Efficient remote preparation of arbitrary two- and three-qubit states via the χ state*

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(Received 26 January 2014; revised manuscript received 5 March 2014; published online 10 July 2014)

The application of χ state are investigated in remote state preparation (RSP). By constructing useful measurement bases with the aid of Hurwitz matrix equation, we propose several RSP schemes of arbitrary two- and three-qubit states via the χ state as the entangled resource. It is shown that the original state can be successfully prepared with the probability 100% and 50% for real coefficients and complex coefficients, respectively. For the latter case, the special ensembles with unit success probability are discussed by the permutation group. It is worth mentioning that the novel measurement bases have no restrictions on the coefficients of the prepared state, which means that the proposed schemes are more applicable.

Keywords: χ state, remote state preparation, Hurwitz matrix equation, measurement basis, permutation group

PACS: 03.67.Hk, 03.67.–a, 03.65.–w **DOI:** 10.1088/1674-1056/23/9/090308

1. Introduction

Secure transmission of a quantum state is a critical task in quantum information processing. Thanks to the previously shared entanglement resource (QR), the task can be achieved without physically transmitting the state itself. A celebrated protocol is quantum teleportation (QT)[1] which transfers an unknown qubit from a sender to a distant receiver by dual usage of local operation and classical communication. In the case that the sender knows about the original state completely, Lo[2] proposed another intriguing method called remote state preparation (RSP) which uses the same QR as in QT but with simpler local operation and reduced classical communication cost (CCC) for some special ensembles of states. It has been shown that the CCC for RSP can be decreased to 1 classical bit (cbit) for a qubit chosen from equatorial circles on a Bloch sphere, while in standard QT two cbits are needed.[3] One of the most outstanding features of RSP distinguished from QT is that RSP exhibits a tradeoff between the required entanglement and the CCC due to the sender's prior knowledge of the original state.[4] In virtue of its unique advantage, various theoretical aspects have been investigated for the generalization of RSP[5–30] and some experimental implementations have been reported.[31–35]

As its central role in the field of quantum information, much attention has been paid to the study of the entanglement. Yeo *et al.*[36] studied a genuine maximally entangled four-qubit χ state. Different from a four-particle GHZ or W state under stochastic local operations and classical communication, it has many interesting entanglement properties. Moreover, as a peculiar entangled resource the χ state has been widely applied in teleportation, superdense coding,[36] quantum secure direct communication,[37] quantum information splitting,[38,39] quantum steganography,[40] etc. Although many entangled resources such as EPR pair, GHZ-calss state, W-class sate, cluster-type state or Brown-type state have often been exploited for carrying out RSP, the usefulness of the χ state is seldom investigated.

This paper aims at the application of the χ state in RSP. In Section 2, we first construct suitable measurement bases with the aid of Hurwitz matrix equation, which have no restrictions on the coefficients of the prepared state. Then two schemes with the success probability 100% and 50% are put forward to remotely prepare an arbitrary two-qubit state in both the real and complex spaces, respectively. Moreover, some special complex coefficient ensembles with unit success probability are discussed by the permutation group. Remote preparation of an arbitrary three-qubit state is considered in Section 3. Section 4 is the conclusions.

2. Remote preparation of an arbitrary two-qubit state

The sender Alice needs to prepare a two-qubit state

$$x_0|00\rangle + x_1|01\rangle + x_2|10\rangle + x_3|11\rangle \tag{1}$$

for the distant receiver Bob, where the coefficients x_j $(j = 0,\ldots,3)$ satisfy $\sum_{j=0}^{3} |x_j|^2 = 1$. Alice knows the parameters x_0,

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61201253 and 61303039) and the Fundamental Research Funds for the Central Universities of China (Grant No. 2682014CX095).

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x_1, x_2, x_3 exactly, while Bob does not have any information about the desired state.

The construction of the measurement basis plays a key role in RSP. In the following, we take advantage of Hurwitz matrix equation to devise useful measurement basis.

Lemma 1[41] Suppose that an $n\times n$ real matrix set $\{\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_n\}$ satisfies condition

$$\boldsymbol{A}_i\boldsymbol{A}_j^{\mathrm{T}}+\boldsymbol{A}_j\boldsymbol{A}_i^{\mathrm{T}}=2\delta_{ij}E,\ \ 1\le i,j\le n, \tag{2}$$

which is called a real Hurwitz matrix equation. Then $n=\{1,2,4,8\}$ and the canonical form is unique up to a real equivalent.

When $n=4$, the canonical form of the 4×4 real matrices is

$$\begin{aligned}&\boldsymbol{P}_1=\boldsymbol{E}_4,\ \ \boldsymbol{P}_2=\mathrm{diag}(\boldsymbol{J}_2,-\boldsymbol{J}_2),\\ &\boldsymbol{P}_3=\begin{pmatrix}0&-\boldsymbol{E}\\ \boldsymbol{E}&0\end{pmatrix},\ \ \boldsymbol{P}_4=\begin{pmatrix}0&\boldsymbol{J}_2\\ \boldsymbol{J}_2&0\end{pmatrix}.\end{aligned} \tag{3}$$

