Derivation of quantum Chernoff metric with perturbation expansion method*

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We investigate a measure of distinguishability defined by the quantum Chernoff bound, which naturally induces the quantum Chernoff metric over a manifold of quantum states. Based on a quantum statistical model, we alternatively derive this metric by means of perturbation expansion. Moreover, we show that the quantum Chernoff metric coincides with the infinitesimal form of the quantum Hellinger distance, and reduces to the variant version of the quantum Fisher information for the single-parameter case. We also give the exact form of the quantum Chernoff metric for a qubit system containing a single parameter.

Keywords: quantum Chernoff metric, Hellinger distance, perturbation expansion

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1. Introduction

Hypothesis testing is a conventional tool in information theory and classical statistics,[1–4] which refers to discriminate two different probability distributions based on the data of observation. To quantify the discrimination performance, the average error probability is defined, and its minimum bound was first given by Chernoff in terms of an asymptotic limit of many copies.[5] This bound is called the Chernoff bound and is inherently identified as a measure of distinguishability between two probability distributions.[6,7]

Moving to the quantum settings, the hypotheses of interest are the density metrics representing the states of a quantum system. Suppose that a quantum system may be prepared in one of two different states. To discriminate those two states, one has to perform a measurement on the quantum system, and then according to the outcomes of the measurement, one makes a conclusive guess of which state the system should be in. The quantum analog of the error probability is directly generalized from the classical one by choosing the optimal measurement, while the corresponding minimal error probability was not solved until recently. Nussbaum and Szkola first settled this problem, and they gave the quantum Chernoff bound (QCB) for the symmetric hypothesis testing in the asymptotic limit.[8] Subsequently, Audenaert *et al.* gave a complete proof of the QCB for the asymptotic case.[9] Now there is growing interest in the study and the practical application concerning the QCB, such as the discrimination of quantum states,[10–14] quantum operations,[15,16] quantum reading,[17–19] quantum illumination,[20,21] and quantum phase transition.[22–24]

The QCB naturally leads to a distance function over the manifold of quantum states[25] due to possessing a well-defined operational meaning. Additionally, it has been proven that, for two infinitesimally close states, the QCB may reduce to the Riemannian metric over the space of quantum states, which is known as the quantum Chernoff metric (QCM).[9,25] Audenaert *et al.* derived this metric by using the matrix algebra approach.[9]

In this paper, we alternatively derive the QCM based on the theory of perturbation expansion, which has been applied in the derivation of the Bures metric.[26–28] Via a quantum statistical model, the QCM is derived by an expansion of the density matrix up to the second order. We also find that the QCM coincides with the infinitesimal form of the quantum Hellinger distance introduced by Luo.[29] For the single-parameter case, the QCM reduces to the variant version of the quantum Fisher information (QFI) in the theory of statistical estimation.[30–33] In particular, we obtain the exact form of the QCM for a two-dimensional system containing only a single parameter.

The paper is organized as follows. In Section 2, we briefly review the definition of the probability error and the Chernoff bound. In Section 3, we give the detailed derivation of the QCM. In Section 4, the relation between the QCM and the quantum Hellinger metric is discussed. The exact form of the QCM for the qubit states is also given. Finally, conclusions are given in Section 5.

2. Hypothesis testing and error probability

We start by considering a quantum system that may be prepared in one of two states represented by ρ_0 and ρ_1. Here state ρ_0 is the null hypothesis and ρ_1 is the alternative hypothesis. Generally, the two hypotheses are treated asymmetrically.

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We supply the prior probabilities π_0 and π_1, which are positive quantities satisfying $\pi_0+\pi_1=1$. Now, the main task is to decide which one is the true state of the quantum system by performing a measurement. The general decision-making procedure is expressed as follows. We choose a generalized measurement which is described by a set of positive operator-valued measures (POVM) $\{\hat{E}_x\}$ associated with $\sum_{x\in\chi}\hat{E}_x=I$. Here, the quantity x denotes the result of the measurement and χ is the set of all possible results. After performing the POVM on the quantum system, the probability distribution of the outcome is

$$p_i(x)=\mathrm{Tr}\left(\rho_i\hat{E}_x\right),\quad i=0,1. \tag{1}$$

Then the probability of making an error decision is given by[7]

$$\begin{aligned}P_{\mathrm{e}}(p_0,p_1) &= \frac{1}{2}\sum_{x\in\chi}\min\{\pi_0p_0(x),\pi_1p_1(x)\}\\ &= \frac{1}{2}-\frac{1}{2}\sum_{x\in\chi}|\pi_1p_1(x)-\pi_0p_0(x)|,\end{aligned} \tag{2}$$

which is known as the probability of error between two probability distributions. By using Eq. (1), equation (2) can be further represented as

$$P_{\mathrm{e}}(p_0,p_1)=\frac{1}{2}-\frac{1}{2}\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right|, \tag{3}$$

where $\Gamma\equiv\pi_1\rho_1-\pi_0\rho_0$ is the Helstrom matrix.[1–3]

