k \le 3)$, $x_{(i,3)}$ are frozen variables. By definition, $\mathcal{C}_2$ is a monoidal categorification of $\mathcal{A}$ if and only if $\mathcal{C}_2$ is a monoidal categorification of $\mathcal{A}_2$.

Thus, the above proposition is equivalent to the following statement:

Theorem 1 (Main result)

$$x_{(1,1)} \mapsto [W^{(1)}_{1,q^2}],\ x_{(1,2)} \mapsto [W^{(1)}_{2,q^2}],\ x_{(1,3)} \mapsto [W^{(1)}_{3,q^0}],$$

$$x_{(2,1)} \mapsto [W^{(2)}_{1,q^3}],\ x_{(2,2)} \mapsto [W^{(2)}_{2,q^1}],\ x_{(2,3)} \mapsto [W^{(2)}_{3,q^1}],$$

$$x_{(3,1)} \mapsto [W^{(3)}_{1,q^2}],\ x_{(3,2)} \mapsto [W^{(3)}_{2,q^2}],\ x_{(3,3)} \mapsto [W^{(3)}_{3,q^0}]$$

extends to a ring isomorphism ι from the cluster algebra $\mathcal{A}_2$ to the Grothendieck ring R_2 of $\mathcal{C}_2$, where $W^{(i)}_{k,r(i,k)}$ are KR modules of $\mathcal{C}_2$. Furthermore, $\mathcal{C}_2$ is a monoidal categorification of $\mathcal{A}_2$.

This is just the case of $g = sl_4$ for $\ell = 2$ in the conjecture given by Hernandez and Leclerc.[15]

In the case of $\ell = 1$, for every simply laced g, the cluster type of Grothendieck ring R_1 of $\mathcal{C}_1$ coincide with the root system of g. However, this does not work for $\ell \ge 2$. In the case of $\ell \ge 2$, one needs to judge the compatibility of any two cluster variables. When a quiver is bipartite of type ADE, two cluster variables are compatible if and only if the corresponding two almost positive roots are compatible. Therefore, Proposition 1 plays an important role to obtain Theorem 2.

3. Conclusions and discussion

We mainly prove one case of Hernandez and Leclerc's conjecture.[15] This is a full subcategory of the finite-dimensional representations of affine quantum group $U_q(\widehat{A_3})$, which is a monoidal categorification of a certain cluster algebra.

Some other conclusions are obtained in the proof. Since there are 45 cluster variables in the cluster algebra of type E_6 with 3 frozen variables, the category $\mathcal{C}_2$ has 45 real prime simple objects. The class of any simple object of $\mathcal{C}_2$ in R_2 is a cluster monomial. Besides, for every exchange relation of cluster algebra $\mathcal{A}$, there exists an exact sequence of $\mathcal{C}_2$ corresponding to it. The set of such exact sequences includes the T systems and coincides with extended T systems defined in Ref. [18].

For example, for exchange relation $x[-\alpha_0]x[\alpha_0] = x[-\alpha_1] + x_7$, the corresponding exact sequence is

$$0 \to L(Y_{1,4}Y_{2,1}) \to L(Y_{1,0}) \otimes L(Y_{1,2}Y_{1,4}) \to L(Y_{1,0}Y_{1,2}Y_{1,4}) \to 0.$$

For exchange relation $x[-\alpha_2]x[\alpha_0+\alpha_1+\alpha_2] = x[\alpha_0+\alpha_1]x_6 + x[\alpha_0]x[-\alpha_3]x[-\alpha_5]$, the corresponding exact sequence is

$$0 \to L(Y_{3,4}) \otimes L(Y_{2,1}Y_{2,3}Y_{2,5}) \to L(Y_{1,4}Y_{2,1}Y_{3,4}) \otimes L(Y_{1,2}Y_{2,5}) \to L(Y_{1,2}Y_{1,4}) \otimes L(Y_{2,5}) \otimes L(Y_{2,1}Y_{3,4}) \to 0.$$

Furthermore, $L(Y_{3,4}) \otimes L(Y_{2,1}Y_{2,3}Y_{2,5}) \cong L(Y_{2,1}Y_{2,3}Y_{2,5}Y_{3,4})$, $L(Y_{1,2}Y_{1,4}) \otimes L(Y_{2,5}) \otimes L(Y_{2,1}Y_{3,4}) \cong L(Y_{1,2}Y_{1,4}Y_{2,1}Y_{2,5}Y_{3,4})$.

