Exact solution of Dirac equation for Scarf potential with new tensor coupling potential for spin and pseudospin symmetries using Romanovski polynomials*

A. Suparmi[a)†], C. Cari[a)], and U. A. Deta[b)]

[a)] Physics Department, Faculty of Mathematics and Natural Science, Sebelas Maret University, Indonesia
[b)] Department of Physics, Faculty of Mathematics and Natural Science, State University of Surabaya, Indonesia

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The bound state solutions of Dirac equations for a trigonometric Scarf potential with a new tensor potential under spin and pseudospin symmetry limits are investigated using Romanovski polynomials. The proposed new tensor potential is inspired by superpotential form in supersymmetric (SUSY) quantum mechanics. The Dirac equations with trigonometric Scarf potential coupled by a new tensor potential for the pseudospin and spin symmetries reduce to Schrödinger-type equations with a shape invariant potential since the proposed new tensor potential is similar to the superpotential of trigonometric Scarf potential. The relativistic wave functions are exactly obtained in terms of Romanovski polynomials and the relativistic energy equations are also exactly obtained in the approximation scheme of centrifugal term. The new tensor potential removes the degeneracies both for pseudospin and spin symmetries.

Keywords: Scarf potential, new tensor potential, spin and pseudospin symmetries, Romanovski polynomials

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1. Introduction

The exact analytical solutions of Dirac equations play important roles in relativistic quantum mechanics since they provide all important information of the system investigated. The Dirac equations for central potentials coupled with Yukawa-type and Coulomb-type tensor potentials and its modifications have been solved exactly and have been applied in quantum chemistry and high energy physics. To describe the motion of spin half particles, Dirac equations which have exact solutions under the approximation scheme of the centrifugal term for various potentials with tensor potentials have been explored.[1–10] From the observation, the expressions of the tensor coupling potentials under the approximation scheme of centrifugal term are similar to the expressions of the components of the given potentials.

A new tensor potential is proposed due to the inspiration of the algebraic structure of supersymmetric (SUSY) quantum mechanics whose super partner potential is composed of the square of the superpotential and its derivative. The proposed new tensor potential is trigonometric cotangent plus a cosecant potential which is similar to the superpotential form of trigonometric Scarf potential. However, this new tensor potential is so specific that it cannot be widely applied as a Coulomb-type tensor potential. However, it is worthy of being explored to provide a new description of the motion of spin-half particles. Dirac equations for central/non-central potentials such as Hulthen potential, Morse potential, attractive exponential potential, Poschl–Teller potential, asymetrical Hartmann potential, Tietz–Hua potential, Makarov potential, four parameters diatomic potential, and ring-shaped potential have been solved mostly by Nikiforov–Uvarof (NU) method,[4,9–13] factorization methods and SUSY QM,[14–16] hypergeometric and confluent hypergeometric method[17,18,18,20–22] and asymptotic iteration method[23,24] in the limit of spin and pseudospin symmetries. However, there are only a few potentials solved exactly such as Coulomb and harmonics oscillator potentials with a Coulomb-type tensor potential, other potentials are solvable only for s-wave. For l-wave, the Dirac equations for central potentials are only solved approximately due to the contribution of the centrifugal term. The approximation scheme of the centrifugal term was proposed by Greene and Aldrich[25] and this approximation works well for trigonometric, hyperbolic and exponential potentials.

The Dirac equations for some potentials have been solved in the cases of spin symmetry and pseudospin symmetry.[12,19,26–33] The spin symmetry occurs when the difference between repulsive vector potential with the attractive scalar potential is equal to constant, while the pseudospin symmetry arises when the sum of the scalar potential with a vector potential is equal to a constant. The concept of spin symmetry has been applied to the spectrum of meson and antinucleon,[34] and to the spin symmetry breaking for the spin doublet which was used to explain the traditional magic number in atomic nuclei.[35,36] The pseudospin symmetric concept has been used to explain the quasidegeneracy of the nucleon doublets,[37] exotic nuclei,[38] super-deformation in nulei,[39]

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†Corresponding author. E-mail: suparmi@uns.ac.id
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and to establish an affective nuclear shell-model scheme.[40] Moreover, the pseudospin symmetric case has been intensively studied using SUSY QM and similarity renormalization group (SRG) to find the relativistic energy and the corresponding wave function to explain the origin of the relativistic feature of pseudospin symmetry and the mechanism of broken pseudospin symmetry.[41–44] Both spin and pseudospin symmetry concepts play an important role in explaining the magic number change in exotic nuclei.[36]