The probability of error between two density matrices is defined as

$$P_{\mathrm{e}}(\rho_0,\rho_1)=\frac{1}{2}-\frac{1}{2}\max_{\{\hat{E}_x\}}\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right|, \tag{4}$$

where the maximum is over all POVM's. Finding the optimal value of $P_{\mathrm{e}}(p_0,p_1)$ over all the measurements is equivalent to finding the maximal value of the term $\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right|$ in Eq. (3). Since Γ is a Hamiltonian operator, we assume that a spectrum decomposition of Γ is $\sum_i\gamma_i|\gamma_i\rangle\langle\gamma_i|$ with γ_i denoting the eigenvalues. Thus, we have

$$\begin{aligned}\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right| &= \sum_{x\in\chi}\left|\sum_i\gamma_i\langle\gamma_i|\hat{E}_x|\gamma_i\rangle\right|\\ &\le \sum_{x\in\chi}\sum_i\left|\gamma_i\right|\langle\gamma_i|\hat{E}_x|\gamma_i\rangle\\ &= \sum_i\left|\gamma_i\right|\langle\gamma_i|\sum_{x\in\chi}\hat{E}_x|\gamma_i\rangle\\ &= \sum_i\left|\gamma_i\right|\\ &\equiv \mathrm{Tr}\left|\Gamma\right|,\end{aligned} \tag{5}$$

where the inequality in the second line follows from the Cauchy–Schwarz inequality, and $\mathrm{Tr}|\cdot|$ in the last line denotes the trace operation of a matrix which is diagonal in the same basis with eigenvalues that are the absolute values. From the above derivation, we can see that the quantity $\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right|$ is bounded by $\mathrm{Tr}|\Gamma|$, i.e.,

$$\sum_{x\in\chi}\left|\mathrm{Tr}\left(\Gamma\hat{E}_x\right)\right|\le\mathrm{Tr}\left|\Gamma\right|,\ \ \forall\ \ \{\hat{E}_x\}. \tag{6}$$

This upper bound can be achieved by some POVM $\{\hat{E}_x\}$, such as the projectors onto an orthonormal eigenbasis of Γ.[1–3,7] By combining Eqs. (4) and (6), the probability of error for two density matrices can be expressed as

$$P_{\mathrm{e}}(\rho_0,\rho_1)=\frac{1}{2}-\frac{1}{2}\mathrm{Tr}\left|\pi_1\rho_1-\pi_0\rho_0\right|. \tag{7}$$

When we consider a large number of identical copies of $\rho_0^{\otimes N}$ and $\rho_1^{\otimes N}$, the above equation becomes

$$P_{\mathrm{e}}(\rho_0,\rho_1)=\frac{1}{2}-\frac{1}{2}\mathrm{Tr}\left|\pi_1\rho_1^{\otimes N}-\pi_0\rho_0^{\otimes N}\right|. \tag{8}$$

According to the Bayes' decision rule, the optimal probability of error in such a decision problem must decrease towards zero in the asymptotic limit $N\to\infty$. Thus after a finite sampling, the minimal error probability turns out to have the following form:

$$P_{\mathrm{e,min}}\sim\exp\left(-N\xi_{\mathrm{QCB}}\right). \tag{9}$$

The particular value ξ_{QCB} of this exponential is defined as the quantum Chernoff bound. Before deriving the QCB, let us introduce a crucial inequality

$$\mathrm{Tr}\left(A^sB^{1-s}\right)\ge\frac{1}{2}\mathrm{Tr}\left(A+B-|A-B|\right) \tag{10}$$

for all $0\le s\le1$, where A and B are positive operators. The proof of this inequality can be found in Ref. [9]. Let $A=\pi_1\rho_1^{\otimes N}$ and $B=\pi_0\rho^{\otimes N}$, then the upper bound of the error exponent in the asymptotic limit is obtained as[8,9]

$$\xi_{\mathrm{QCB}}=\lim_{n\to\infty}-\frac{\log P_{\mathrm{e,min}}}{n}=-\log\min_{0\le s\le1}\mathrm{Tr}\left(\rho_0^{1-s}\rho_1^s\right). \tag{11}$$