When $n=8$, the canonical form of the 8×8 real matrices is

$$\begin{aligned}&\boldsymbol{Q}_1=\boldsymbol{E}_8,\ \ \boldsymbol{Q}_2=\mathrm{diag}(\boldsymbol{J}_2,-\boldsymbol{J}_2,-\boldsymbol{J}_2,\boldsymbol{J}_2),\\ &\boldsymbol{Q}_3=\begin{pmatrix}0&-\boldsymbol{E}&0&0\\ \boldsymbol{E}&0&0&0\\ 0&0&0&\boldsymbol{E}\\ 0&0&-\boldsymbol{E}&0\end{pmatrix},\ \ \boldsymbol{Q}_4=\begin{pmatrix}0&\boldsymbol{J}_2&0&0\\ \boldsymbol{J}_2&0&0&0\\ 0&0&0&-\boldsymbol{J}_2\\ 0&0&-\boldsymbol{J}_2&0\end{pmatrix},\\ &\boldsymbol{Q}_5=\begin{pmatrix}0&-\boldsymbol{E}\\ \boldsymbol{E}&0\end{pmatrix},\ \ \boldsymbol{Q}_6=\begin{pmatrix}0&0&\boldsymbol{J}_2&0\\ 0&0&0&\boldsymbol{J}_2\\ \boldsymbol{J}_2&0&0&0\\ 0&\boldsymbol{J}_2&0&0\end{pmatrix},\\ &\boldsymbol{Q}_7=\begin{pmatrix}0&0&0&-\boldsymbol{J}_1\\ 0&0&\boldsymbol{J}_1&0\\ 0&-\boldsymbol{J}_1&0&0\\ \boldsymbol{J}_1&0&0&0\end{pmatrix},\\ &Q_8=\begin{pmatrix}0&0&0&-\boldsymbol{J}_3\\ 0&0&\boldsymbol{J}_3&0\\ 0&-\boldsymbol{J}_3&0&0\\ \boldsymbol{J}_3&0&0&0\end{pmatrix},\end{aligned} \tag{4}$$

with 2×2 matrices

$$\boldsymbol{J}_1=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix},\ \ \boldsymbol{J}_2=\begin{pmatrix}0&-1\\ 1&0\end{pmatrix},\ \ \boldsymbol{J}_3=\begin{pmatrix}0&1\\ 1&0\end{pmatrix}. \tag{5}$$

Here $\boldsymbol{E}$ denotes the identity matrix. The 4×4 matrices $\boldsymbol{P}_1,\ldots,\boldsymbol{P}_4$ can be viewed as four elements in the Caykey–Dikson algebra, in other words, they may be viewed as four imaginary units $1,e_1,e_2,e_3$. Similarly, the 8×8 matrices $\boldsymbol{Q}_1,\boldsymbol{Q}_2,\ldots,\boldsymbol{Q}_8$ can be viewed as octonion.

Using the matrices $\boldsymbol{P}_1,\ldots,\boldsymbol{P}_4$, one can construct a complex matrix

$$\boldsymbol{A}=\sum_{j=1}^{4}\varepsilon_j z\boldsymbol{P}_j^{\mathrm{T}}=\begin{pmatrix}\boldsymbol{z}\boldsymbol{P}_1^{\mathrm{T}}\\ \boldsymbol{z}\boldsymbol{P}_2^{\mathrm{T}}\\ \boldsymbol{z}\boldsymbol{P}_3^{\mathrm{T}}\\ \boldsymbol{z}\boldsymbol{P}_4^{\mathrm{T}}\end{pmatrix}=\begin{pmatrix}z_1&z_2&z_3&z_4\\ -z_2&z_1&z_4&-z_3\\ -z_3&-z_4&z_1&z_2\\ -z_4&z_3&-z_2&z_1\end{pmatrix}, \tag{6}$$

where ε_j $(j=1,\ldots,4)$ is a 4-dimensional unit column vector with its j-th coordinate being 1, and all the other coordinates being zero, $\boldsymbol{z}=(z_1,z_2,z_3,z_4)$ is a complex vector. Notice that only one z_j occurs in each row (column) vector of the matrix $\boldsymbol{A}$. This feature is helpful to design useful measurement basis. Moreover, the matrix $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}\boldsymbol{A}^{\dagger}+\boldsymbol{A}^{*}\boldsymbol{A}^{\mathrm{T}}=2\sum_{j=1}^{4}|z_j|^2\boldsymbol{E}, \tag{7}$$

where † denotes the conjugate transpose of a matrix and * denotes the conjugate of a matrix.

We first consider the case that the coefficients $x_0,\ldots,x_3$ of the prepared state are real in Section 2.1. Then the case of complex coefficients is handled in Section 2.2.

2.1. Real coefficient state

Assume that Alice and Bob share the χ state

$$\begin{aligned}|\chi\rangle_{1234}=&\frac{1}{2}(|00\rangle|\varphi^-\rangle-|01\rangle|\psi^-\rangle\\ &+|10\rangle|\psi^+\rangle+|11\rangle|\varphi^+\rangle)_{1234}\end{aligned} \tag{8}$$

as the quantum channel, where $|\varphi^{\pm}\rangle=(|00\rangle\pm|11\rangle)/\sqrt{2}$, $|\psi^{\pm}\rangle=(|01\rangle\pm|10\rangle)/\sqrt{2}$ are Bell states. Alice processes qubits $1,2$, Bob processes qubits $3,4$.