In this paper, we propose a new tensor coupling potential as a function of trigonometric expression given as

$$U(r) = -a\left(V_2 \cot(ar) + V_3 \csc(ar)\right), \tag{1}$$

where V_2 and V_3 are the strengths of the nucleon forces, and a is the parameter that controls the range of the tensor potential. The negative trigonometric cotangent potential alone is similar to the combination of Coulomb potential with square well potential therefore it is expected that the combination of trigonometric cotangent and cosecant potential is as suitable to be a screening potential as Coulomb-type and Yukawa-type tensor potentials. These tensor potentials were originally used to model strong nucleon–nucleon interactions caused by the exchange in nuclear physics.[45–47] The potential which will be coupled by the new tensor potential that can be solved exactly under the approxomation scheme of centrifugal term is the trigonometric Scarf potential. The Dirac equation with trigonometric Scarf potential together with the new tensor potential reduces to Schrödinger-type equation with a shape invariant potential in the limit of pseudospin and spin symmetries. The relativistic energy and wave functions of Scarf trigonometric potential plus the new tensor potential are analyzed using Romanovski polynomials. The trigonometric Scarf potential is a potential model used to explain strong and electromagnetic interactions and molecule vibrations.

Finite Romanovski polynomial is a traditional method which consists of reducing the Schrödinger equation by an appropriate variable substitution to a form of generalized hypergeometric equation.[48] The polynomial was discovered by Routh[49] and then rediscovered 45 years later by Romanovski.[50] The notion "finite" refers to the observation that for any given set of parameters (i.e. in any potential) only a finite of polynomials appear orthogonal.[51,52] We apply Romanovski polynomial method since this method is simpler than NU method in obtaining the energy spectrum and the wave function although there are limitations in determining the normalization of the wave function. Up to now, only a few researchers used Romanovski polynomials to solve Schrödinger and Dirac equations for certain potentials.[51,53–57]

The rest of this paper is organized as follows. Basic theory of Dirac equation is briefly presented in Section 2. The Romanovski polynomial as a method of analysis is presented in Section 3. Section 4 presents the results and discussion. Finally, a brief conclusion is presented in Section 5.

2. Basic theory

The motion of nucleon with mass M in a repulsive vector potential $V_{\rm V}(r)$ and an attractive scalar potential $V_{\rm S}(r)$ plus a tensor potential $U(r)$ is described by Dirac equation given as[9,12,19,23,28,29]

$$\begin{aligned}&\{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + V_{\rm S}(r)) - {\rm i}\beta\boldsymbol{\alpha}\cdot\boldsymbol{r}U(r)\}\,\psi(\boldsymbol{r})\\ &= \{E - V_{\rm V}(r)\}\,\psi(\boldsymbol{r}),\end{aligned} \tag{2}$$

where E is the relativistic energy and $\boldsymbol{p}$ is the three-dimensional momentum operator, $-{\rm i}\boldsymbol{\nabla}$,

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}, \tag{3}$$

with σ is three-dimensional Pauli matrix, $\boldsymbol{I}$ is the 2×2 identity matrix. By setting $\hbar = 1$, $c = 1$ and writing the Dirac spinor in Eq. (2) as

$$\psi(\boldsymbol{r}) = \begin{pmatrix} \zeta(\boldsymbol{r}) \\ \varphi(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{n\kappa}(r)}{r} Y^{l}_{jm}(\theta,\varphi) \\ {\rm i}\dfrac{G_{n\kappa}(r)}{r} Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{4}$$

where $G_{n\kappa}(r)$ and $F_{n\kappa}(r)$ are the lower and upper components of Dirac spinors, respectively, $Y^{l}_{jm}(\theta,\varphi)$ is spin spherical harmonics, $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ is pseudospin spherical harmonics, l is the orbital quantum number, $\tilde{l}$ is the pseudo orbital quantum number, and m is the projection of the angular momentum on the z axis.

Inserting Eqs. (3) and (4) into Eq. (2), we obtain

$$\left(\frac{\rm d}{{\rm d}r} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}(r) = \left(M + E_{n\kappa} - V_{\rm V}(r) + V_{\rm S}(r)\right) G_{n\kappa}(r), \tag{5}$$

$$\left(\frac{\rm d}{{\rm d}r} - \frac{\kappa}{r} + U(r)\right) G_{n\kappa}(r) = \left(M - E_{n\kappa} + V_{\rm V}(r) + V_{\rm S}(r)\right) F_{n\kappa}(r). \tag{6}$$