It is also referred to as a measure of the distinguishability between two quantum states.[25] When ρ_0 and ρ_1 commute, the QCB of Eq. (11) reduces to the classical analog for two classical distributions p_0 and p_2

$$\xi_{\mathrm{CB}}=-\log\min_{0\le s\le1}\mathrm{Tr}\left(p_0^{1-s}p_1^s\right), \tag{12}$$

which provides an upper bound to the probability of error in hypothesis testing. Correspondingly, it can be formulated as a measure of the statistical distinguishability in the statistical inference problem.[1–4]

3. Quantum Chernoff metric

Below, we only consider the non-logarithmic variety of the QCB of Eq. (11), which is denoted as

$$\xi_{\mathrm{QCB}}=\min_{0\le s\le1}\mathrm{Tr}\left(\rho_0^{1-s}\rho_1^s\right). \tag{13}$$

Note that two identical states have $\xi_{\rm QCB}=1$, and two orthogonal states have $\xi_{\rm QCB}=0$. Thus, a well-defined operational distance may be defined as

$$D_{\rm QCB}(\rho_0,\rho_1) = 1-\xi_{\rm QCB}, \tag{14}$$

which shows that an increasing distance between two quantum states implies the decrease of the asymptotic error probability. For two infinitesimally close quantum states ρ and $\rho+{\rm d}\rho$, the distance above based on the QCB may induce a Riemannian metric over the quantum state space. Audenaert *et al.* derived this metric by applying the matrix algebra.[9] Here, we provide an alternative way to derive the quantum Chernoff metric by using the method of perturbation expansion.

We start by considering a quantum statistical model. A family of quantum states $\rho(\boldsymbol{\lambda})$ are defined on a d-dimensional Hilbert space $\mathscr{H}$, and a state is labeled by a parameter vector $\boldsymbol{\lambda}$ being the coordinate system of an N-dimensional statistical manifold $\mathscr{M}$, with the mapping $\boldsymbol{\lambda}\mapsto\rho(\boldsymbol{\lambda})$.[34,35] Here, we assume that $\rho(\boldsymbol{\lambda})$ is a non-singular density matrix and set $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,...,\lambda_N)^{\rm T}$. Consider two neighbour quantum states $\rho_0=\rho(\boldsymbol{\lambda})$ and $\rho_1=\rho(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})$, where $\delta\boldsymbol{\lambda}$ is a small displacement of parameter $\boldsymbol{\lambda}$. Then, equation (13) can be rewritten as

$$\xi_{\rm QCB}=\min_{0\le s\le 1}{\rm Tr}\left[\rho^{1-s}(\boldsymbol{\lambda})\rho^{s}(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})\right]. \tag{15}$$

By performing a series expansion, we can expand $\rho^s(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})$ at $\boldsymbol{\lambda}$ up to the second order as

$$\rho^s(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})\simeq\rho^s+\frac{\partial\rho^s}{\partial\lambda_i}\delta\lambda_i+\frac{1}{2}\left(\frac{\partial^2\rho^s}{\partial\lambda_i\partial\lambda_j}\right)\delta\lambda_i\delta\lambda_j. \tag{16}$$

Here and after, the Einstein summation convention is used. Substituting the above equation into Eq. (15) yields

$$\begin{aligned}&\xi_{\rm QCB}\\ &=\min_{0\le s\le 1}{\rm Tr}\left[\rho+\rho^{1-s}\frac{\partial\rho^s}{\partial\lambda_i}\delta\lambda_i+\frac{1}{2}\rho^{1-s}\left(\frac{\partial^2\rho^s}{\partial\lambda_i\partial\lambda_j}\right)\delta\lambda_i\delta\lambda_j\right]\\ &=1+\frac{1}{2}\min_{0\le s\le 1}\left[{\rm Tr}\left(\rho^{1-s}\frac{\partial^2\rho^s}{\partial\lambda_i\partial\lambda_j}\right)\delta\lambda_i\delta\lambda_j\right],\end{aligned} \tag{17}$$

where the last equality results from the fact

$${\rm Tr}\left(\rho^{1-s}\frac{\partial\rho^s}{\partial\lambda_i}\right)=0, \tag{18}$$

which is proved as follows.