Construct a matrix

$$\boldsymbol{F}=\begin{pmatrix}y_1&y_2&y_3&y_4\\ -y_2&y_1&y_4&-y_3\\ -y_3&-y_4&y_1&y_2\\ -y_4&y_3&-y_2&y_1\end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}&y_1=\frac{x_0-x_3}{\sqrt{2}},\ \ y_2=-\frac{x_1-x_2}{\sqrt{2}},\\ &y_3=\frac{x_1+x_2}{\sqrt{2}},\ \ y_4=\frac{x_0+x_3}{\sqrt{2}}.\end{aligned} \tag{10}$$

Notice that the matrix $\boldsymbol{A}$ in Eq. (6) is the matrix $\boldsymbol{F}$ when $z_j=y_j, j=1,\ldots,4$. Since $x_0,\ldots,x_3$ are real and $\sum_{j=1}^{4}y_j^2=1$, $\boldsymbol{F}$ is a real orthogonal matrix in this case.

Alice measures her particles (1,2) under the orthogonal basis $\{|\zeta_1\rangle,\ldots,|\zeta_4\rangle\}$, which are related to the computation basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ by

$$\begin{pmatrix}|\zeta_1\rangle\\ |\zeta_2\rangle\\ |\zeta_3\rangle\\ |\zeta_4\rangle\end{pmatrix}=\boldsymbol{F}\begin{pmatrix}|00\rangle\\ |01\rangle\\ |10\rangle\\ |11\rangle\end{pmatrix}. \tag{11}$$

After the measurement, Alice conveys her measurement result to Bob through the classical channel.

With this basis, the shared quantum channel in Eq. (8) can be expressed as

$$|\chi\rangle_{1234}$$

$$
\begin{aligned}
&= |\zeta_1\rangle_{12}(y_1|\varphi^-\rangle - y_2|\psi^-\rangle + y_3|\psi^+\rangle + y_4|\varphi^+\rangle)_{34} \\
&\quad + |\zeta_2\rangle_{12}(-y_2|\varphi^-\rangle - y_1|\psi^-\rangle + y_4|\psi^+\rangle - y_3|\varphi^+\rangle)_{34} \\
&\quad + |\zeta_3\rangle_{12}(-y_3|\varphi^-\rangle + y_4|\psi^-\rangle + y_1|\psi^+\rangle + y_2|\varphi^+\rangle)_{34} \\
&\quad + |\zeta_4\rangle_{12}(-y_4|\varphi^-\rangle - y_3|\psi^-\rangle - y_2|\psi^+\rangle + y_1|\varphi^+\rangle)_{34} \\
&= |\zeta_1\rangle_{12}(x_0|00\rangle + x_1|01\rangle + x_2|10\rangle + x_3|11\rangle)_{34} \\
&\quad + |\zeta_2\rangle_{12}(-x_2|00\rangle + x_3|01\rangle + x_0|10\rangle - x_1|11\rangle)_{34} \\
&\quad + |\zeta_3\rangle_{12}(-x_1|00\rangle + x_0|01\rangle - x_3|10\rangle + x_2|11\rangle)_{34} \\
&\quad + |\zeta_4\rangle_{12}(-x_3|00\rangle - x_2|01\rangle + x_1|10\rangle + x_0|11\rangle)_{34}. \quad (12)
\end{aligned}
$$

It reflects that Bob can always perform a proper unitary operation to recover the initial state whatever Alice's measurement outcome is. The detailed recovery operations are shown in Table 1, where I, X, Y, Z are the Pauli operations.

Table 1. The corresponding recovery unitary operation (RUO) and the the collapsed state (CS) conditioned on Alice's measurement outcome (AMO).

AMO	CS	RUO
$\|\zeta_1\rangle$	$(x_0\|00\rangle + x_1\|01\rangle + x_2\|10\rangle + x_3\|11\rangle)_{34}$	I_3I_4
$\|\zeta_2\rangle$	$(-x_2\|00\rangle + x_3\|01\rangle + x_0\|10\rangle - x_1\|11\rangle)_{34}$	Y_3Z_4
$\|\zeta_3\rangle$	$(x_1\|00\rangle + x_3\|01\rangle - x_0\|10\rangle + x_2\|11\rangle)_{34}$	I_3Y_4
$\|\zeta_4\rangle$	$(x_2\|00\rangle + x_0\|01\rangle - x_3\|10\rangle + x_1\|11\rangle)_{34}$	Y_3X_4

2.2. Complex coefficient state

Besides the χ state, Alice introduces an auxiliary qubit $|0\rangle_A$. The combined system can be rewritten as

$$
\begin{aligned}
&\frac{1}{2}(|000\rangle|\varphi^-\rangle - |001\rangle|\psi^-\rangle + |010\rangle|\psi^+\rangle \\
&+ |011\rangle|\varphi^+\rangle)_{A1234}. \quad (13)
\end{aligned}
$$

We present the new RSP protocol as follows.

S1 Construct a useful measurement basis for the sender Alice.

Alice can choose to measure her particles under whatever related basis since she knows the coefficients $x_0, \ldots, x_3$ of the prepared state. In order to realize the RSP with a high probability, it is necessary to construct a suitable measurement basis.

