Evolution of entanglement between qubits ultra-strongly coupling to a quantum oscillator*

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We investigate the dynamics of two qubits coupled with a quantum oscillator by using the adiabatic approximation method. We take account of the interaction between the qubits and show how the entanglement is affected by the interaction parameter. The most interesting result is that we can prolong the entanglement time or improve the entanglement degree by using an appropriate interaction parameter. As the generation and preservation of entanglement of qubits are crucial for quantum information processing, our research will be useful.

Keywords: Jaynes–Cummings model, ultra-strong coupling, adiabatic approximation

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1. Introduction

A general understanding of entanglement with quantum theory was first introduced by Schrödinger.[1] The behaviors of few-state entangled quantum systems have been widely used in quantum information processing (QIP).[2–4] It is now attracting increasing interest in many areas of physics, such as natural and artificial atoms for quantum computation,[5] quantum simulators,[6] quantum phase transitions,[7] and quantum magnets.[8] The generation and preservation of entanglement of qubits are crucial for QIP. We could implement it by using a continuous-variable (CV) physical system. The CV physical system such as a harmonic oscillator and a light field, which is defined in infinite-dimensional Hilbert space, has attracted considerable attention as the entanglement could be controlled by external agents. It is, thus, evident that the study of the entanglement transfer from a CV system to a qubit system is of primary importance.

The CV interaction can be treated using the Jaynes–Cummings (JC) model.[9] The JC model is an exactly soluble quantum mechanical model for the interaction between two dissimilar quantum systems such as a single-mode radiation field and a two-level atom. This model has acquired further significance because it contains subtle features of quantum entanglement[10] that are interaction strength dependent. The JC model has been widely used in many physical fields, such as quantum optics,[11–13] condensed mater physics,[14] mechanical oscillators,[15–18] and quantum information protocols.[19] Superconducting circuits are the most popular realization of the JC model in the last decade.[20–23]

There have been some previous theoretical discussions that also use the JC model for related problems.[24–31] The study by Son *et al.*[24] gave an example of the time-dependent entanglement. Later studies showed that the field entanglement in the JC cavity could be employed for transfer into networks.[25–27] That work was extended by Lee *et al.*[28] and zhou and wang.[29] The evolution of entanglement in a qubit–field system was examined by Rendell and Rajagopal[30] and Muhamm and Eberly.[31]

Though the JC model is sufficient to study how to control the quantum qubits, it is an approximation of the Rabi model. The JC model relies upon the rotating-wave approximation (RWA). The RWA works well under the conditions of near resonance and weak coupling between the qubits and the oscillator. The parameter regime in which the coupling strength is strong enough to invalidate the RWA is called the ultrastrong coupling regime. When the coupling between the qubits and the oscillator becomes stronger and stronger, the RWA will break down. The ultrastrong coupling is now increasingly important in fields such as the spontaneous emission[32] and the quantum Zeno effect.[33,34] It also plays a significant role in superconducting circuits,[35] semiconductor quantum dots,[36] and the Bloch–Siegert shift.[37] Some experiments, such as the one with a Cooper-pair box or superconducting loops coupled to a superconducting transmission line, showed that the JC model cannot be used to describe the system's evolution correctly under the strong coupling. Motivated by those experimental developments and the importance of understanding the collective quantum behavior, another approximation was investigated by Irish and Agarwal *et al.*, called the adiabatic approximation.[38–43] This approximation was shown to work best at small qubit frequencies and high coupling strengths. Those studies provided many interesting results for the Rabi model, such as the two qubits' entanglement death and revival phenomena for an initial coherent state of an oscillator.

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Enlightened by those works, we will focus on the preservation of the entanglement in the adiabatic approximation. In this article, we extend Agarwal's work[40] to consider the interaction between two qubits. The two qubits couple with a harmonic oscillator, and they also couple with each other, which is modeled as a spin–spin interaction.[44–48] Thus, we introduce an interaction parameter. We show how the entanglement is affected by the interaction parameter. The system analyzed allows us to control the qubits' entanglement quality and time. Our approach differs further from that of most previous studies by focusing on the effects of (i) ultra-strong coupling and (ii) entanglement time significantly longer than before; the interaction parameter could not only control the entanglement quality but also the entanglement time. This paper is divided into two parts. In the first part, we use the adiabatic approximation to calculate the eigenvalues and eigenfunctions of the Rabi Hamiltonian. In the second part, we discuss the evolution of the entanglement between qubits by using the Rabi model.