The cluster algebras $\mathcal{A}_\ell$ defined in the conjecture are of finite type if and only if ℓ and the type of g satisfy one of the conditions: ℓ is arbitrary and g is of type A_1; $\ell = 1$ and g is of type ADE; g is of type A_n such that $n\ell = 4,\ 6,\ 8$.[15] In this paper, we consider the case of A_3 for $\ell = 2$. The remaining finite types can be proved in the same way. When the cluster algebra $\mathcal{A}_\ell$ is of infinite type, $\mathcal{A}_\ell$ has an infinite number of cluster variables and there exist simple objects of $\mathcal{C}_\ell$ which are not real, our method does not work.

Since the calculations of the detailed expressions of cluster variables and compatible roots are mechanical and lengthy, we omit them here. We refer the readers to the expanded version of the present paper for more mathematical details.[19]

These results will help us study the simple objects of affine quantum groups and the categorifications of cluster algebras in mathematical physics.

Appendix A: Proof of Proposition 1

We prove the proposition by three steps. Firstly, we give a polynomial ring structure on cluster algebra $\mathcal{A}$ and a ring isomorphism from $\mathcal{A}$ to R_2. Secondly, we calculate the truncated q characters of all real prime simple objects of $\mathcal{C}_2$. Finally, we realize the monoidal categorification of $\mathcal{A}$.

Let Q be the root lattice of type E_6. Let $\Phi \subset Q$ be the root system, the vertices of the Dynkin diagram are labelled as Proposition 1. Denote the set of simple roots by Π, and the set of almost positive roots by $\Phi_{\ge -1}$.

Ring isomorphism

For every cluster variable $\square/\prod x_i^{d_i}$, there exists a unique almost positive root $\sum d_i\alpha_i$, where α_i $(0 \le i \le 5)$ is the simple root corresponding to i $(0 \le i \le 5)$.[4] Hence, we denote $x\left[\sum d_i\alpha_i\right]$ by the variable $\square/\prod x_i^{d_i}$.

Theorem 2 The cluster algebra $\mathcal{A}$ is equal to the polynomial ring $\mathbb{Z}[a,a',b,c,c',d,d',e,f]$, where $a = x_0, a' = x_4, b =$

$x_5, c = x\big[\alpha_0+\alpha_1\big], c' = x\big[\alpha_3+\alpha_4\big], d = x\big[\alpha_0+\alpha_1+\alpha_2+\alpha_5\big], d' = x\big[\alpha_2+\alpha_3+\alpha_4+\alpha_5\big], e = x\big[\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\big], f = x\big[\alpha_2+\alpha_5\big]$.

Proof The detailed polynomial expressions of cluster variables in a, a', b, c, c', d, d', e, f are mechanical and lengthy, so we omit them. Furthermore, these nine generators are algebraically independent. Hence, $\mathcal{A} = \mathbb{Z}[a,a',b,c,c',d,d',e,f]$.

Obviously, the assignment

$$\begin{aligned}
&x\big[-\alpha_0\big] \mapsto [L(Y_{1,0})], && x\Big[\sum_{i=0}^{2}\alpha_i+\alpha_5\Big] \mapsto [L(Y_{1,2})], && x\big[\alpha_3+\alpha_4\big] \mapsto [L(Y_{1,4})],\\
&x\big[-\alpha_4\big] \mapsto [L(Y_{3,0})], && x\Big[\sum_{i=2}^{5}\alpha_i\Big] \mapsto [L(Y_{3,2})], && x\big[\alpha_0+\alpha_1\big] \mapsto [L(Y_{3,4})],\\
&x\big[\alpha_2+\alpha_5\big] \mapsto [L(Y_{2,1})], && x\Big[\sum_{i=0}^{5}\alpha_i\Big] \mapsto [L(Y_{2,3})], && x\big[-\alpha_5\big] \mapsto [L(Y_{2,5})],
\end{aligned}$$

extends to a ring isomorphism ι from $\mathcal{A}$ to R_2.