Firstly, construct a 4×8 matrix

$$
\boldsymbol{M} = (\boldsymbol{F} \;\; \boldsymbol{F}^*), \quad (14)
$$

where the 4×4 submatrices matrix $\boldsymbol{F}$ (see Eq. (9)) is closely related to four-element multiplication matrix. As for the case of complex coefficients x_0, x_1, x_2, x_3, $\boldsymbol{F}$ is a complex matrix and satisfies

$$
\boldsymbol{F}\boldsymbol{F}^\dagger + \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{E}_4, \quad (15)
$$

as $\sum_{j=1}^{4} |y_j|^2 = 1$. It means all the row vectors in the matrix $\boldsymbol{M}$ form a unit orthogonal vector set.

Thus, a 4×8 matrix $\boldsymbol{N}$ dependent on $y_j, j = 1, \ldots, 4$ can always be found such that $\begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{N} \end{pmatrix}$ is an 8×8 unitary matrix. An explicit expression of such a matrix $\boldsymbol{N}$ is given below. Notice that

$$
\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \lambda \boldsymbol{E}_4, \quad (16)
$$

where $\lambda = \sum_{j=1}^{4} y_j^2 = \sum_{j=0}^{3} x_j^2$. Then the matrix $\boldsymbol{N}$ can be taken as

$$
\left(\boldsymbol{F}^*, -\frac{\lambda^*}{\lambda}\boldsymbol{F}\right). \quad (17)
$$

Here $-\lambda^*/\lambda$ is equal to 1 when $\lambda = 0$.

Finally, with the help of 8×8 unitary matrix

$$
\begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{N} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} & \boldsymbol{F}^* \\ \boldsymbol{F}^* & -\frac{\lambda^*}{\lambda}\boldsymbol{F} \end{pmatrix},
$$

we can construct a set of completely orthogonal basis $|\zeta_1\rangle, \ldots, |\zeta_8\rangle$ in the 8-dimensional Hilbert space $\mathbb{C}^8$ as follows:

$$
\begin{pmatrix} |\zeta_1\rangle \\ |\zeta_2\rangle \\ |\zeta_3\rangle \\ |\zeta_4\rangle \\ |\zeta_5\rangle \\ |\zeta_6\rangle \\ |\zeta_7\rangle \\ |\zeta_8\rangle \end{pmatrix} = \begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{N} \end{pmatrix}^* \begin{pmatrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{pmatrix}. \quad (18)
$$

S2 Alice performs a three-particle projective measurement on her qubits $(\boldsymbol{A}, 1, 2)$ with respect to the basis $\{|\zeta_1\rangle, \ldots, |\zeta_8\rangle\}$ and informs Bob the measurement outcome by classical communication.

S3 Bob performs the recovery operations on his qubits to convert the collapsed state into the prepared state.

Let us expand the state $|\Psi\rangle_{A1234}$ in Eq. (13) by the measurement basis in the form

$$
\begin{aligned}
|\Psi\rangle_{A1234} &= \frac{1}{2}[|\zeta_1\rangle_{A12} f(x_0, x_1, x_2, x_3)_{34} \\
&\quad + |\zeta_2\rangle_{A12} f(-x_2, x_3, x_0, x_1)_{34} \\
&\quad + |\zeta_3\rangle_{A12} f(-x_1, x_0, -x_3, x_2)_{34} \\
&\quad + |\zeta_4\rangle_{A12} f(-x_3, -x_2, x_1, x_0)_{34} \\
&\quad + |\zeta_5\rangle_{A12} f(x_0^*, x_1^*, x_2^*, x_3^*)_{34} \\
&\quad + |\zeta_6\rangle_{A12} f(-x_2^*, x_3^*, x_0^*, -x_1^*)_{34} \\
&\quad + |\zeta_7\rangle_{A12} f(-x_1^*, x_0^*, -x_3^*, x_2^*)_{34} \\
&\quad + |\zeta_8\rangle_{A12} f(-x_3^*, -x_2^*, x_1^*, x_0^*)_{34}]. \quad (19)
\end{aligned}
$$

For convenience, denote $f(u_0, u_1, u_2, u_3) = u_0|00\rangle + u_1|01\rangle + u_2|10\rangle + u_3|11\rangle$. The above equation reveals that the desired state can be reconstructed by Bob only if Alice's measurement outcome lies in $\{|\zeta_1\rangle, \ldots, |\zeta_4\rangle\}$. As a summary, Bob's

recovery operations on his qubits $(3,4)$ conditioned on Alice's measurement outcomes are shown in Table 1. It is possible that Alice's measurement outcome lies in $\{|\zeta_5\rangle,\ldots,|\zeta_8\rangle\}$ since each measurement outcome occurs with an equal probability of $1/8$, which means that Bob is not able to recover the original state due to lack of the knowledge of $x_j, j=0,\ldots,3$.

Therefore, the total success probability of the scheme is $4\times 1/8 = 50\%$ in the general condition.

A question naturally arises: can the two-qubit state be prepared with higher success probability if its coefficients are some special values? The following classification criterion gives the answer.