2. Spectrum of $\hat{H}$

The Rabi Hamiltonian is written as

$$H^{\mathrm{R}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^+\hat{a} + \hbar\omega\frac{\beta}{2}(\hat{a}^+ + \hat{a})(\hat{\sigma}_+ + \hat{\sigma}_-), \quad (1)$$

where $\hat{\sigma}_z$ and $\hat{\sigma}_+ + \hat{\sigma}_- = \hat{\sigma}_x$ are the usual Pauli matrices in the Hilbert space of the qubit, and $\hat{a}^+$ and $\hat{a}$ refer to the creation and the annihilation operators of an interacting mode of a harmonic oscillator. In Ref. [40], Agarwal made a generalization of Eq. (1) in which the $\hat{\sigma}$ operators are replaced by the two-qubit counterparts

$$\hat{H} = \hbar\omega_0\hat{S}_z + \hbar\omega\hat{a}^+\hat{a} + \hbar\omega\beta(\hat{a} + \hat{a}^+)\hat{S}_x, \quad (2)$$

where

$$\hat{S}_z = \frac{1}{2}(\hat{\sigma}_z^1 + \hat{\sigma}_z^2), \quad \hat{S}_x = \frac{1}{2}(\hat{\sigma}_x^1 + \hat{\sigma}_x^2). \quad (3)$$

On this basis, we add an interaction term $\hbar\omega_0\gamma\hat{\sigma}_z^1\hat{\sigma}_z^2$, which represents the inter-qubit coupling. We obtain

$$\begin{aligned}\hat{H} &= H_0 + H_1 \\ &= \hbar\omega\hat{a}^+\hat{a} + \hbar\omega\beta(\hat{a} + \hat{a}^+)\hat{S}_x \\ &\quad + \hbar\omega_0\hat{S}_z + \hbar\omega_0\gamma\hat{\sigma}_z^1\hat{\sigma}_z^2, \end{aligned} \quad (4)$$

where

$$\begin{aligned}\hat{H}_0 &= \hbar\omega[\hat{a}^+\hat{a} + \beta(\hat{a} + \hat{a}^+)\hat{S}_x], \\ \hat{H}_1 &= \hbar\omega_0(\hat{S}_z + \gamma\hat{\sigma}_z^1\hat{\sigma}_z^2),\end{aligned}$$

and γ is the strength of the interaction between the two qubits.

The first task is to establish a new set of basis states. As $\omega_0 \ll \omega$, we can neglect the self-energy of the two-level system $\hat{H}_1$. Then we can find the eigenspectrum of $\hat{H}_0$

$$\hat{H}_0 \mid \Phi\rangle = E \mid \Phi\rangle. \quad (5)$$

The eigenstate form of $\mid \Phi\rangle$ is $\mid \Phi\rangle = \mid j,m\rangle \mid \phi_m\rangle$, where $\mid j,m\rangle$ denotes the eigenstate of $\hat{S}_x$, and $\mid \phi_m\rangle$ are the oscillator eigenstates of $\hat{H}_0$ by replacing $\hat{S}_x$ to $\mid j,m\rangle$.[39,40]

As $\hat{\sigma}_z^i \mid e\rangle = \mid e\rangle$ and $\hat{\sigma}_z^i \mid g\rangle = - \mid g\rangle$, we can define the states $\mid j,m\rangle$ as

$$\begin{aligned}\mid 1,1\rangle &= \frac{1}{2}(\mid ee\rangle + \mid eg\rangle + \mid ge\rangle + \mid gg\rangle), \\ \mid 1,0\rangle &= \frac{1}{\sqrt{2}}(\mid ee\rangle - \mid gg\rangle), \\ \mid 0,0\rangle &= \frac{1}{\sqrt{2}}(- \mid eg\rangle + \mid ge\rangle), \\ \mid 1,-1\rangle &= \frac{1}{2}(\mid ee\rangle - \mid eg\rangle - \mid ge\rangle + \mid gg\rangle), \end{aligned} \quad (6)$$

with eigenvalue m; $\mid \phi_m\rangle$ satisfy the eigenvalue equation

$$\hbar\omega[\hat{a}^+\hat{a} + m\beta(\hat{a} + \hat{a}^+)] \mid \phi_m\rangle = E \mid \phi_m\rangle. \quad (7)$$