Manipulating Eqs. (5) and (6), we obtain Dirac equations for pseudospin and spin symmetries, respectively, given as

$$\begin{aligned}&\left\{\frac{{\rm d}^2}{{\rm d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r)\right\} G_{n\kappa}(r)\\ &+ \left\{\frac{{\rm d}U}{{\rm d}r} + \Delta(r)\left(M - E_{n\kappa} + \Sigma(r)\right)\right\} G_{n\kappa}(r)\\ &= \left(M + E_{n\kappa}\right)\left(M - E_{n\kappa} + \Sigma(r)\right) G_{n\kappa}(r),\end{aligned} \tag{7}$$

$$\begin{aligned}&\left\{\frac{{\rm d}^2}{{\rm d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r)\right\} F_{n\kappa}(r)\\ &- \left\{\frac{{\rm d}U}{{\rm d}r} + \Sigma(r)\left(M + E_{n\kappa} - \Delta(r)\right)\right\} F_{n\kappa}(r)\\ &= \left(M + E_{n\kappa} - \Delta(r)\right)\left(M - E_{n\kappa}\right) F_{n\kappa}(r),\end{aligned} \tag{8}$$

where $\Sigma(r) = V_{\rm V}(r) + V_{\rm S}(r)$ is the sum of scalar and vector potentials, and $\Delta(r) = V_{\rm V}(r) - V_{\rm S}(r)$ is the difference between vector and scalar potentials.

In the case of pseudospin symmetry, according to Eq. (7),

we have $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ that gives

$$\kappa=-\tilde{l}=-(j+1/2)\rightarrow\tilde{l}=l+1,\quad j=l+1/2, \tag{9}$$

$$\kappa=\tilde{l}+1=(j+1/2)\rightarrow\tilde{l}=l-1,\quad j=l-1/2, \tag{10}$$

for $\kappa<0$ and for $\kappa>0$, respectively, κ is the spin–orbit quantum number. For pseudospin symmetry the sum and the difference between vector and scalar potentials are given as

$$\Sigma(r)=C_{\rm ps},\quad \Delta(r)=V(r), \tag{11}$$

where $C_{\rm ps}$ is constant. Equation (7) is Schrödinger-type equation with the effective potential

$$\begin{aligned}V_{\rm eff} &= \frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa}{r}U(r)+U^2(r)\\ &\quad -\frac{{\rm d}U}{{\rm d}r}-V(r)\left(M-E_{n\kappa}+C_{\rm ps}\right)\\ &= \phi^2(r)-\phi'(r)-V(r)\left(M-E_{n\kappa}+C_{\rm ps}\right)\end{aligned} \tag{12}$$

is shape invariant since the effective potential in Eq. (12) is the combination of two potentials, $V_{\rm L}=\phi^2(r)-\phi'(r)$ and $V_{\rm R}=V(r)\left(M-E_{n\kappa}+C_{\rm ps}\right)$ with $\phi(r)=U(r)-\kappa/r$ is similar to the superpotential of the trigonometric Scarf potential in the approximation scheme of centrifugal term.

For spin symmetry, we have $\kappa(\kappa+1)=l(l+1)$ that leads to

$$\kappa=-(l+1)=-\left(j+\frac{1}{2}\right)\rightarrow j=l+\frac{1}{2},\quad \kappa<0, \tag{13}$$

$$\kappa=l=\left(j+\frac{1}{2}\right)\rightarrow j=l-\frac{1}{2},\quad \kappa>0. \tag{14}$$

The difference and the sum of vector and scalar potentials for spin symmetry are

$$\Delta(r)=C_{\rm s},\quad \Sigma(r)=V(r). \tag{15}$$

Similar to the argumentation of the pseudospin symmetry, the Schrödinger-type equation in Eq. (8) has a shape invariant potential with the effective potential $V_{\rm eff}$ given as

$$\begin{aligned}V_{\rm eff} &= \frac{\kappa(\kappa+1)}{r^2}-\frac{2\kappa}{r}U(r)+U^2(r)\\ &\quad +\frac{{\rm d}U}{{\rm d}r}+V(r)\left(E_{n\kappa}+M-C_{\rm s}\right)\\ &= \phi^2(r)+\phi'(r)+V(r)\left(E_{n\kappa}+M-C_{\rm s}\right).\end{aligned} \tag{16}$$

Both Dirac equations for pseudospin and spin symmetries in Eqs. (7) and (8) are solved using Romanovski polynomials.