Theorem 1 Two-qubit state can be prepared under the proposed scheme deterministically if and only if the coefficients $x_j, j=0,\ldots,3$ satisfy

$$(x_0^*, x_1^*, x_2^*, x_3^*) = \mathrm{e}^{\mathrm{i}\tau_p} p(\pm x_0, \pm x_1, \pm x_2 \pm x_3). \tag{20}$$

Here τ_p is a real constant depending on the permutation $p \in S_4$ with S_4 being the permutation group on the four letters.[42]

From Eq. (20) and $\sum_{j=0}^{3} |x_j|^2 = 1$, one can calculate the special ensembles of the coefficients that the two-qubit state can be prepared with unit success probability. For example, if Alice's measurement result is $|\zeta_5\rangle_{A12}$, she informs Bob the outcome. As a consequence, Bob knows that his qubits $(3,4)$ are now in the state $|f(x_0^*, x_1^*, x_2^*, x_3^*)\rangle$ which can be rewritten as $|f(p(\pm x_0, \pm x_1, \pm x_2, \pm x_3)\rangle$ up to a global phase $\mathrm{e}^{\mathrm{i}\tau_p}$. Therefore, Bob can reconstruct the prepared state by performing the corresponding unitary operations.

3. Remote preparation of an arbitrary three-qubit state

In this section, we proceed to show the utility of χ state for the remote preparation of an arbitrary three-qubit state

$$\begin{aligned} &t_0|000\rangle + t_1|001\rangle + t_2|010\rangle + t_3|011\rangle \\ &+ t_4|100\rangle + t_5|101\rangle + t_6|110\rangle + t_7|111\rangle, \end{aligned} \tag{21}$$

where the coefficients $t_j, j=0,\ldots,7$ satisfy $\sum_{j=0}^{7} |t_j|^2 = 1$.

With the assistance of the octonion matrices $\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_8$, one can construct a complex matrix

$$\begin{aligned} \boldsymbol{B} &= \sum_{j=1}^{8} \varepsilon_j z \boldsymbol{Q}_j^{\mathrm{T}} \\ &= \begin{pmatrix} z\boldsymbol{Q}_1^{\mathrm{T}} \\ z\boldsymbol{Q}_2^{\mathrm{T}} \\ \vdots \\ z\boldsymbol{Q}_8^{\mathrm{T}} \end{pmatrix} \\ &= \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ -z_2 & z_1 & z_4 & -z_3 & z_6 & -z_5 & -z_8 & z_7 \\ -z_3 & -z_4 & z_1 & z_2 & z_7 & z_8 & -z_5 & -z_6 \\ -z_4 & z_3 & -z_2 & z_1 & z_8 & -z_7 & z_6 & -z_5 \\ -z_5 & -z_6 & -z_7 & -z_8 & z_1 & z_2 & z_3 & z_4 \\ -z_6 & z_5 & -z_8 & z_7 & -z_2 & z_1 & -z_4 & z_3 \\ -z_7 & z_8 & z_5 & -z_6 & -z_3 & z_4 & z_1 & -z_2 \\ -z_8 & -z_7 & z_6 & z_5 & -z_4 & -z_3 & z_2 & z_1 \end{pmatrix}, \end{aligned} \tag{22}$$

where $\varepsilon_j, j=1,\ldots,8$ is an 8-dimensional unit column vector with the j-th coordinate being 1, all the other coordinates are zero, and $\boldsymbol{z} = (z_1, z_2, \ldots, z_8)$ is a complex vector. A notable feature is that only one z_j occurring in each row (column) vector of the matrix $\boldsymbol{B}$. Moreover, $\boldsymbol{B}$ satisfies

$$\boldsymbol{B}\boldsymbol{B}^{\dagger} + \boldsymbol{B}^{*}\boldsymbol{B}^{\mathrm{T}} = 2\sum_{j=1}^{8} |z_j|^2 \boldsymbol{E}. \tag{23}$$

In Section 3.1, we investigate the case where the coefficients $t_0,\ldots,t_7$ of the prepared state are real. Complex coefficients are considered in Section 3.2.

3.1. Real coefficient state

Alice and Bob share the χ state and one Bell state $|\varphi^+\rangle$ as the quantum channel. The combined state is

$$\begin{aligned} &|\chi\rangle_{1234}|\varphi^+\rangle_{56} \\ &= \frac{1}{2\sqrt{2}}(|000\rangle|\varphi^-\rangle|0\rangle + |001\rangle|\varphi^-\rangle|1\rangle - |010\rangle|\psi^-\rangle|0\rangle \\ &\quad - |011\rangle|\psi^-\rangle|1\rangle + |100\rangle|\psi^+\rangle|0\rangle + |101\rangle|\psi^+\rangle|1\rangle \\ &\quad + |110\rangle|\varphi^+\rangle|0\rangle + |111\rangle|\varphi^+\rangle|1\rangle)_{125346}, \end{aligned} \tag{24}$$

where particles (1,2,5) are held by Alice, and the remaining (3,4,6) by Bob.

Construct a matrix

$$\boldsymbol{G} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ -s_2 & s_1 & s_4 & -s_3 & s_6 & -s_5 & -s_8 & s_7 \\ -s_3 & -s_4 & s_1 & s_2 & s_7 & s_8 & -s_5 & -s_6 \\ -s_4 & s_3 & -s_2 & s_1 & s_8 & -s_7 & s_6 & -s_5 \\ -s_5 & -s_6 & -s_7 & -s_8 & s_1 & s_2 & s_3 & s_4 \\ -s_6 & s_5 & -s_8 & s_7 & -s_2 & s_1 & -s_4 & s_3 \\ -s_7 & s_8 & s_5 & -s_6 & -s_3 & s_4 & s_1 & -s_2 \\ -s_8 & -s_7 & s_6 & s_5 & -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned} &s_1 = \frac{t_0 - t_6}{\sqrt{2}}, s_2 = \frac{t_1 - t_7}{\sqrt{2}}, s_3 = -\frac{t_2 - t_4}{\sqrt{2}}, s_4 = -\frac{t_3 - t_5}{\sqrt{2}}, \\ &s_5 = \frac{t_2 + t_4}{\sqrt{2}}, s_6 = \frac{t_3 + t_5}{\sqrt{2}}, s_7 = \frac{t_0 + t_6}{\sqrt{2}}, s_8 = \frac{t_1 + t_7}{\sqrt{2}}. \end{aligned} \tag{26}$$