Assume that the spectrum decomposition of $\rho(\boldsymbol{\lambda})$ is given by

$$\rho(\boldsymbol{\lambda})=\sum_n p_n|n\rangle\langle n|, \tag{19}$$

where p_n and $|n\rangle$ stand for the n-th eigenvalue and eigenstate of ρ, respectively, satisfying $0\le p_n\le 1$ and $\sum_n p_n=1$. Note that both the eigenvalues and the eigenstates depend on the parameter vector $\boldsymbol{\lambda}$. By substituting Eq. (19) into Eq. (18), we obtain the following chain of equalities:

$$\begin{aligned}{\rm Tr}\left(\rho^{1-s}\frac{\partial\rho^s}{\partial\lambda_i}\right)&=\sum_n p_n^{1-s}\left[\frac{\partial p_n^s}{\partial\lambda_i}+p_n^s\left(\langle\partial_{\lambda_i}n|n\rangle+\langle n|\partial_{\lambda_i}n\rangle\right)\right]\\ &=s\frac{\partial{\rm Tr}(\rho)}{\partial\lambda_i}\\ &=0,\end{aligned} \tag{20}$$

where the first equality is given by the identity $\langle\partial_{\lambda_i}n|n\rangle+\langle n|\partial_{\lambda_i}n\rangle=0$, resulting from the normalization condition $\langle n|n\rangle=1$, and the second equality is obtained by considering the existence of a factor p_n^{s-1} with $s-1\le 0$ for a non-singular density matrix. Making a partial derivation on both sides of Eq. (18) with respect to λ_j, we obtain

$${\rm Tr}\left(\rho^{1-s}\frac{\partial\rho^s}{\partial\lambda_i\partial\lambda_j}\right)=-{\rm Tr}\left(\frac{\partial\rho^{1-s}}{\partial\lambda_j}\frac{\partial\rho^s}{\partial\lambda_i}\right). \tag{21}$$

By combining Eqs. (17) and (21), $\xi_{\rm QCB}$ can be represented as

$$\begin{aligned}\xi_{\rm QCB}&=1+\frac{1}{2}\min_{0\le s\le 1}\delta\lambda_i{\rm Tr}\left(\rho^{1-s}\frac{\partial^2\rho^s}{\partial\lambda_i\partial\lambda_j}\right)\delta\lambda_j\\ &=1-\frac{1}{2}\max_{0\le s\le 1}\delta\lambda_i{\rm Tr}\left(\frac{\partial\rho^{1-s}}{\partial\lambda_i}\frac{\partial\rho^s}{\partial\lambda_j}\right)\delta\lambda_j\\ &=1-\frac{1}{2}\max_{0\le s\le 1}\delta\boldsymbol{\lambda}^{\rm T}\cdot\boldsymbol{M}\cdot\delta\boldsymbol{\lambda},\end{aligned} \tag{22}$$

where $\boldsymbol{M}$ is an $N\times N$ matrix with elements $M_{ij}={\rm Tr}\left(\frac{\partial\rho^{1-s}}{\partial\lambda_i}\frac{\partial\rho^s}{\partial\lambda_j}\right)$. It is desirable to find the maximum of $\delta\boldsymbol{\lambda}^{\rm T}\cdot\boldsymbol{M}\cdot\delta\boldsymbol{\lambda}$. By using the simplified symbol $\partial_i\equiv\partial/\partial\lambda_i$, we have

$$\begin{aligned}&\max_{0\le s\le 1}\delta\boldsymbol{\lambda}^{\rm T}\cdot M\cdot\delta\boldsymbol{\lambda}\\ &=\max_{0\le s\le 1}\delta\lambda_i\delta\lambda_j\sum_{nm}\langle n|\partial_i\rho^{1-s}|m\rangle\langle m|\partial_j\rho^s|n\rangle\\ &=\max_{0\le s\le 1}\delta\lambda_i\delta\lambda_j\Big[\sum_n\frac{\partial p_n^{1-s}}{\partial\lambda_i}\frac{\partial p_n^s}{\partial\lambda_j}+\\ &\quad\sum_{m\ne n}\left(p_n^{1-s}-p_m^{1-s}\right)\left(p_n^s-p_m^s\right)\langle n|\partial_i m\rangle\langle\partial_j m|n\rangle\Big]\\ &=\max_{0\le s\le 1}\Big\{s(1-s)\sum_n p_n^{-1}\left(\sum_i\delta\lambda_i\frac{\partial p_n}{\partial\lambda_i}\right)^2+\\ &\quad\sum_{m\ne n}\left(p_n^{1-s}-p_m^{1-s}\right)\left(p_n^s-p_m^s\right)\Big|\sum_i\delta\lambda_i\langle n|\partial_i m\rangle\Big|^2\Big\}.\end{aligned} \tag{23}$$