3. Method of analysis

The method used to solve the Dirac equations in the limit of spin and pseudospin symmetries is the Romanovski polynomials since the Dirac equations reduce to one-dimensional Schrödinger-type equations in these limits. The one-dimensional second-order differential equation satisfied by Romanovski polynomials is developed based on the hypergeometric differential equation. The one-dimensional Schrödinger equation of potential of interest reduces to the differential equation of Romanovski polynomials by appropriate variable and wave function substitutions. The one-dimensional Schrödinger equation is given as

$$-\frac{\hbar^2}{2M}\frac{\partial^2\Psi(x)}{\partial x^2}+V(x)\Psi(x)=E\Psi(x), \tag{17}$$

where $V(x)$ is an effective potential which is a mostly shape invariant potential. By suitable variable substitution $x=f(s)$ in Eq. (17), one gets the generalized hypergeometric-type equation as

$$\frac{\partial^2\Psi(s)}{\partial s^2}+\frac{\tilde{\tau}(s)}{\sigma(s)}\frac{\partial\Psi(s)}{\partial s}+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s)=0, \tag{18}$$

with $\sigma(s)$ and $\tilde{\sigma}(s)$ mostly polynomials of order two, $\tilde{\tau}(s)$ is polynomial of order one, s, $\sigma(s)$, $\tilde{\sigma}(s)$, and $\tilde{\tau}(s)$ can have any real or complex values.[57] Setting a new wave function

$$\Psi_n(r)=g_n(s)=(1+s^2)^{\frac{\beta}{2}}{\rm e}^{\frac{-\alpha}{2}\tan^{-1}s}D_n^{(\beta,\alpha)}(s) \tag{19}$$

in Eq. (18), we obtain a hypergeometric-type differential equation which can be solved using finite Romanovski polynomials[53,58] given as

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0, \tag{20}$$

where

$$\sigma(s)=as^2+bs+c, \tag{21}$$

$$\tau=fs+h, \tag{22}$$

$$-[n(n-1)+2n(1-p)]=\lambda=\lambda_n, \tag{23}$$

$$y_n=R_n^{(p,q')}(s)=D_n^{(\beta,\alpha)}(s). \tag{24}$$

For Romanovski polynomials, the values of parameters in Eqs. (21)–(23) are

$$a=1,\quad b=0,\quad c=1,\quad f=2(1-p),\quad h=q',\quad p>0. \tag{25}$$

Inserting Eqs. (21)–(25) into Eq. (20), we obtain

$$\begin{aligned}&(1+s^2)\frac{\partial^2R_n^{(p,q')}}{\partial s^2}+\left\{2s(-p+1)+q'\right\}\frac{\partial R_n^{(p,q')}(s)}{\partial s}\\ &-\left\{n(n-1)+2n(1-p)\right\}R_n^{(p,q')}(s)=0,\end{aligned} \tag{26}$$

which is a second-order differential equation satisfied by Romanovski polynomials. Equation (20) is described in the textbook by Nikiforov–Uvarov[58] where it is cast into self-adjoint form and its weight function, $w(s)$, satisfies Pearson differential equation

$$\frac{{\rm d}(\sigma(s)w(s))}{{\rm d}s}=\tau(s)w(s). \tag{27}$$

By inserting Eqs. (21)–(23), and (25) into Eq. (27) and solving it, we obtain the weight function as

$$w^{(p,q')}(s)=(1+s^2)^{-p}{\rm e}^{q'\tan^{-1}(s)}. \tag{28}$$

The corresponding polynomials are classified according to the weight function, and are built up from the Rodrigues represen-

tation given as

$$y_n = \frac{B_n}{w(s)} \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left\{ \left(as^2 + bs + c\right)^n w(s) \right\}, \tag{29}$$

where B_n is a normalization constant, and for $\sigma(s) > 0$ and $w(s) > 0$, $y_n(s)$'s are normalized polynomials. The orthogonal integral of Romanovski polynomials with respect to the weight function $w(s)$ within a given interval (s_1, s_2), given as[51,53]

$$\int_{s_1}^{s_2} w(s) y_n(s) y_{n'}(s) \mathrm{d}s = \delta_{nn'}. \tag{30}$$

This weight function in Eq. (28) was first reported by Routh[49] and then by Romanovski.[50] The polynomials expressed in Eq. (29) are named after Romanovski and will be denoted by $R_n^{(p,q)}(s)$. Due to the decrease of the weight function by s^{-2p}, the integral of the type

$$\int_{-\infty}^{\infty} w^{(p,q')} R_n^{(p,q')}(s) R_{n'}^{(p,q')}(s) \mathrm{d}s \tag{31}$$

will be convergent only if

$$n' + n < 2p - 1. \tag{32}$$

This means that only a finite number of Romanovski polynomials are orthogonal, and the orthogonality integral of the polynomial is expressed similar to Eq. (30) where $y_n = R_n^{(p,q')}(s)$.