Denoting $m\beta$ by β_m and using the displacement operator $\hat{D}(\alpha) = \exp[\alpha(\hat{a}^+ - \hat{a})]$, we can obtain the state $\mid \Phi_m^N\rangle = D(-\beta_m) \mid N\rangle \equiv \mid N_m\rangle$, where $N = 0, 1, \ldots$. Thus, we obtain the eigenstates of $\hat{H}_0$

$$\mid \Phi\rangle \longrightarrow \mid \Phi_{j,m,N}\rangle = \mid j,m\rangle \mid N_m\rangle, \quad (8)$$

and eigenvalues

$$E_{N,m} = \hbar\omega(N - \beta_m^2). \quad (9)$$

These approximate eigenstates and energies constitute the displaced oscillator basis, which will be used in the subsequent calculations. This approximation has been used by Irish[39] and Agarwal[40] to study similar problems.

Under the above consideration, we can use basis $\mid 1,m\rangle \otimes \mid N_m\rangle$ to obtain the block diagonal form of Hamiltonian $\hat{H}$

$$\hat{H} = \hbar\omega \begin{pmatrix} H_0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & H_1 & \ldots & 0 & 0 & \ldots \\ \ldots & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & H_N & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \quad (10)$$

For each N, in the order of $\mid 1,1\rangle \mid N_1\rangle, \mid 1,0\rangle \mid N_0\rangle, \mid 1,-1\rangle \otimes \mid N_{-1}\rangle$ for rows and columns, the Hamiltonian has the form

$$\hat{H}_N = \hbar\omega \begin{pmatrix} N - \beta^2 & \Omega_N & \Omega_N' \\ \Omega_N & N + \dfrac{\omega_0}{\omega} & \Omega_N \\ \Omega_N' & \Omega_N & N - \beta^2 \end{pmatrix}. \quad (11)$$

The off-diagonal terms Ω_N and Ω_N' are given by

$$\Omega_N = \sqrt{\frac{1}{2}}\frac{\omega_0}{\omega}\langle N_1 \mid N_0\rangle = \sqrt{\frac{1}{2}}\frac{\omega_0}{\omega}\mathrm{e}^{-\beta^2/2}L_N(\beta^2), \quad (12)$$

$$\Omega_N' = \frac{\omega_0}{\omega}\gamma\langle N_1 \mid N_{-1}\rangle = \frac{\omega_0}{\omega}\gamma\mathrm{e}^{-2\beta^2}L_N(2\beta^2), \quad (13)$$

where $L_N(x)$ is the Laguerre polynomial. This is in contrast to ignoring the interaction between the two qubits case where there is only one off-diagonal term in $\hat{H}_N$.[40] The normalized eigenfunctions and eigenvalues of $\hat{H}_N$ are

$$
\begin{aligned}
&|E_N^0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad |E_N^{\pm}\rangle = \frac{1}{l_{\pm}}\begin{pmatrix}1\\y_{\pm}\\1\end{pmatrix},\\
&y_{\pm} = \frac{\beta^2+\gamma\omega_0/\omega-\Omega'_N \pm \sqrt{(\beta^2+\gamma\omega_0/\omega-\Omega'_N)^2+8\Omega_N^2}}{2\Omega_N},\\
&l_{\pm}^2 = 2+y_{\pm}^2,\\
&E_N^0 = \hbar\omega[N-\beta^2-\Omega'_N],\\
&E_N^{\pm} = \frac{\hbar\omega}{2}\Bigg[2N-\beta^2+\gamma\frac{\omega_0}{\omega}+\Omega'_N\\
&\qquad \pm\sqrt{(\beta^2+\gamma\frac{\omega_0}{\omega}-\Omega'_N)^2+8\Omega_N^2}\Bigg]. \qquad (14)
\end{aligned}
$$

We can see that off-diagonal term Ω'_N affects $|E_N^{\pm}\rangle$, E_N^0, and $E_N^{\pm}$.

3. Entanglement dynamics

The evolution of entanglement between several qubits coupled to a single mode or many oscillator modes has been studied extensively. In some cases,[31,49–52] the interaction between the qubits and the oscillator mode(s) were treated within the RWA. In other cases,[40,53,54] they were treated in regimes where the RWA was invalid. However, in all those studies, the qubits were noninteracting. In the case of interacting qubits, new time scales and qualitatively new features arise, mandating an extension of the previous results. It may be important in areas of quantum information processing, where the coherent control of entanglement is vital.