Evidently, the first term in the last equality of Eq. (23) takes the maximum at $s=1/2$, and the second term obeys the following inequality:

$$\begin{aligned}&\sum_{m\ne n}\left(p_n+p_m-p_n^{1-s}p_m^s-p_n^sp_m^{1-s}\right)\Big|\sum_i\delta\lambda_i\langle n|\partial_{\lambda_i}m\rangle\Big|^2\\ &\le\sum_{m\ne n}\left(p_n+p_m-2\sqrt{p_np_m}\right)\Big|\sum_i\delta\lambda_i\langle n|\partial_{\lambda_i}m\rangle\Big|^2\\ &=\sum_{m\ne n}\left(\sqrt{p_n}-\sqrt{p_m}\right)^2\Big|\sum_i\delta\lambda_i\langle n|\partial_{\lambda_i}m\rangle\Big|^2,\end{aligned} \tag{24}$$

where the inequality in the second line follows from the fact

$$p_n^{1-s}p_m^s + p_n^s p_m^{1-s} = \sqrt{p_n p_m}\left(p_n^{1/2-s}p_m^{s-1/2} + p_n^{s-1/2}p_m^{1/2-s}\right) \geq 2\sqrt{p_n p_m}, \quad (25)$$

and the equal sign holds when $s = 1/2$. Therefore, both terms reach the maxima at $s = 1/2$, which leads to the term $\delta\boldsymbol{\lambda}^{\mathrm{T}} \cdot \boldsymbol{M} \cdot \delta\boldsymbol{\lambda}$ taking the maximum at the same point.

According to the above results, the quantum Chernoff distance of Eq. (14) for two neighboring quantum states reads

$$\mathrm{d}s^2_{\mathrm{QCB}} = 1 - \mathrm{Tr}\left[\sqrt{\rho(\boldsymbol{\lambda})}\sqrt{\rho(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})}\right] = \sum_{ij}\mathscr{M}_{ij}\delta\lambda_i\delta\lambda_j, \quad (26)$$

where we have set $\mathscr{M}_{ij} = M_{ij}/2$ and

$$M_{ij} = \sum_n \frac{1}{4p_n}\frac{\partial p_n}{\partial\lambda_i}\frac{\partial p_n}{\partial\lambda_j} + \sum_{m\neq n}(\sqrt{p_n}-\sqrt{p_m})^2 \times \langle n|\partial_{\lambda_i}m\rangle\langle\partial_{\lambda_j}m|n\rangle. \quad (27)$$

The quantity $\mathscr{M}_{ij}$ in Eq. (26) is the so-called quantum Chernoff metric, which is first given in Ref. [9]. Equation (27) divides the QCM into two terms. The first term is indeed related to the Fisher metric, which is the unique monotone metric on the statistical manifold[36] and the second term is contributed by the noncommutativity of $\rho(\boldsymbol{\lambda})$ and $\rho(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})$.

Up to now, we have derived the QCM by directly applying the perturbation expansion method. It is well known that the QCM, which belongs to the family of monotone metrics proposed by Petz,[36] describes a geometrical structure of the quantum state space. Its properties are discussed in detail in Refs. [9] and [25]. Besides, the QCM can be operationally defined as a measure of the statistical distinguishability on the space of the density operators in the quantum information geometry.

Some remarks are presented as follows. Our derivation here is based on the quantum statistical model in which an infinitesimal variation of the parameters produces a small change in the state of the quantum system. Based on this model, Braunstein and Caves mapped the problem of distinguishing two infinitesimally close states to that of estimating the parameter optimally.[35] They found that the Bures metric used as a measure of the distinguishability is simply proportional to the QFI defined by the symmetric logarithmic derivative operator which provides the optimal accuracy of the unbiased estimation (the Cramér–Rao bound).[34,35] Therefore, a question arises as to whether the QCM can be used to describe the precision of estimation by following the above idea? This question has attracted much attention. For instance, Tsang proposed a quantum version of the Ziv–Zakai bound which is defined by the minimal error probability.[37]

Moreover, we may assume that the quantum statistical model here can be physically realized as a set of quantum states associated with a family of Hamiltonians smoothly depending on the set of parameters $\boldsymbol{\lambda}$, e.g., coupling constants. Zanardi *et al.* established a universal differential-geometric framework to study the quantum critical phenomena featured by some quantum systems.[28] They demonstrated that the QCM is a well-defined indicator of quantum phase transition, as it presents a singularity at the critical point.[22–24]