Truncated q characters of real prime simple objects

For each $\beta \in \Phi_{\geq -1}$, we attach a simple object $S(\beta)$ of $\mathcal{C}_2$.

$$\begin{aligned}
&S(-\alpha_0) = L(Y_{1,0}), && S\Big(\sum_{i=0}^{2}\alpha_i+\alpha_5\Big) = L(Y_{1,2}), && S(\alpha_3+\alpha_4) = L(Y_{1,4}),\\
&S(-\alpha_4) = L(Y_{3,0}), && S\Big(\sum_{i=2}^{5}\alpha_i\Big) = L(Y_{3,2}), && S(\alpha_0+\alpha_1) = L(Y_{3,4}),\\
&S(\alpha_2+\alpha_5) = L(Y_{2,1}), && S\Big(\sum_{i=0}^{5}\alpha_i\Big) = L(Y_{2,3}), && S(-\alpha_5) = L(Y_{2,5}),\\
&S(-\alpha_1) = L(Y_{1,4}Y_{2,1}), && S(-\alpha_2) = L(Y_{1,4}Y_{2,1}Y_{3,4}), && S(-\alpha_3) = L(Y_{2,1}Y_{3,4}),\\
&S(\alpha_0) = L(Y_{1,2}Y_{1,4}), && S(\alpha_4) = L(Y_{3,2}Y_{3,4}), && S(\alpha_5) = L(Y_{2,1}Y_{2,3}),\\
&S(\alpha_1) = L(Y_{1,0}Y_{3,4}), && S(\alpha_2) = L(Y_{2,1}Y_{2,5}), && S(\alpha_3) = L(Y_{1,4}Y_{3,0}).
\end{aligned}$$

For any positive roots $\alpha = \sum_{i\in\widetilde{I}} a_i\alpha_i \in \Phi_{>0} - \{\Pi,\ \alpha_0+\alpha_1,\ \alpha_3+\alpha_4,\ \alpha_2+\alpha_5\}$, α can be uniquely written as

(i) if $a_1 = a_2 \geq a_3$, $\alpha = a_3(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) + (a_1-a_3)(\alpha_0+\alpha_1+\alpha_2+\alpha_5) + \sum_{i=0,4,5} b_i(-\alpha_i)$ for some $b_i \in \mathbb{N}$,

(ii) if $a_1 \leq a_2 = a_3$, $\alpha = a_1(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) + (a_3-a_1)(\alpha_2+\alpha_3+\alpha_4+\alpha_5) + \sum_{i=0,4,5} b_i(-\alpha_i)$ for some $b_i \in \mathbb{N}$,

(iii) if $a_1 = a_3 < a_2$, $\alpha = \lfloor a_2/2\rfloor(\alpha_0+\alpha_1+\alpha_2+\alpha_5) + \lfloor a_2/2\rfloor(\alpha_2+\alpha_3+\alpha_4+\alpha_5) + (a_1-\lfloor a_2/2\rfloor)(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) + \sum_{i=0,4,5} b_i(-\alpha_i)$ for some $b_i \in \mathbb{N}$, where $\lfloor a_2/2\rfloor$ denotes the biggest integer is smaller than $a_2/2$.

Correspondingly, we define $S(\alpha)$ as

(i) if $a_1 = a_2 \geq a_3$, $S(\alpha) = Y_{1,0}^{b_0}Y_{1,2}^{a_1-a_3}Y_{2,3}^{a_3}Y_{2,5}^{b_5}Y_{3,0}^{b_4}Y_{3,2}^{0}$,

(ii) if $a_1 \leq a_2 = a_3$, $S(\alpha) = Y_{1,0}^{b_0}Y_{1,2}^{0}Y_{2,3}^{a_1}Y_{2,5}^{b_5}Y_{3,0}^{b_4}Y_{3,2}^{a_3-a_1}$,

(iii) if $a_1 = a_3 < a_2$, $S(\alpha) = Y_{1,0}^{b_0}Y_{1,2}^{\lfloor a_2/2\rfloor}Y_{2,3}^{a_1-\lfloor a_2/2\rfloor}Y_{2,5}^{b_5}Y_{3,0}^{b_4}Y_{3,2}^{\lfloor a_2/2\rfloor}$.