We can begin the illustration with a configuration without the correlation between the oscillator and the qubits. We prepare the qubits in one of the σ_z Bell states and the oscillator in an undisplaced coherent state $|\alpha\rangle$

$$
\begin{aligned}
|\xi(0)\rangle &= \frac{1}{\sqrt{2}}(|e,e\rangle+|g,g\rangle)\,|\alpha\rangle\\
&= \frac{1}{\sqrt{2}}(|1,1\rangle+|1,-1\rangle)\,|\alpha\rangle. \qquad (15)
\end{aligned}
$$

When $N \gg \beta$, we can obtain

$$
|N\rangle \approx \hat{D}(\mp\beta)\,|N\rangle = |N_{\pm 1}\rangle. \qquad (16)
$$

So equation (15) becomes

$$
\begin{aligned}
|\xi(0)\rangle &= \frac{1}{\sqrt{2}}\sum_{N=0}^{\infty}\frac{\mathrm{e}^{-|\alpha|^2/2}\alpha^N}{\sqrt{N!}}(|1,1\rangle\,|N_1\rangle\\
&\quad + |1,-1\rangle\,|N_{-1}\rangle). \qquad (17)
\end{aligned}
$$

According to Eq. (14), we have

$$
|\xi(0)\rangle = \sum_{N=0}^{\infty}\frac{\mathrm{e}^{-|\alpha|^2/2}\alpha^N}{\sqrt{2}\sqrt{N!}}\left(\frac{2}{l_+}|E_N^+\rangle+\frac{2}{l_-}|E_N^-\rangle\right). \qquad (18)
$$

Using Eqs. (14) and (16), we can obtain

$$
\begin{aligned}
&|\xi(t)\rangle\\
&= \sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{2}\sqrt{N!}}\frac{\mathrm{e}^{-|\alpha|^2/2}}{y_+-y_-}\Bigg[\Bigg(\frac{-l_+y_-}{2}\mathrm{e}^{-\mathrm{i}E_N^+t/\hbar}\\
&\quad+\frac{l_-y_+}{2}\mathrm{e}^{-\mathrm{i}E_N^-t/\hbar}\Bigg)(|1,1\rangle+|1,-1\rangle)\\
&\quad+(l_+\mathrm{e}^{-\mathrm{i}E_N^+t/\hbar}-l_-\mathrm{e}^{-\mathrm{i}E_N^-t/\hbar})\,|1,0\rangle\Bigg]\,|N\rangle. \qquad (19)
\end{aligned}
$$

Then we use the eigenbasis σ_z to compact in Eq. (20)

$$
\begin{aligned}
&|\xi(t)\rangle\\
&= \sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{2}\sqrt{N!}}\frac{\mathrm{e}^{-|\alpha|^2/2}}{y_+-y_-}\Bigg[\Bigg((\sqrt{2}-y_-)\mathrm{e}^{-\mathrm{i}E_N^+t/\hbar}\\
&\quad+(y_+-\sqrt{2})\mathrm{e}^{-\mathrm{i}E_N^-t/\hbar}\Bigg)\,|ee\rangle+(-(y_-+\sqrt{2})\mathrm{e}^{-\mathrm{i}E_N^+t/\hbar}\\
&\quad+(y_++\sqrt{2})\mathrm{e}^{-\mathrm{i}E_N^-t/\hbar})\,|gg\rangle\Bigg]\,|N\rangle. \qquad (20)
\end{aligned}
$$

We will stay within the assumptions of $\beta \in (0,0.25)$, $\omega_0 \le 0.25\omega$, and $\Omega_N \gg \beta^2$[40] and further make the following assumptions:

$$
\gamma \in (-1,1), \quad \Omega'_N \gg \beta^2. \qquad (21)
$$

With the above assumptions, we can obtain the simplification $y_{\pm} = \pm\sqrt{2}$. So equation (20) simplifies to

$$
\begin{aligned}
|\xi(t)\rangle &= \frac{\mathrm{e}^{-|\alpha|^2/2}}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{\alpha^N}{\sqrt{N!}}(\mathrm{e}^{-\mathrm{i}E_N^+t/\hbar}\,|e,e\rangle\\
&\quad+\mathrm{e}^{-\mathrm{i}E_N^-t/\hbar}\,|g,g\rangle)\,|N\rangle. \qquad (22)
\end{aligned}
$$