The heart of the Romanovski polynomials method is in obtaining Eq. (26) from one-dimensional Schrödinger equation. By using an appropriate transformation of variable, $r = f(s)$ in Eq. (17), we obtain a generalized hypergeometric equation expressed in Eq. (18). By inserting a new wave function expressed in Eq. (19) into Eq. (18), we get a new equation in the form of Eq. (20) with β and α parameters. By comparing Eq. (26) with the new equation, the relations between β and p, α and q' are obtained. The Romanovski polynomials obtained from Rodrigues formula expressed in Eq. (29) and the weight function in Eq. (28) is expressed as

$$\begin{aligned} R_n^{(p,q')}(s) &= D_n^{(\beta,\alpha)}(s) \\ &= \frac{1}{(1+s^2)^{-p} \mathrm{e}^{q' \tan^{-1}(s)}} \\ &\quad \times \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left\{ (1+s^2)^n (1+s^2)^{-p} \mathrm{e}^{q' \tan^{-1}(s)} \right\}. \end{aligned} \tag{33}$$

If the wave function of the n-th level in Eq. (19) is rewritten as

$$\Psi_n(r) = \frac{1}{\sqrt{\frac{\mathrm{d}f(s)}{\mathrm{d}s}}} (1+s^2)^{-p/2} \mathrm{e}^{q' \tan^{-1}(s)/2} R_n^{(p,q')}(s), \tag{34}$$

then the orthogonality integral of the wave functions in Eq. (34) gives rise to orthogonality integral of the finite Romanovski polynomials, given as

$$\int_0^{\infty} \Psi_n(r) \Psi_{n'}(r) \mathrm{d}r = \int_0^{\infty} w^{(p,q')} R_n^{(p,q')}(s) R_{n'}^{(p,q')}(s) \mathrm{d}s. \tag{35}$$

In this case, the values of p and q' are not n-dependence and n is the degree of polynomials.

4. Results and discussion

4.1. Solution of pseudospin symmetry

The trigonometric Scarf potential coupled with the new tensor potential is given as

$$V(r) = a^2 \left\{ \frac{V_0}{\sin^2(ar)} - \frac{V_1 \cos(ar)}{\sin^2(ar)} \right\}, \tag{36}$$

where V_0 and V_1 describe the depth of the potential and are positive, a is a positive parameter which controls the range of the potential, and $0 < r < \infty$. By inserting Eqs. (1) and (36) into Eq. (7), we obtain

$$\begin{aligned} &\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa a}{r} \left(V_2 \cot(ar) + V_3 \csc(ar)\right) \right. \\ &\left. - a^2 \left(V_2 \cot(ar) + V_3 \csc(ar)\right)^2 + a^2 V_2 \csc^2(ar) \right\} G_{n\kappa}(r) \\ &+ a^2 V_3 \csc(ar) \cot(ar) G_{n\kappa}(r) \\ &+ a^2 \left(\frac{V_0}{\sin^2(ar)} - \frac{V_1 \cos(ar)}{\sin^2(ar)} \right) \left(M - E_{n\kappa} + C_{\mathrm{ps}}\right) G_{n\kappa}(r) \\ &= \left(M + E_{n\kappa}\right) \left(M - E_{n\kappa} + C_{\mathrm{ps}}\right) G_{n\kappa}(r), \end{aligned} \tag{37}$$

which cannot be solved exactly except we use an approximation to the $1/r^2$ term. For small a, the approximation of $1/r^2$ is given by[25]

$$\frac{1}{r^2} \cong \frac{a^2}{\left(\sin^2(ar)\right)}. \tag{38}$$

Inserting Eq. (38) into Eq. (37), we obtain

$$\begin{aligned} &\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{a^2 \left[\kappa(\kappa-1) + 2\kappa V_3 + V_3^2 + V_2^2 - V_2 - V_0 \gamma_{\mathrm{ps}}\right]}{\sin^2(ar)} \right\} G_{n\kappa}(r) \\ &- \frac{a^2 (2\kappa V_2 + 2V_2 V_3 - V_3 + V_1 \gamma_{\mathrm{ps}}) \cos(ar)}{\sin^2(ar)} G_{n\kappa}(r) \\ &= \left\{ \left(M + E_{n\kappa}\right) \left(M - E_{n\kappa} + C_{\mathrm{ps}}\right) - a^2 V_2^2 \right\} G_{n\kappa}(r). \end{aligned} \tag{39}$$