For $i \in I_0 = \{1,3\}$, we denote $F_i = L(Y_{i,0}Y_{i,2}Y_{i,4})$; for $i = 2$, we denote $F_2 = L(Y_{2,1}Y_{2,3}Y_{2,5})$.

The eighteen Kirillov–Reshetikhin modules of $\mathcal{C}_2$ are minuscule. Their q characters can be obtained by the FM algorithm.

The following eight objects $L(Y_{2,5}Y_{1,2})$, $L(Y_{1,4}Y_{2,1})$, $L(Y_{1,0}Y_{2,3}Y_{2,5})$, $L(Y_{1,4}Y_{2,1}Y_{3,4})$, $L(Y_{1,0}Y_{3,4})$, $L(Y_{1,0}Y_{1,2}Y_{2,5})$, $L(Y_{2,3}Y_{1,0})$, and $L(Y_{2,1}Y_{2,5})$ are also minuscule.

Let us introduce the notation

$$P \cap Q = \{\text{the monomials are contained in } P \text{ and } Q\}$$

for $P, Q \in \mathbb{Z}[Y_{i,r}^{\pm};\ i \in I, r \in \mathbb{Z}]$.

Consider $L(Y_{1,0}Y_{2,3}Y_{3,0})$,

$$\begin{aligned}
&N(Y_{1,0}Y_{2,3}Y_{3,0})_{\leq 4}\\
&= Y_{1,0}Y_{3,0}Y_{2,3}(1+A_{1,1}^{-1}+A_{2,4}^{-1}+A_{3,1}^{-1}+A_{1,1}^{-1}A_{2,4}^{-1}\\
&\quad +A_{1,1}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}+A_{1,1}^{-1}A_{3,1}^{-1}\\
&\quad +A_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1}+A_{2,4}^{-1}A_{3,1}^{-1}+A_{1,1}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}+A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}\\
&\quad +2A_{1,1}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}+A_{1,1}^{-1}A_{2,2}^{-2}A_{2,4}^{-1}A_{3,1}^{-1})\\
&\leq \chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0})).
\end{aligned}$$

For $i = 1$, $j = 3$, or $i = 3$, $j = 1$, we have

$$\begin{aligned}
&\chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0}))_{\leq 4}\\
&\leq \chi_q(L(Y_{i,0}Y_{2,3}))_{\leq 4}\chi_q(L(Y_{j,0}))_{\leq 4},\\
&(\chi_q(L(Y_{1,0}Y_{2,3}))_{\leq 4}\chi_q(L(Y_{3,0}))_{\leq 4})\\
&\bigcap(\chi_q(L(Y_{3,0}Y_{2,3}))_{\leq 4}\chi_q(L(Y_{1,0}))_{\leq 4})
\end{aligned}$$

$$= N(Y_{1,0}Y_{2,3}Y_{3,0})_{\leq 4}.$$

Hence, $\chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0}))_{\leq 4} = N(Y_{1,0}Y_{2,3}Y_{3,0})_{\leq 4}$.

In the same way, the remaining real prime simple objects can be calculated. Therefore,

$$\iota(x[\beta]) = [S(\beta)], \quad \iota(x_6) = [F_2],$$
$$\iota(x_7) = [F_1], \quad \iota(x_8) = [F_3], \qquad (\beta \in \Phi_{\geq -1}).$$

For any $\alpha, \beta \in \Phi_{\geq -1}$, Fomin and Zelevinsky associated a nonnegative integer $(\alpha||\beta)$.[20] Two roots $\alpha, \beta \in \Phi_{\geq -1}$ are compatible if $(\alpha\|\beta) = 0$. For $\alpha \in \Phi_{\geq -1}$, denote the set of almost positive roots, which are compatible with α, by C_α. A description of compatible subsets of type E_6 are lengthy, one can see the details in our paper.[19]