The reduced density matrix for the qubits is obtained by tracing out the oscillator states

$$
\begin{aligned}
&\rho^{1,2}(t)\\
&= \sum_N\langle N\,|\,\xi(t)\rangle\langle\xi(t)\,|\,N\rangle\\
&= \frac{1}{2}(|ee\rangle\langle ee|+|gg\rangle\langle gg|)\\
&\quad+\frac{1}{2}\left(\sum_{N=0}^{\infty}S(t,2\omega_0,\gamma)\,|gg\rangle\langle ee|+\mathrm{H.c.}\right), \qquad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
&S(t,2\omega_0,\gamma) = \sum_N p_{\mathrm{coh}}\cos\omega_0(2\langle N_1\,|\,N_0\rangle+\gamma\langle N_1\,|\,N_{-1}\rangle)t,\\
&p_{\mathrm{coh}} = \frac{\mathrm{e}^{-|\alpha|^2}|\alpha|^{2N}}{N!}. \qquad (24)
\end{aligned}
$$

When $t > 0$, we can see that the reduced state of the qubits becomes mixed. For two qubits that are in an arbitrary mixed state, we can use the concurrence to quantify the

entanglement.[55] The density matrix $\rho^{1,2}(t)$ is an example of the so called X-matrix.[56] The concurrence for an X-matrix is

$$C(t) = \left| \sum_{k=0}^{\infty} S(t, 2\omega_0, \gamma) \right|. \tag{25}$$

We evaluate the above sum approximately. If $|\alpha|$ is large, we can replace the Poisson distribution $p_{\rm coh}$ with a Gaussian one

$$p_{\rm coh} = \frac{{\rm e}^{-|\alpha|^2}|\alpha|^{2N}}{N!} \longrightarrow \frac{1}{\sqrt{2\pi|\alpha|^2}}\,{\rm e}^{-\frac{(N-|\alpha|^2)^2}{2|\alpha|^2}}. \tag{26}$$

Then if $|\alpha\beta| \ll 1$, we retain the first three terms of the Laguerre polynomial in Eq. (24) to obtain

$$\begin{aligned} 2\omega_0\langle N_1 \mid N_0\rangle &\approx 2\omega_0\,{\rm e}^{-x/2}\left(1 - Nx + \frac{N(N-1)}{4}x^2\right), \\ \omega_0\gamma\langle N_1 \mid N_{-1}\rangle &\approx \gamma\omega_0\,{\rm e}^{-2x}\left(1 - 2Nx + N(N-1)x^2\right), \end{aligned} \tag{27}$$

where $x = \beta^2$. Substituting this approximation into Eq. (24) yields

$$\begin{aligned} &S(t, 2\omega_0, \gamma) \\ &= {\rm Re}\sum_N p_{\rm coh}\exp\left[{\rm i}\tau\left(1 - Nx + \frac{N(N-1)}{4}x^2\right)\right. \\ &\quad \left. + {\rm i}\tau'\left(1 - 2Nx + N(N-1)x^2\right)\right], \end{aligned} \tag{28}$$

where $\tau = 2\omega_0 t\,{\rm e}^{-x/2}$ and $\tau' = \gamma\omega_0 t\,{\rm e}^{-2x}$. According to the Poisson sum formula, we have

$$\begin{aligned} &S(t, 2\omega_0, \gamma) \\ &= {\rm Re}\left[\sum_{k=-\infty}^{\infty}\bar{S}(t, 2\omega_0, \gamma) + \frac{1}{2}p_{\rm coh}(0)\exp(\tau + \tau')\right], \end{aligned} \tag{29}$$

where

$$\begin{aligned} \bar{S}(t, 2\omega_0, \gamma) &= \int_0^{\infty} {\rm d}n\, p_{\rm coh}\,{\rm e}^{2{\rm i}\pi k n} \\ &\quad \times \exp\left[{\rm i}\tau\left(1 - Nx + \frac{N(N-1)}{4}x^2\right)\right. \\ &\quad \left. + {\rm i}\tau'\left(1 - 2Nx + N(N-1)x^2\right)\right]. \end{aligned} \tag{30}$$