Setting

$$A_{\mathrm{ps}} = \left[\kappa(\kappa-1) + 2\kappa V_3 + V_3^2 + V_2^2 q - V_2 q - V_0 \gamma_{\mathrm{ps}}\right], \tag{40}$$

$$B_{\mathrm{ps}} = (V_3 - 2\kappa V_2 - 2V_2 V_3 - V_1 \gamma_{\mathrm{ps}}), \tag{41}$$

$$E'_{\mathrm{ps}} = \left\{ \frac{\left(M + E_{n\kappa}\right) \left(M - E_{n\kappa} + C_{\mathrm{ps}}\right)}{a^2} - V_2^2 \right\}, \tag{42}$$

$$\gamma_{\mathrm{ps}} = \left(M - E_{n\kappa} + C_{\mathrm{ps}}\right), \tag{43}$$

in Eq. (39), we obtain a one-dimensional Schrödinger-type equation given as

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{a^2 A_{\mathrm{ps}}}{\sin^2(ar)} + \frac{a^2 B_{\mathrm{ps}} \cos(ar)}{\sin^2(ar)} \right\} G_{n\kappa}(r) = a^2 E'_{\mathrm{ps}} G_{n\kappa}(r). \tag{44}$$

By setting $\cos(ar) = \mathrm{i}x$ in Eq. (44), we obtain

$$\left\{ \left(1+x^2\right) \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + \frac{A_{\mathrm{ps}}}{(1+x^2)} - \frac{B_{\mathrm{ps}} \mathrm{i}x}{(1+x^2)} + E'_{\mathrm{ps}} \right\} G_{n\kappa}(x) = 0. \tag{45}$$

To solve Eq. (45) using Romanovski polynomials, we set a new wave function which is similar to Eq. (19) as

$$G_{n\kappa}(x) = \left(1+x^2\right)^{\frac{\beta}{2}} \mathrm{e}^{-\frac{\alpha}{2} \tan^{-1} x} D_n^{(\beta,\alpha)}(x). \tag{46}$$

From Eqs. (45) and (46), we get

$$\left(1+x^2\right)\frac{\partial^2 D_n^{(\beta,\alpha)}(x)}{\partial x^2}+\left[x\left(2\beta+1\right)-\alpha\right]\frac{\partial D_n^{(\beta,\alpha)}(x)}{\partial x}$$
$$-\left\{\frac{\beta\alpha x-\frac{1}{2}\alpha x-\frac{\alpha^2}{4}+\beta^2-\beta-A_{\rm ps}+B_{\rm ps}\,{\rm i}x}{\left(1+x^2\right)}-E'_{\rm ps}-\beta_{\rm ps}^2\right\}$$
$$\times D_n^{(\beta,\alpha)}(x)=0, \tag{47}$$

that reduces to

$$\left(1+x^2\right)\frac{\partial^2 D_n^{(\beta_{\rm ps},\alpha_{\rm ps})}(x)}{\partial x^2}+\left[x\left(2\beta_{\rm ps}+1\right)-\alpha_{\rm ps}\right]\frac{\partial D_n^{(\beta_{\rm ps},\alpha_{\rm ps})}(x)}{\partial x}$$
$$-\left(-E'_{\rm ps}-\beta_{\rm ps}^2\right)D_n^{(\beta_{\rm ps},\alpha_{\rm ps})}(x)=0, \tag{48}$$

after we set

$$\beta_{\rm ps}\alpha_{\rm ps}-\frac{1}{2}\alpha_{\rm ps}+B_{\rm ps}\,{\rm i}=0,\quad -\frac{\alpha_{\rm ps}^2}{4}+\beta_{\rm ps}^2-\beta_{\rm ps}-A_{\rm ps}=0. \tag{49}$$

Equation (47) is the differential equation satisfied by Romanovski polynomials.

The values of $\alpha_{\rm ps}$ and $\beta_{\rm ps}$ obtained from Eq. (49) are given as

$$\alpha_{\rm ps}=-{\rm i}\left\{\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}+\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}\right\}, \tag{50}$$

$$\left(\beta_{\rm ps}-1/2\right)=\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}. \tag{51}$$

By comparing Eqs. (26) and (48), we obtain

$$p=-\beta+1/2,\quad \alpha_{\rm ps}=-q', \tag{52}$$

$$\beta_{\rm ps}-1/2=\sqrt{-E'_{\rm ps}}-n-1/2. \tag{53}$$

The relativistic energy equation obtained from Eqs. (42), (51), and (53) is

$$\left\{\frac{\left(M+E_{n\kappa}\right)\left(E_{n\kappa}-M-C_{\rm ps}\right)}{a^2}+V_2^2\right\}$$
$$=\left\{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}+n+1/2\right\}^2. \tag{54}$$

The relativistic energy spectra calculated from relativistic energy equation in Eq. (54) are presented in Table 1 and it shows that the degeneracy occurs for a pair of states $(n,l,j+1/2)$ with $(n,l+2,j-1/2)$. The degeneracy is removed by the presence of the trigonometric cotangent and cosecant tensor potential, as shown in the 5th and 9th columns. For $\kappa<0$, the presence of the tensor potential decreases the relativistic energy while for $\kappa>0$ the tensor potential increases the relativistic energy.