4. Quantum Chernoff metric, quantum Hellinger distance, and quantum Fisher information

In this section, we discuss the relation between the QCM and the quantum Hellinger distance, as well as the quantum Fisher information. We begin with introducing the classical Hellinger distance, which is frequently used to describe the distance between two probability distributions P and Q,

$$H(P,Q) = \sum_{i=1}^n(\sqrt{p_i}-\sqrt{q_i})^2 = 2 - 2\sum_{i=1}^n\sqrt{p_i q_i}, \quad (28)$$

where $p_i \in P$ and $q_i \in Q$. Its quantum version is defined by directly replacing the probabilities with the density matrices, and the summation with a trace[29]

$$D_{\mathrm{H}}(\rho_0,\rho_1) = \mathrm{Tr}\left[(\sqrt{\rho_0}-\sqrt{\rho_1})^2\right] = 2\left[1 - A(\rho_0,\rho_1)\right], \quad (29)$$

where

$$A(\rho_0,\rho_1) = \mathrm{Tr}(\sqrt{\rho_0}\sqrt{\rho_1}) \quad (30)$$

is the so-called quantum affinity. On the manifold of quantum states labeled by the vector of parameters, $\boldsymbol{\lambda}$, the quantum Hellinger distance for two neighbor states coincides with Eq. (26) up to a factor of 2, i.e.,

$$\mathrm{d}s^2_{\mathrm{H}} = 2\,\mathrm{d}s^2_{\mathrm{QCB}}. \quad (31)$$

It indicates that the QCM is equivalent to the infinitesimal form of the quantum Hellinger distance. In addition, it has been clarified that the QCM is equivalent to the Wigner–Yanase metric.[25,29]

When the quantum states only depend on a single parameter λ, the QCM of Eq. (27) is represented by

$$\mathscr{I}_\lambda = \sum_n\frac{(\partial_\lambda p_n)^2}{8p_n} + \frac{1}{2}\sum_{m\neq n}(\sqrt{p_n}-\sqrt{p_m})^2|\langle n|\partial_\lambda m\rangle|^2. \quad (32)$$

For a pure state $\psi(\lambda)$, the above expression simply reduces to the following form:

$$\mathscr{I}_\lambda = \langle\partial_\lambda\psi|\partial_\lambda\psi\rangle - |\langle\psi|\partial_\lambda\psi\rangle|^2. \quad (33)$$

For the sake of clarity, we have used the symbol $\mathscr{I}_\lambda$ denoting the QCM with respect to the single parameter λ. From Eqs. (32) and (33), we find that the quantity $\mathscr{I}_\lambda$ is the variant version of the QFI in the theory of statistical estimation,

which is actually the Wigner–Yanase skew information.[30] A more detailed discussion of $\mathscr{I}_\lambda$ is given in Refs. [31] and [38].

Below, we give the exact form of Eq. (32) for an arbitrary single-qubit state containing a single parameter. Assume that an arbitrary qubit density matrix $\rho(\lambda)$ labeled by a single parameter λ can be decomposed as

$$\rho(\lambda) = p_0|\psi_0\rangle\langle\psi_0| + p_1|\psi_1\rangle\langle\psi_1|. \tag{34}$$

Then equation (32) may be represented as

$$\begin{aligned}\mathscr{I}_\lambda &= \frac{1}{8p_0p_1}\left[p_1(\partial_\lambda p_0)^2 + p_0(\partial_\lambda p_1)^2\right] \\ &\quad + (\sqrt{p_0}-\sqrt{p_1})^2|\langle\psi_0|\partial_\lambda\psi_1\rangle|^2 \\ &= \frac{(\partial_\lambda p_0)^2}{8\det(\rho)} + \left(1-2\sqrt{\det(\rho)}\right)|\langle\psi_0|\partial_\lambda\psi_1\rangle|^2, \end{aligned}\tag{35}$$

where we have used the relations $p_0+p_1=1$, $\partial_\lambda p_0 = -\partial_\lambda p_1$, and $\det(\rho) = p_0p_1$.

Furthermore, by taking the derivative of both sides of Eq. (34) with respect of parameter λ, we arrive at

$$\begin{aligned}&\partial_\lambda\rho \\ &= \partial_\lambda p_0|\psi_0\rangle\langle\psi_0| + \partial_\lambda p_1|\psi_1\rangle\langle\psi_1| + p_0|\partial_\lambda\psi_0\rangle\langle\psi_0| \\ &\quad + p_1|\partial_\lambda\psi_1\rangle\langle\psi_1| + p_0|\psi_0\rangle\langle\partial_\lambda\psi_0| + p_1|\psi_1\rangle\langle\partial_\lambda\psi_1|. \end{aligned}\tag{36}$$