Notice the matrix $\boldsymbol{B}$ in Eq. (22) is the matrix $\boldsymbol{G}$ if $z_j = s_j, j = 1,\ldots,8$. Now the matrix $\boldsymbol{G}$ is a real orthogonal matrix as $\sum_{j=1}^{8} s_j^2 = 1$ for real coefficients $t_0,\ldots,t_7$.

Design a set of orthogonal basis $|\xi_j\rangle, j = 1,\ldots,8$ with

$$\begin{pmatrix} |\xi_1\rangle \\ |\xi_2\rangle \\ |\xi_3\rangle \\ |\xi_4\rangle \\ |\xi_5\rangle \\ |\xi_6\rangle \\ |\xi_7\rangle \\ |\xi_8\rangle \end{pmatrix} = \boldsymbol{G} \begin{pmatrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{pmatrix}. \tag{27}$$

Under this basis, the joint system can be reorganized into

$$\begin{aligned} &|\chi\rangle_{1234}|\varphi^+\rangle_{56} \\ &= \frac{1}{2}[|\xi_1\rangle_{125} g(t_0,t_1,t_2,t_3,t_4,t_5,t_6,t_7)_{346} \\ &\quad + |\xi_2\rangle_{125} g(-t_1,t_0,t_3,-t_2,t_5,-t_4,-t_7,t_6)_{346} \\ &\quad + |\xi_3\rangle_{125} g(-t_4,-t_5,t_6,t_7,t_0,t_1,-t_2,-t_3)_{346} \\ &\quad + |\xi_4\rangle_{125} g(t_3,-t_2,t_1,-t_0,t_7,-t_6,t_5,-t_4)_{346} \\ &\quad + |\xi_5\rangle_{125} g(-t_2,-t_3,t_0,t_1,-t_6,-t_7,t_4,t_5)_{346} \\ &\quad + |\xi_6\rangle_{125} g(-t_5,t_4,t_7,-t_6,-t_1,t_0,t_3,-t_2)_{346} \\ &\quad + |\xi_7\rangle_{125} g(-t_6,t_7,-t_4,t_5,t_2,-t_3,t_0,-t_1)_{346} \\ &\quad + |\xi_8\rangle_{125} g(-t_7,-t_6,-t_5,-t_4,t_3,t_2,t_1,t_0)_{346}], \end{aligned} \tag{28}$$

where $g(v_0,\ldots,v_7) = v_0|000\rangle + v_1|001\rangle + v_2|010\rangle + v_3|011\rangle + v_4|100\rangle + v_5|101\rangle + v_6|110\rangle + v_7|111\rangle$.

It can be seen that conditioned on Alice's measurement results Bob can always transform his collapsed particles to the original state by performing suitable unitary operations in Table 2.

Table 2. Corresponding recovery unitary operation (RUO) and the collapsed state (CS) depending on Alice's measurement outcome (AMO).

AMO	CS	RUO
$\|\xi_1\rangle$	$g(t_0,t_1,t_2,t_3,t_4,t_5,t_6,t_7)_{346}$	$I_3I_4I_6$
$\|\xi_2\rangle$	$g(-t_1,t_0,t_3,-t_2,t_5,-t_4,-t_7,t_6)_{346}$	$Z_3Z_4Y_6$
$\|\xi_3\rangle$	$g(-t_4,-t_5,t_6,t_7,t_0,t_1,-t_2,-t_3)_{346}$	$Y_3Z_4I_6$
$\|\xi_4\rangle$	$g(t_3,-t_2,t_1,-t_0,t_7,-t_6,t_5,-t_4)_{346}$	$I_3X_4Y_6$
$\|\xi_5\rangle$	$g(-t_2,-t_3,t_0,t_1,-t_6,-t_7,t_4,t_5)_{346}$	$I_3Y_4I_6$
$\|\xi_6\rangle$	$g(-t_5,t_4,t_7,-t_6,-t_1,t_0,t_3,-t_2)_{346}$	$X_3Z_4Y_6$
$\|\xi_7\rangle$	$g(-t_6,t_7,-t_4,t_5,t_2,-t_3,t_0,-t_1)_{346}$	$Y_3X_4Z_6$
$\|\xi_8\rangle$	$g(-t_7,-t_6,-t_5,-t_4,t_3,t_2,t_1,t_0)_{346}$	$Y_3X_4X_6$

3.2. Complex coefficient state

Assume the prior shared entangled resources are the χ state and a three-particle GHZ state $|\mathrm{GHZ}\rangle$, which can be expressed as

$$\begin{aligned} &|\chi\rangle_{1234}|GHZ\rangle_{567} \\ &= \frac{1}{2\sqrt{2}}(|0000\rangle|\varphi^-\rangle|0\rangle + |0011\rangle|\varphi^-\rangle|1\rangle - |0100\rangle|\psi^-\rangle|0\rangle \\ &\quad - |0111\rangle|\psi^-\rangle|1\rangle + |1000\rangle|\psi^+\rangle|0\rangle + |1011\rangle|\psi^+\rangle|1\rangle \\ &\quad + |1100\rangle|\varphi^+\rangle|0\rangle + |1111\rangle|\varphi^+\rangle|1\rangle)_{1256347}. \end{aligned} \tag{29}$$

The particles (1,2,5,7) are held by Alice, and the particles (3,4,6) by Bob.