Using the Gaussian integral (we extend the lower limit of the integral to $n = -\infty$ when $|\alpha|^2 \gg 1$), we obtain

$$\bar{S}(t, 2\omega_0, \gamma) = \frac{\exp(\Phi_{\rm Re} + {\rm i}\Phi_{\rm Im})}{(1 + (yf/2 + 2y'f)^2)^{1/4}}, \tag{31}$$

where

$$\Phi_{\rm Re} = \frac{|\alpha|^2}{2(1 + (yf/2 + 2y'f)^2)}[1 - g^2 + (yf + 4y'f)g] - \frac{|\alpha|^2}{2}, \tag{32}$$

$$\begin{aligned} \Phi_{\rm Im} &= \frac{|\alpha|^2}{2(1 + (yf/2 + 2y'f)^2)}\left[(1 - g^2)\left(\frac{yf}{2} + 2y'f\right) - 2g\right] \\ &\quad - \frac{\theta}{2} + \tau + \tau', \end{aligned} \tag{33}$$

and we have defined

$$\begin{aligned} &y = \tau x,\ \ y' = \tau' x,\ \ f = |\alpha|^2 x, \\ &\theta = \tan^{-1}(yf/2 + 2y'f)^2, \\ &g = y + 2y' + yx/4 + y'x - 2\pi k. \end{aligned} \tag{34}$$

Using $x \ll 1$, we can obtain $\tau' = \gamma\tau/2$ and $y' = \gamma y/2$. So we can see that $\bar{S}(t, 2\omega_0, \gamma)$ will be maximum when the time is around $y = 2k\pi/(1+\gamma)$. With this observation, we can simplify the expression for $\Phi_{\rm Re}$ by neglecting the terms that are of the order of yx, $(y - 2\pi k)f^2$, f^3, $(y - 2\pi k)^2 f$ and higher powers of these terms to obtain

$$\begin{aligned} \Phi_{\rm Re} &= -\frac{|\alpha|^2}{2(1 + ((1+2\gamma)\pi k f/(1+\gamma))^2)} \\ &\quad \times \left[(1+\gamma)y - 2\pi k\left(1 + \frac{1+2\gamma}{2+2\gamma}f\right)\right]^2. \end{aligned} \tag{35}$$

In the same way, we can obtain

$$\Phi_{\rm Im} = -\frac{\tan^{-1}(\pi k f)^2}{2} + \frac{1}{x}[-y[1 + f(1+\gamma)] + 2\pi k f]. \tag{36}$$

Thus, the form of $\bar{S}(t, 2\omega_0, \gamma)$ is

$$\begin{aligned} \bar{S}(t, 2\omega_0, \gamma) &= \exp\left(\frac{-((1+\gamma)\tau - \tau_k)^2|\,\alpha|^2 x^2}{2(1 + ((1+2\gamma)\pi k f/(1+\gamma))^2)} + {\rm i}\Phi_{\rm Im}\right) \\ &\quad \times \frac{1}{(1 + ((1+2\gamma)\pi k f/(1+\gamma))^2)^{1/4}}, \end{aligned} \tag{37}$$

where

$$\tau_k = 2\pi k\left(1 + \frac{1+2\gamma}{2+2\gamma}f\right)/x. \tag{38}$$

Obviously, the amplitude of $\bar{S}(t, 2\omega_0, \gamma)$ is much larger than $p_{\rm coh}(0)$. Thus, we can neglect the term $p_{\rm coh}(0)\exp({\rm i}(\tau + \tau'))/2$ in Eq. (29). We just need the positive time. So $k \geq 0$ in Eq. (29). Finally, we simplify the form of $S(t, 2\omega_0, \gamma)$ to

$$S(t, 2\omega_0, \gamma) = {\rm Re}\left[\sum_{k=0}^{\infty}\bar{S}(t, 2\omega_0, \gamma)\right]. \tag{39}$$

Thus, we can obtain the concurrence for $\rho^{1,2}(t)$ as

$$\begin{aligned} C(t) &= \sum_{k=0}^{\infty}\exp\left(\frac{-((1+\gamma)\tau - \tau_k)^2|\,\alpha|^2 x^2}{2(1 + ((1+2\gamma)\pi k f/(1+\gamma))^2)}\right) \\ &\quad \times \frac{1}{(1 + ((1+2\gamma)\pi k f/(1+\gamma))^2)^{1/4}}. \end{aligned} \tag{40}$$