Monoidal categorification

The unique cluster expansion of $\gamma \in Q$ is the expression $\gamma = \sum_{\alpha \in \Phi_{\geq -1}} m_\alpha \alpha$, where $m_\alpha \in \mathbb{Z}_{\geq 0}$, and $m_\alpha m_\beta = 0$ whenever $(\alpha\|\beta) \neq 0$.[20]

For any $\alpha \in \Phi_{\geq -1}$, we have attached a simple object $S(\alpha) = L(m_\alpha)$ for some highest weight monomial m_α. Set $Y^\alpha = m_\alpha$, then for $\gamma = \sum_{\alpha \in \Phi_{\geq -1}} n_\alpha \alpha$, define $Y^\gamma = \prod_{\alpha \in \Phi_{\geq -1}} (Y^\alpha)^{n_\alpha}$.

Let V be a simple object in $\mathcal{C}_2$, and its highest weight monomial $m_V = \prod_{i \in I} Y_{i,\xi_i}^{a_i} Y_{i,\xi_i+2}^{b_i} Y_{i,\xi_i+4}^{c_i}$ for some $a_i, b_i, c_i \in \mathbb{Z}_{\geq 0}$. Then m_V can be written as

$$\begin{aligned} m &= \prod_{i \in I} (Y_{i,\xi_i} Y_{i,\xi_i+2} Y_{i,\xi_i+4})^{\min(a_i,b_i,c_i)} Y^\gamma \\ &= \prod_{i \in I} (Y_{i,\xi_i} Y_{i,\xi_i+2} Y_{i,\xi_i+4})^{\min(a_i,b_i,c_i)} \prod_{\alpha \in \Phi_{\geq -1}} (Y^\alpha)^{n_\alpha} \end{aligned}$$

for some unique cluster expansion $\gamma = \sum_{\alpha \in \Phi_{\geq -1}} n_\alpha \alpha \in Q$. The factorization of m_V is unique.

Theorem 3 $\mathcal{C}_2$ is a monoidal categorification of $\mathcal{A}$.

Proof In order to prove this theorem, we then need to prove that

$$L(m_V) \cong \bigotimes_{i \in I} F_i^{\otimes \min(a_i,b_i,c_i)} \bigotimes_{\alpha \in \Phi_{\geq -1}} S(\alpha)^{\otimes n_\alpha}. \tag{1}$$

Hernandez proved that for simple objects $S_1, S_2, \ldots, S_N$ of $\mathcal{C}$, the tensor product $S_1 \otimes \cdots \otimes S_N$ is simple if and only if $S_i \otimes S_j$ is simple for any $i < j$.[21] Therefore, to check Eq. (1) reduces to check the following cases: (i) for every $\alpha \in \Phi_{\geq -1}$, $S(\alpha)$ is prime; (ii) for every $i \in I$, F_i is prime; (iii) $F_i \otimes S(\alpha)$ and $F_i \otimes F_j$ are simple for each $\alpha \in \Phi_{\geq -1}$ and $i, j \in I$; (iv) if $(\alpha\|\beta) = 0$, then $S(\alpha) \otimes S(\beta)$ is simple.

Since we have given explicit formulas of $\chi_q(F_i)_{\leq 4} (i \in I)$ and $\chi_q(S(\alpha))_{\leq 4}$ for all $\alpha \in \Phi_{\geq -1}$. Thus, cases (i) and (ii) follow.

There is another method. Recently, Chari *et al.* explored the relationship between self extensions of simple objects of $\mathcal{C}$ and the property of a simple object being prime. By Theorem 2, we can quickly check that $S(\alpha)(\alpha \in \Phi_{\geq -1})$ and $F_i (i \in I)$ are prime.[22]

Now we copy with the case (iii).