The process of the protocol is described as follows.

S1 Give a novel measurement basis.

Firstly, construct an 8×16 matrix

$$\boldsymbol{R} = (\boldsymbol{G}\ \ \boldsymbol{G}^*), \tag{30}$$

where $\boldsymbol{G}$ is defined by Eq. (25).

Then choose an 8×16 matrix $\boldsymbol{S}$ dependent on $s_j, j = 1,\ldots,8$ such that $\begin{pmatrix} \boldsymbol{R} \\ \boldsymbol{S} \end{pmatrix}$ is a 16×16 unitary matrix. In fact, such a matrix $\boldsymbol{S}$ can always be found since

$$\boldsymbol{G}\boldsymbol{G}^\dagger + \boldsymbol{G}^*\boldsymbol{G}^{\mathrm{T}} = \boldsymbol{E}_8. \tag{31}$$

Here we present an explicit expression of the matrix $\boldsymbol{S}$. Notice that

$$\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}} = \eta\boldsymbol{E}_8, \tag{32}$$

with $\eta = \frac{1}{2}\sum_{j=0}^{7} t_j^2$. The matrix $\boldsymbol{S}$ can be taken as

$$\boldsymbol{S} = \left(\boldsymbol{G}^*, -\frac{\eta^*}{\eta}\boldsymbol{G}\right), \tag{33}$$

where $-\eta^*/\eta$ is defined by 1 in the situation $\eta = 0$.

Finally, construct the complete orthogonal basis $\{|\xi_1\rangle,\ldots,|\xi_{16}\rangle\}$, which is related to the computation basis $\{|0000\rangle,\ldots,|1111\rangle\}$ by the following equation:

$$\begin{pmatrix} |\xi_1\rangle \\ |\xi_2\rangle \\ |\xi_3\rangle \\ |\xi_4\rangle \\ |\xi_5\rangle \\ |\xi_6\rangle \\ |\xi_7\rangle \\ |\xi_8\rangle \\ |\xi_9\rangle \\ |\xi_{10}\rangle \\ |\xi_{11}\rangle \\ |\xi_{12}\rangle \\ |\xi_{13}\rangle \\ |\xi_{14}\rangle \\ |\xi_{15}\rangle \\ |\xi_{16}\rangle \end{pmatrix} = \begin{pmatrix} \boldsymbol{R} \\ \boldsymbol{S} \end{pmatrix}^* \begin{pmatrix} |0000\rangle \\ |0011\rangle \\ |0100\rangle \\ |0111\rangle \\ |1000\rangle \\ |1011\rangle \\ |1100\rangle \\ |1111\rangle \\ |0001\rangle \\ |0010\rangle \\ |0101\rangle \\ |0110\rangle \\ |1001\rangle \\ |1010\rangle \\ |1101\rangle \\ |1110\rangle \end{pmatrix}. \tag{34}$$

S2 Alice performs a four-particle projective measurement on her particles $(1,2,5,7)$ under the measurement basis $|\xi_1\rangle,\ldots,|\xi_{16}\rangle$ and tells Bob her measurement result via the classical communication.

S3 Bob recovers the prepared three-qubit state on his qubits $(3,4,6)$ by performing appropriate unitary operation.

In the form of the measurement basis, the joint system $|\Psi\rangle_{1256347}$ in Eq. (29) can be expanded as