Now, we discuss the effect of parameter γ on the concurrence for $\rho^{1,2}(t)$. From Eqs. (37) and (39), we can see that $S(t, 2\omega_0, \gamma)$ exhibits collapse and revival with $\bar{S}(t, 2\omega_0, \gamma)$. In other words, the entanglement between the qubits exhibits collapse and revival. The revival time and the height of $S(t, 2\omega_0, \gamma)$ are

$$t_k^{\rm rev} = \frac{\pi k}{\omega_0\beta^2(1+\gamma)}\left(1 + \frac{1+2\gamma}{2+2\gamma}|\alpha\beta|^2\right), \tag{41}$$

$$h_k = \frac{1}{(1+(1+2\gamma/(1+\gamma))^2 k^2\pi^2|\alpha\beta|^4)^{1/4}}. \quad (42)$$

From Eq. (37), we can see that the width of the primary revival ($k=0$) is

$$\delta\tau_0 = \frac{1}{|\alpha|\beta^2}, \quad (43)$$

and the width of the k-th revival is

$$\delta\tau_k = \delta\tau_0\sqrt{1+\left(\frac{1+2\gamma}{1+\gamma}\right)^2 k^2\pi^2|\alpha\beta|^4}. \quad (44)$$

It is important to note that the k-th revival time becomes longer with increasing γ when $\gamma\in(0,1)$. When $\gamma\in(-1,-0.5)$, the smaller the parameter γ, the higher the height of the k-th concurrence. These are evident in Fig. 1. The interaction between two atoms affecting the entanglement was also discussed in Ref. [57]. For the cavity initially in the coherent and thermal equilibrium states, the entanglement exhibits collapse and revival. The period of collapse–revival becomes long as the qubit–qubit coupling and the average number of photons increase. This is similar to our results. However, in Ref. [57], the qubits frequencies equal to the oscillator frequency ($\omega_0=\omega$). We are concerned here with the regime where the qubits are quasi-degenerate, with frequencies much smaller than the oscillator frequency ($\omega_0\ll\omega$). This is in contrast to the previous work.

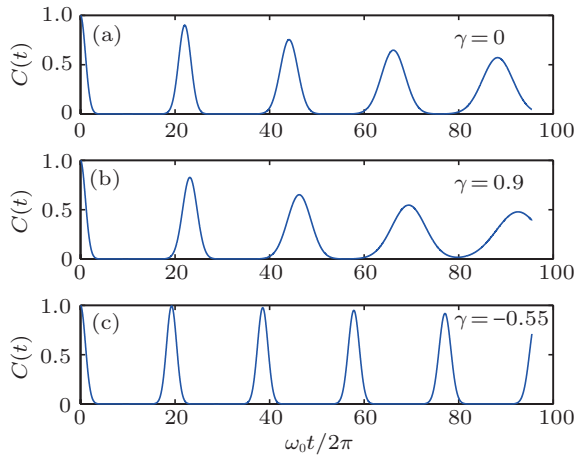


Fig. 1. (color online) The entanglement dynamics between the two qubits for $\omega_0=0.15\omega$, $\beta=0.16$, $\alpha=3$. Here (a) $\gamma=0$, (b) $\gamma=0.9$, and (c) $\gamma=-0.55$. The revival time and the height of concurrence can be longer or higher respectively by changing parameter γ.

The above discussion can be applied in areas of quantum information processing where the coherent control of entanglement is vital. We can choose an appropriate parameter γ, according to the actual requirements, to prolong the entanglement time or improve the entanglement degree.

4. Brief discussion of K-qubit system and quantum phase transitions

4.1. The K-qubit system

The above analysis is restricted to the two-qubit case. In this section, we will give a brief discussion of K qubits interacting with a high frequency oscillator. The Hamiltonian governing the dynamics of the K-qubit system is

$$\hat{H} = \hbar\omega_0\hat{S}_z + \hbar\omega\hat{a}^+\hat{a} + \hbar\omega\beta(\hat{a}+\hat{a}^+)\hat{S}_x + \hbar\omega_0\gamma\sum_{i\neq j}^{k}\hat{\sigma}_z^i\hat{\sigma}_z^j, \quad (45)$$

where

$$\hat{S}_z = \frac{1}{2}\sum_{i=1}^{k}\hat{\sigma}_z^i,\ \ \hat{S}_x = \frac{1}{2}\sum_{i=1}^{k}\hat{\sigma}_x^i. \quad (46)$$