It is noted that $\partial_\lambda\rho$ is still Hermitian. From the above equation, we find

$$(\partial_\lambda\rho)_{00} = \langle\psi_0|\partial_\lambda\rho|\psi_0\rangle = \partial_\lambda p_0, \tag{37}$$

$$(\partial_\lambda\rho)_{11} = \langle\psi_1|\partial_\lambda\rho|\psi_1\rangle = \partial_\lambda p_1 = -\partial_\lambda p_0, \tag{38}$$

$$(\partial_\lambda\rho)_{01} = \langle\psi_0|\partial_\lambda\rho|\psi_1\rangle = (p_1-p_0)\langle\psi_0|\partial_\lambda\psi_1\rangle. \tag{39}$$

By submitting Eqs. (37)–(39) into Eq. (35), we obtain

$$\begin{aligned}\mathscr{I}_\lambda &= \frac{(\partial_\lambda p_0)^2}{8\det(\rho)} + \frac{(1-2\sqrt{\det(\rho)})}{1-4\det(\rho)}|(\partial_\lambda\rho)_{01}|^2 \\ &= \frac{(\partial_\lambda p_0)^2}{8\det(\rho)} + \frac{1}{1+2\sqrt{\det(\rho)}}|(\partial_\lambda\rho)_{01}|^2. \end{aligned}\tag{40}$$

One can see that the expression of Eq. (32) is extremely similar to that of the Bures metric.[38] Moreover, in Ref. [39], it was shown that the explicit formula of the Bures metric for a single-qubit system is determined by $X = \mathrm{Tr}[(\partial_\lambda\rho)^2]$ and $Y = \mathrm{Tr}[(1-\rho)(\partial_\lambda\rho)(1-\rho)(\partial_\lambda\rho)]/\det(\rho)$. Therefore, we infer that the QCM may also be determined by those two terms. We further express those two terms in the form

$$X = 2\left[(\partial_\lambda p_0)^2 + |(\partial_\lambda\rho)_{01}|^2\right], \tag{41}$$

$$Y = \left[\frac{1}{\det(\rho)} - 2\right](\partial_\lambda p_0)^2 + 2|(\partial_\lambda\rho)_{01}|^2, \tag{42}$$

with the help of Eqs. (37)–(39). By inversely solving Eqs. (41) and (42), we can obtain

$$(\partial_\lambda p_0)^2 = \frac{\det(\rho)(X-Y)}{4\det(\rho)-1}, \tag{43}$$

$$|(\partial_\lambda\rho)_{01}|^2 = \frac{X(2\det(\rho)-1)+2\det(\rho)Y}{2(4\det(\rho)-1)}. \tag{44}$$

Submitting Eqs. (43) and (44) into Eq. (40) finally yields

$$\begin{aligned}\mathscr{I}_\lambda &= \alpha(\rho)\mathrm{Tr}\left[(\partial_\lambda\rho)^2\right] \\ &\quad + \beta(\rho)\mathrm{Tr}[(1-\rho)(\partial_\lambda\rho)(1-\rho)(\partial_\lambda\rho)], \end{aligned}\tag{45}$$

where the coefficients are given by

$$\alpha(\rho) = \frac{8\det(\rho)+2\sqrt{\det(\rho)}-3}{8(4\det(\rho)-1)\left(1+2\sqrt{\det(\rho)}\right)}, \tag{46}$$

$$\beta(\rho) = \frac{8\det(\rho)-2\sqrt{\det(\rho)}-1}{8\det(\rho)(4\det(\rho)-1)\left(1+2\sqrt{\det(\rho)}\right)}. \tag{47}$$

This result greatly facilitates the computation of $\mathscr{I}_\lambda$ and has been used in our recent work[38] for geometrically discussing the dynamics of the QFI under decoherence. We obtain the formula of Eq. (45) by taking account of the formal similarity between $\mathscr{I}_\lambda$ and the Bures metric. Equation (45) is only valid for two-dimensional systems. For a d-dimensional ($d>2$) system, the explicit formula for $\mathscr{I}_\lambda$ is left for further study. Fortunately, the explicit formula of the Bures metric for finite-dimensional density matrices has already been given by Dittmann in Ref. [39]. This may help us to derive the exact form of $\mathscr{I}_\lambda$ for $d>2$ cases.

5. Conclusion

In this paper, we have investigated a measure of distinguishability defined by the quantum Chernoff bound, which allows us to quantify the asymptotic behavior of the error in the context of discriminating two different sources of quantum states. For two infinitesimally close states, this distinguishing measure produces a Riemannian metric over the space of quantum states. Using the quantum statistical model, we alternatively proved this metric by applying the method of perturbation expansion. Moreover, we found that the quantum Chernoff metric is equivalent to the infinitesimal form of the quantum Hellinger distance. When the quantum states only depend on a single parameter, the quantum Chernoff metric reduces to the variant version of the quantum Fisher information (i.e., the skew information) in the theory of statistical inference. Finally, we derived the exact form of the quantum Chernoff metric for a qubit state containing a single parameter. Our results may contribute to the understanding of the quantum Chernoff metric and the quantum parameter estimation.