Without the presence of the tensor potential, the relativistic energy spectra as a function of the width of the potential, a, calculated from Eq. (54) are presented in Table 2. The graph of relativistic energy as a function of the width of potential are shown in Fig. 1(a) for $n=0$ and Fig. 1(b) for $n=1$. Figures 1(a) and 1(b) show that for s and p waves, the smaller the values of a are, the lower the energies become, but the energies are higher for d and f waves. From Eqs. (28), (33), (50)–(52), we obtain the weight function and the Romanovski polynomials given as

$$w^{\left(-\beta_{\rm ps}+1/2,-\alpha_{\rm ps}\right)}(x)=\left(1+x^2\right)^{\beta_{\rm ps}-1/2}{\rm e}^{-{\rm i}\alpha_{\rm ps}\tan^{-1}(x)}, \tag{55}$$

$$R_n^{\left(-\beta_{\rm ps}+1/2,-\alpha_{\rm ps}\right)}(x)$$
$$=\frac{1}{\left(1+x^2\right)^{\beta_{\rm ps}-1/2}{\rm e}^{-{\rm i}\alpha_{\rm ps}\tan^{-1}(x)}}$$
$$\times\frac{{\rm d}^n}{{\rm d}x^n}\left\{\left(1+x^2\right)^{\beta_{\rm ps}-1/2+n}{\rm e}^{-{\rm i}\alpha_{\rm ps}\tan^{-1}(x)}\right\}. \tag{56}$$

The lower component of Dirac spinor obtained from Eqs. (46), (50), (51), and (56) is given as

$$G_{n\kappa}(x)=\left(1+x^2\right)^{-\frac{\beta_{\rm ps}}{2}+\frac{1}{2}}{\rm e}^{{\rm i}\frac{\alpha_{\rm ps}}{2}\tan^{-1}(x)}$$
$$\times\frac{{\rm d}^n}{{\rm d}x^n}\left\{\left(1+x^2\right)^{\beta_{\rm ps}-1/2+n}{\rm e}^{-{\rm i}\frac{\alpha_{\rm ps}}{2}\tan^{-1}(x)}\right\}. \tag{57}$$

By using the hyperbolic–trigonometric relationship given as

$$\tan^{-1}(-{\rm i}\cos(ar))=-{\rm i}\tanh(\cos(ar))$$
$$=-\frac{\rm i}{2}\ln\left(\frac{1+\cos(ar)}{1-\cos(ar)}\right), \tag{58}$$

and using Eqs. (50), (51), (54), and (55), we obtain the weight function given as

$$w^{\left(-\beta_{\rm ps}+1/2,-\alpha_{\rm ps}\right)}(r)=\left(1-\cos(ar)\right)^{-\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}}$$
$$\times\left(1+\cos(ar)\right)^{\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}}. \tag{59}$$

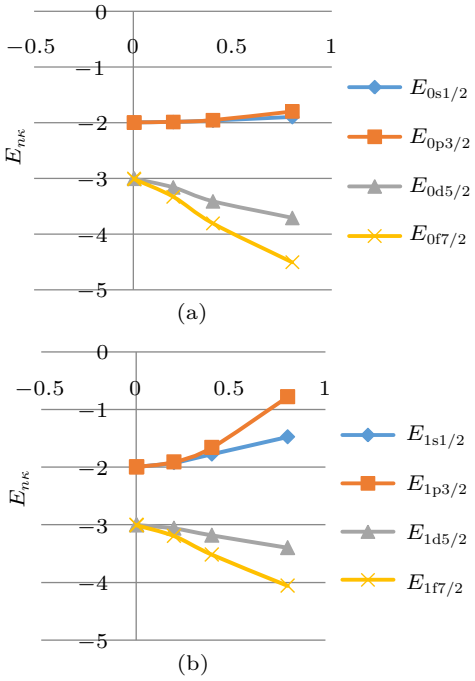


Fig. 1. (color online) The graphs of energy spectra of Scarf potential as a function of potential width, a, for pseudospin symmetry, with tensor potential $V_2=V_3=0$ for $n=0$ (a) and for $n=1$ (b).