Following the same calculation as before, we can obtain the eigenfunctions of $\hat{H}$ when $\omega_0=0$, which are $|\Phi\rangle = |j,m\rangle\otimes|\phi_m\rangle$. For simplicity, we take K to be even, then we have $j=0,1,\ldots,K/2$, with $m=-j,\ldots,j$. The eigenvalue of $\hat{H}$ corresponding to state $|\Phi\rangle = |j,m\rangle|\phi_m\rangle$ is the same as before, $E_{N,m}=\hbar\omega(N-\beta_m^2)$, which is independent of K. The matrix elements of the perturbation term, $\hbar\omega_0\hat{S}_z$ and $\hbar\omega_0\gamma\sum_{i\neq j}^{k}\hat{\sigma}_z^i\hat{\sigma}_z^j$, in the $|\Phi\rangle = |j,m\rangle|\phi_m\rangle$ basis are

$$\begin{aligned}&\hbar\omega_0\langle j,m|\left(\hat{S}_z+\gamma\sum_{i\neq j}^{k}\hat{\sigma}_z^i\hat{\sigma}_z^j\right)|j',m'\rangle\langle N_m|N'_{m'}\rangle\\ &= \hbar\omega_0\delta_{j,j'}\langle N_m|N_{m'}\rangle\Big[\frac{1}{2}(\delta_{m'+1,m}\sqrt{(j-m)(j+m+1)}\\ &\quad+\delta_{m'-1,m}\sqrt{(j+m)(j-m+1)})\\ &\quad+\frac{\gamma}{2}(\delta_{m'+2,m}\sqrt{(j-m)(j+m+2)}\\ &\quad+\delta_{m'-2,m}\sqrt{(j+m)(j-m+2)})\Big]. \end{aligned} \quad (47)$$

Under the adiabatic approximation, for a given j, the Hamiltonian has a block diagonal form in the $|\Phi\rangle = |j,m\rangle|\phi_m\rangle$ basis, with each block corresponding to a particular value of N. Then we can obtain the eigenvalues and the eigenfunctions of each of these block diagonal matrices. This is only a brief discussion of the K-qubit system; the detailed analysis and entanglement dynamics of the K qubits may be our future work.

4.2. Quantum phase transitions

The interaction between two atoms also affects the quantum phase transition.[58–61] Now we give a brief discussion about quantum phase transitions. According to the above discussion of the K-qubit system, we can obtain the ground-state energy's second derivative ($k=4$) and show it in Fig. 2. As shown in Fig. 2, there is no second quantum phase transition in the system. When we decrease γ, the result does not change, which differs to that in Ref. [58]. In Ref. [58], Chen

et al. showed that the first-order and the second-order quantum phase transitions may occur in finite systems, and both the first-order and the second-order QPTs may occur at the same point when the interactions increase. The difference in the interaction form may be one of the reasons, which may also be our future work.

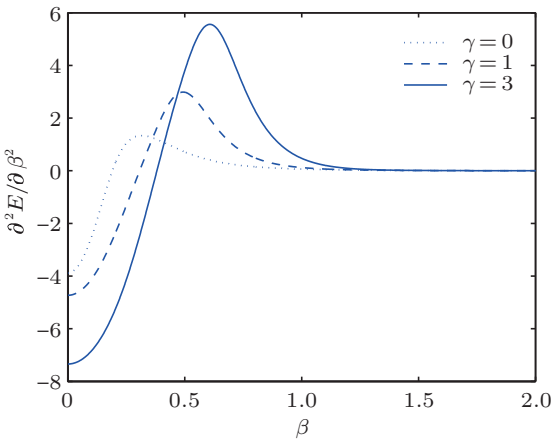


Fig. 2. (color online) The ground-state energy's second derivatives for different atomic interactions ($\gamma=0$, $\gamma=1$, and $\gamma=3$).

5. Conclusion

In this report, we extended the JC model for two qubits interacting with an oscillator beyond the RWA limit to include the interaction between the two qubits. We worked within the adiabatic approximation introduced by Irish *et al*.[39] and Agarwal's *et al*.[40] We were able to compare the spectra of $\hat{H}$. Then, we concentrated on the qubit–qubit entanglement in this study. We controlled the entanglement time by a novel choice of parameter γ. This is very interesting and may have wide applications in quantum information processing.

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