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References

[1] Helstrom C W 1976 *Quantum Detection and Estimation Theory*
[2] Holevo A S 1982 *Probabilistic and Statistical Aspects of Quantum Theory*
[3] Malley J D and Hornstein J 1993 *Statistical Science* **8** 433
[4] Cover T M and Thomas J A 2006 *Elements of Information Theory*
[5] Chernoff H 1925 *Ann. Math. Stat.* **23** 493
[6] Fuchs C A 1996 arXiv:quant-ph 9601020
[7] Fuchs C A and van de Graaf J 1999 *IEEE Trans. Inf. Theory* **45** 1216
[8] Nussbaum M and Szkola A 2009 *Annals of Statistics* **37** 1040
[9] Audenaert K M R, Calsamiglia J, Muñoz-Tapia R, Bagan E, Masanes L, Acin A and Verstraete F 2007 *Phys. Rev. Lett.* **98** 160501
[10] Pirandola S and Lloyd S 2008 *Phys. Rev. A* **78** 012331
[11] Boca M, Ghiu I, Marian P and Marian T A 2009 *Phys. Rev. A* **79** 014302
[12] Calsamiglia J, de Vicente J I, Muñoz-Tapia R and Bagan E 2010 *Phys. Rev. Lett.* **105** 080504
[13] Sentís G, Bagan E, Calsamiglia J and Muñoz-Tapia R 2010 *Phys. Rev. A* **82** 042312
[14] Higgins B L, Doherty A C, Bartlett S D, Pryde G J and Wiseman H M 2011 *Phys. Rev. A* **83** 052314
[15] Duan R Y, Feng Y and Ying M S 2009 *Phys. Rev. Lett.* **103** 210501
[16] Invernizzi C, Paris M G A and Pirandola S 2011 *Phys. Rev. A* **84** 022334
[17] Pirandola S 2011 *Phys. Rev. Lett.* **106** 090504
[18] Spedalieri G, Lupo C, Mancini S, Braunstein S L and Pirandola S 2012 *Phys. Rev. A* **86** 012315
[19] Nair R 2011 *Phys. Rev. A* **84** 032312
[20] Tan S H, Erkmen B I, Giovannetti V, Guha S, Lloyd S, Maccone L, Pirandola S and Shapiro J H 2008 *Phys. Rev. Lett.* **101** 253601
[21] Shapiro J H 2009 *Phys. Rev. A* **80** 022320
[22] Zanardi P, Venuti L C and Giorda P 2007 *Phys. Rev. A* **76** 062318
[23] Abasto D F, Jacobson N T and Zanardi P 2008 *Phys. Rev. A* **77** 022327
[24] Invernizzi C and Paris M G A 2010 *J. Mod. Opt.* **57** 198
[25] Calsamiglia J, Muñoz-Tapia R, Masanes L, Acin A and Bagan E 2008 *Phys. Rev. A* **77** 032311
[26] Hübner M 1992 *Phys. Lett. A* **163** 239
[27] Sommers H J and Zyczkowski K 2003 *J. Phys. A: Math. Gen.* **36** 10083
[28] Zanardi P, Giorda P and Cozzini M 2007 *Phys. Rev. Lett.* **99** 100603
[29] Luo S L and Zhang Q 2004 *Phys. Rev. A* **69** 032106
[30] Wigner E P and Yanase M M 1963 *Proc. Natl. Acad. Sci. USA* **49** 910
[31] Luo S L 2003 *Phys. Rev. Lett.* **91** 180403
[32] Sun H G, Liu W F and Li C J 2011 *Chin. Phys. B* **20** 090301
[33] Sun H G, Zhang L H, Liu W F and Li C J 2012 *Chin. Phys. B* **21** 010301
[34] Paris M G A 2009 *Int. J. Quan. Info.* **7** 125
[35] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439
[36] Petz D 1996 *Linear Algebra Appl.* **244** 81
[37] Tsang M 2012 *Phys. Rev. Lett.* **108** 230401
[38] Zhong W, Sun Z, Ma J, Wang X G and Nori F 2013 *Phys. Rev. A* **87** 022337
[39] Dittmann J 1999 *J. Phys. A: Math. Theor.* **32** 2663