For every $i, j \in I$, $\chi_q(F_i \otimes F_j)_{\leq 4} = Y_{i,\xi_i} Y_{i,\xi_i+2} Y_{i,\xi_i+4} Y_{j,\xi_j} Y_{j,\xi_j+2} Y_{j,\xi_j+4}$, hence $F_i \otimes F_j$ is simple.

$$\begin{aligned} &N(Y_{i,\xi_i} Y_{i,\xi_i+2} Y_{i,\xi_i+4} Y^\alpha)_{\leq 4} \\ &= Y_{i,\xi_i} Y_{i,\xi_i+2} Y_{i,\xi_i+4} N(Y^\alpha)_{\leq 4} = \chi_q(F_i)_{\leq 4} \chi_q(L(Y^\alpha)_{\leq 4}, \end{aligned}$$

so $F_i \otimes S(\alpha)$ is simple.

Finally, we deal with the case (iv).

We have described explicitly the pair of compatible roots of E_6 in Ref. [19].

For every $\alpha \in \Phi_{\geq -1}$, $\beta \in C_\alpha$, we know that $\chi_q(L(Y^\alpha Y^\beta)) \leq \chi_q(S(\alpha) \otimes S(\beta))$. In order to prove that $S(\alpha) \otimes S(\beta) \cong L(Y^\alpha Y^\beta)$. We only need to check that all the dominant monomials of $\chi_q(S(\alpha) \otimes S(\beta))$ belong to $\chi_q(L(Y^\alpha Y^\beta))$.

For most of the cases, we can check that each dominant monomial occurs in some $\varphi_J(Y^\alpha Y^\beta M) \leq \chi_q(L(Y^\alpha Y^\beta))$, where $J \subseteq I$, and M is a J-dominant monomial. For such cases, we omit them, or see also Ref. [19]. We list the special cases only.

(i) $\alpha = -\alpha_0$, $Y^\alpha = Y_{1,0}$, $\beta = \alpha_1$, $Y^\beta = Y_{1,0}Y_{3,4}$, denote $m = Y^\alpha Y^\beta = Y_{1,0}^2 Y_{3,4}$. The only dominant monomials in $\chi_q(S(\alpha) \otimes S(\beta))_{\leq 4}$ are m and $mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,3}^{-1}$.

It is easy to see that $N(m)_{\leq 4} + mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,3}^{-1} = \chi_q(S(\alpha) \otimes S(\beta))_{\leq 4}$. If $\chi_q(L(m))_{\leq 4}$ does not contain $mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,3}^{-1}$, then $L(m)$ is minuscule and $\chi_q(L(m)) = \mathrm{FM}(m)$. However, $mA_{1,1}^{-2}A_{2,2}^{-1}A_{3,3}^{-1} \leq N(m)$ is not contained in $\mathrm{FM}(m)$.

(ii) $\alpha = \alpha_2$, $Y^\alpha = Y_{2,1}Y_{2,5}$, $\beta = \alpha_2 + \alpha_5$, $Y^\beta = Y_{2,1}$, denote by $m = Y^\alpha Y^\beta$. The only dominant monomials in $\chi_q(S(\alpha) \otimes S(\beta))_{\leq 4}$ are m and $mA_{1,3}^{-1}A_{3,3}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}$.

If $mA_{1,3}^{-1}A_{3,3}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}$ is not contained in $\chi_q(L(Y^\alpha Y^\beta))_{\leq 4}$, then $L(Y^\alpha Y^\beta)$ is minuscule, and $\chi_q(L(Y^\alpha Y^\beta)) = \mathrm{FM}(Y^\alpha Y^\beta) = \sum \lambda_m \varphi_1(m)$. In other words, any monomial in $\chi_q(L(m))$ must be contained in some $\varphi_1(m')$, where $m' \in \mathcal{M}_{1,+}$ and $m' \leq \chi_q(L(m))$. However, the monomial $A_{2,2}^{-2}A_{1,3}^{-2}A_{3,3}^{-2}A_{2,4}^{-1}$ does not satisfy this property. This is a contradiction.

(iii) $\alpha = \alpha_1 + \alpha_2$, $\beta = \alpha_2 + \alpha_5$. The situation is similar to (2). We omit it.

(iv) $\alpha = \alpha_2 + \alpha_5$, $\beta = \alpha_2 + \alpha_3 + \alpha_4$. The situation is similar to (2). We omit it.