$$\begin{aligned}
&|\Psi\rangle_{1257346}\\
=\ &\frac{1}{4}[|\xi_1\rangle_{1257}g(t_0,t_1,t_2,t_3,t_4,t_5,t_6,t_7)_{346}\\
&+|\xi_2\rangle_{1257}g(-t_1,t_0,t_3,-t_2,t_5,-t_4,-t_7,t_6)_{346}\\
&+|\xi_3\rangle_{1257}g(-t_4,-t_5,t_6,t_7,t_0,t_1,-t_2,-t_3)_{346}\\
&+|\xi_4\rangle_{1257}g(t_3,-t_2,t_1,-t_0,t_7,-t_6,t_5,-t_4)_{346}\\
&+|\xi_5\rangle_{1257}g(-t_2,-t_3,t_0,t_1,-t_6,-t_7,t_4,t_5)_{346}\\
&+|\xi_6\rangle_{1257}g(-t_5,t_4,t_7,-t_6,-t_1,t_0,t_3,-t_2)_{346}\\
&+|\xi_7\rangle_{1257}g(-t_6,t_7,-t_4,t_5,t_2,-t_3,t_0,-t_1)_{346}\\
&+|\xi_8\rangle_{1257}g(-t_7,-t_6,-t_5,-t_4,t_3,t_2,t_1,t_0)_{346}\\
&+|\xi_9\rangle_{1257}g(t_0^*,t_1^*,t_2^*,t_3^*,t_4^*,t_5^*,t_6^*,t_7^*)_{346}\\
&+|\xi_{10}\rangle_{1257}g(-t_1^*,t_0^*,t_3^*,-t_2^*,t_5^*,-t_4^*,-t_7^*,t_6^*)_{346}\\
&+|\xi_{11}\rangle_{1257}g(-t_4^*,-t_5^*,t_6^*,t_7^*,t_0^*,t_1^*,-t_2^*,-t_3^*)_{346}\\
&+|\xi_{12}\rangle_{1257}g(t_3^*,-t_2^*,t_1^*,-t_0^*,t_7^*,-t_6^*,t_5^*,-t_4^*)_{346}\\
&+|\xi_{13}\rangle_{1257}g(-t_2^*,-t_3^*,t_0^*,t_1^*,-t_6^*,-t_7^*,t_4^*,t_5^*)_{346}\\
&+|\xi_{14}\rangle_{1257}g(-t_5^*,t_4^*,t_7^*,-t_6^*,-t_1^*,t_0^*,t_3^*,-t_2^*)_{346}\\
&+|\xi_{15}\rangle_{1257}g(-t_6^*,t_7^*,-t_4^*,t_5^*,t_2^*,-t_3^*,t_0^*,-t_1^*)_{346}\\
&+|\xi_{16}\rangle_{1257}g(-t_7^*,-t_6^*,-t_5^*,-t_4^*,t_3^*,t_2^*,t_1^*,t_0^*)_{346}]. \quad (35)
\end{aligned}$$

Due to lack of values $t_j, j=0,\ldots,7$, Bob can recover the prepared state only when Alice's measurement outcome lies in $\{|\xi_1\rangle,\ldots,|\xi_8\rangle\}$. The recovery transformation is summarized in Table 2. Therefore, the total success probability is $8\times 1/16 = 50\%$ in general.

However, as for the measurement outcome $|\xi_9\rangle,\ldots,|\xi_{16}\rangle$, the collapsed state can also be transformed into the prepared state if these coefficients take special values.

Theorem 2 The arbitrary three-qubit state can be prepared deterministically under the proposed scheme if and only if the coefficients $t_j, j=0,\ldots,7$ satisfy

$$\begin{aligned}
&(t_0^*,\cdots,t_7^*)\\
=\ &\mathrm{e}^{\mathrm{i}\tau_p}p(\pm t_0,\pm t_1,\pm t_2,\pm t_3,\pm t_4,\pm t_5,\pm t_6,\pm t_7), \quad (36)
\end{aligned}$$

where τ_p is a real constant which depends on the permutation $p\in S_8$ with S_8 being the permutation group on eight letters.[42]

From the above equation and the condition $\sum_{j=0}^{7}|t_j|^2=1$, the special ensembles of coefficients can be calculated. An example is given below to illustrate that for the special values, Bob can convert each collapsed state to the prepared state even if Alice's measurement outcome lies in $\{|\xi_9\rangle,\ldots,|\xi_{16}\rangle\}$. Suppose Alice gets the measurement outcome $|\xi_9\rangle_{1257}$. As a consequence, Bob's qubits collapse into $|g(t_0^*,\ldots,t_7^*)\rangle_{346} = |g(p(\pm t_0,\pm t_1,\pm t_2,\pm t_3,\pm t_4,\pm t_5,\pm t_6,\pm t_7))\rangle_{346}$ up to a global phase $\mathrm{e}^{\mathrm{i}\tau_p}$. Thus, Bob can perform appropriate unitary operations to obtain the prepared state.

4. Conclusions

Although χ state has many useful applications in quantum information, it is seldom exploited for the remote state preparation. To investigate its application, two RSP schemes for an arbitrary two-qubit state with real and complex coefficients are proposed with χ state as the entangled resource. The success probability is 100% and 50%, respectively. Moreover, some special complex ensembles that can be deterministically prepared are investigated by the permutation group. Similar discussion is made for preparing an arbitrary three-qubit state.

Luo *et al.*[29] have proved that the perfect RSP of a state with real coefficients is equivalent to the Hurwitz theorem from some correspondence. However, for general complex coefficients one cannot use the four element and octonion element matrices directly because they are limited to construct the real orthogonal matrix but not the unitary matrix. In this case, with the aid of the Hurwitz matrix equation we construct the useful measurement basis. Generally speaking, previous RSP schemes for four and eight complex coefficient states that the success probabilities are not less than 50% can be separated into two categories. One succeeds with 50% probability. In this type of RSP, the measurement bases must satisfy several conditions on the coefficients of the prepared state. However, the novel measurement bases in the proposed schemes have no restriction on the coefficients, which means that our schemes are more applicable. The other is deterministic. In this type of RSP, the sender needs to use the information splitting and divide the complex coefficients $p+q_i(=r\mathrm{e}^{\mathrm{i}\theta})$ of the original state into r and θ before constructing the measurement basis. Also, before performing the second-step projective measurement, the sender or the receiver needs to perform an additional unitary operation conditioned on the first-step measurement outcome. Compared with the deterministic RSP proposals for arbitrary two- and three-qubit states,[21–26] the proposed schemes have the following features. Despite of one half of success probability, the measurement is one-step completed. Moreover, the sender needs not to perform the information splitting. From this point of view, our schemes can be looked as a desired transmission protocol.

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