(v) $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $Y^\alpha = Y_{1,0}Y_{2,3}Y_{2,5}Y_{3,0}$, $\beta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$, $Y^\beta = Y_{1,0}Y_{2,3}^2Y_{2,5}Y_{3,0}$, denote $m = Y^\alpha Y^\beta$. The only dominant monomials in $\chi_q(S(\alpha) \otimes S(\beta))_{\leq 4}$ are m, $mA_{2,4}^{-1}$, $mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1}$, $2mA_{1,1}^{-1}A_{3,1}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}$, and $mA_{1,1}^{-2}A_{2,2}^{-2}A_{2,4}^{-1}A_{3,1}^{-2}$.

We can check that all the monomials excluding $mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1}$ of $\chi_q(S(\alpha)\otimes S(\beta))_{\leq 4}$ are contained in $\chi_q(L(m))_{\leq 4}$. If $mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1}$ is not contained in $\chi_q(L(m))$, then $\chi_q(L(m))_{\leq 4} = \chi_q(S(\alpha)\otimes S(\beta))_{\leq 4} - mA_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1} = \sum \lambda_m \varphi_1(m)$. However, there does not exist such monomial m' such that $mA_{1,1}^{-2}A_{2,2}^{-1}A_{3,1}^{-1} \leq \varphi_1(m')$.

(iv) $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5$, $Y^{\alpha} = Y_{1,0}Y_{1,2}Y_{2,5}Y_{3,0}Y_{3,2}$, $\beta = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$, $Y^{\beta} = Y_{1,0}Y_{1,2}Y_{2,3}Y_{2,5}^2Y_{3,0}Y_{3,2}$, denote

$$
\begin{aligned}
m &= Y^{\alpha}Y^{\beta},\\
m_1 &= mA_{1,1}^{-1}A_{1,3}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}A_{3,3}^{-1},\\
m_2 &= mA_{1,1}^{-1}A_{1,3}^{-2}A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}A_{3,3}^{-1},\\
m_3 &= mA_{1,1}^{-1}A_{1,3}^{-1}A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}A_{3,3}^{-2},\\
m_4 &= mA_{1,1}^{-1}A_{1,3}^{-2}A_{2,2}^{-1}A_{2,4}^{-1}A_{3,1}^{-1}A_{3,3}^{-2},\\
m_5 &= mA_{1,1}^{-1}A_{1,3}^{-2}A_{2,2}^{-1}A_{2,4}^{-2}A_{3,1}^{-1}A_{3,3}^{-2}.
\end{aligned}
$$

Denote the set of monomials of $\chi_q(S(\alpha)\otimes S(\beta))_{\leq 4}$ excluding $m_i (1 \leq i \leq 5)$ by $\mathcal{L}$. We can check that every monomial of $\mathcal{L}$ is contained in $\chi_q(L(m))_{\leq 4}$.

If all m_i are not contained in $\chi_q(L(m))_{\leq 4}$, and $\chi_q(L(m)) = \sum \lambda_m \varphi_1(m)$. Considering the monomial $A_{1,1}^{-1}A_{1,3}^{-1}m_1$, obviously, for any $m' \in \mathcal{L}$, $A_{1,1}^{-1}A_{1,3}^{-1}m_1$ does not occur in $\varphi_1(m')$. Only for m_1, we have $A_{1,1}^{-1}A_{1,3}^{-1}m_1 \leq \varphi_1(m_1)_{\leq 4} = m_1(1 + A_{1,3}^{-1} + A_{1,1}^{-1}A_{1,3}^{-1})$. Thus, $m_1 + m_2$ is contained in $\chi_q(L(m))_{\leq 4}$.

Similarly, we can check that m_3 and m_5 are contained in $\chi_q(L(m))_{\leq 4}$. Since m_4 is not dominant, we have proved that all dominant monomials of $\chi_q(S(\alpha)\otimes S(\beta))_{\leq 4}$ are contained in $\chi_q(L(m))_{\leq 4}$.

This finishes the proof of Theorem 4.

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