Table 1. Energy spectra for trigonometric Scarf potential with/without a new tensor potential for $V_0 = 4$ fm^{-1}, $V_1 = 3$ fm^{-1}, $a = 0.05$ fm^{-1}, $M = 3$ fm^{-1}, and $C_{\rm ps} = -5$ fm^{-1}.

l	$n,\kappa<0$	$(l, j=l+1/2)$	$E_{n\kappa}<0$ $V_2\&V_3=0$	$E_{n\kappa}<0$ $V_2=0.6, V_3=0.8$	$n,\kappa>0$	$(l+2, j=l-1/2)$	$E_{n\kappa}<0$ V_2 & $V_3=0$	$E_{n\kappa}<0$ $V_2=0.6, V_3=0.8$
0	0, −1	$0s_{1/2}$	−1.98997	−1.99674	0, 2	$0d_{3/2}$	−1.98997	−1.98058
1	0, −2	$0p_{3/2}$	−1.97773	−1.98834	0, 3	$0f_{5/2}$	−1.97773	−1.96487
2	0, −3	$0d_{5/2}$	−1.96107	−1.97518	0, 4	$0g_{7/2}$	−1.96107	−1.94500
3	0, −4	$0f_{7/2}$	−1.94039	−1.95767	0, 5	$0h_{9/2}$	−1.94039	−1.92142
0	1, −1	$1s_{1/2}$	−1.97756	−1.98768	1, 2	$1d_{3/2}$	−1.97756	−1.96427
1	1, −2	$1p_{3/2}$	−1.96091	−1.97485	1, 3	$1f_{5/2}$	−1.96091	−1.94457
2	1, −3	$1d_{5/2}$	−1.94022	−1.95741	1, 4	$1g_{7/2}$	−1.94022	−1.92107
3	1, −4	$1f_{7/2}$	−1.91595	−1.93599	1, 5	$1h_{9/2}$	−1.91595	−1.89428

Table 2. Energy spectra for trigonometric Scarf potential without a new tensor potential for $V_0 = 4$ fm^{-1}, $V_1 = 3$ fm^{-1}, $a = 0.05$ fm^{-1}, $M = 3$ fm^{-1}, $C_{\rm ps} = -5$ fm^{-1}, and $V_2 = V_3 = 0$ and variation of a.

l	n,κ	$(n,\ j=l+1/2, l-1/2)$	$a=0.8$	$a=0.4$	$a=0.2$	$a=0.004$	$a=0.002$
0, 2	0, −1, 2	$0s_{1/2}, 0d_{3/2}$	−1.90220	−1.96622	−1.99047	−1.99960	−1.99990
1, 3	0, −2, 3	$0p_{3/2}, 0f_{5/2}$	−1.79943	−1.95768	−1.98986	−1.99960	−1.99990
2, 4	0, −3, 4	$0d_{5/2}, 0g_{7/2}$	−3.71219	−3.41398	−3.16317	−3.00851	−3.00215
3, 5	0, −4, 5	$0f_{7/2}, 0h_{9/2}$	−4.50626	−3.80900	−3.33236	−3.01927	−3.00491
0, 2	1, −1, 2	$1s_{1/2}, 1d_{3/2}$	−1.47631	−1.77590	−1.92411	−1.99643	−1.99910
1, 3	1, −2, 3	$1p_{3/2}, 1f_{5/2}$	−0.78088	−1.66083	−1.91152	−1.99640	−1.99910
2, 4	1, −3, 4	$1d_{5/2}, 1g_{7/2}$	−3.39784	−3.18219	−3.06088	−3.00278	−3.00070
3, 5	1, −4, 5	$1f_{7/2}, 1h_{9/2}$	−4.05649	−3.51751	−3.19406	−3.00997	−3.00252

The ground state of lower component of Dirac spinor for any κ state obtained from Eqs. (57) and (58) is

$$G_{0\kappa}(r) = (1-\cos(ar))^{-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}+\frac{1}{4}} \times (1+\cos(ar))^{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}+\frac{1}{4}}. \quad (60)$$

The first excited state of the lower component of Dirac spinor obtained from Eq. (57) when $n = 1$ given as

$$G_{1\kappa} = (1-\cos(ar))^{-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}+\frac{1}{4}} \times (1+\cos(ar))^{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}+\frac{1}{4}} \times {\rm i}\left\{-\left(\beta_{\rm ps}+1/2\right)2\cos(ar)+\frac{B_{\rm ps}}{\left(\beta_{\rm ps}-1/2\right)}\right\}. \quad (61)$$

The graphs of the ground state wave function of the Dirac spinor lower component as a function κ for pseudospin symmetry drawn from Eq. (60) without the presence of the tensor potential is shown in Fig. 2 and with the presence of tensor potential in Fig. 3, for fixed $a = 0.05$ fm^{-1}. The graphs show that the larger the absolute value of κ is, the wider the pulse width is. From Figs. 2 and 3, we can see that the tensor coupling causes the pulse width of the waves decrease for negatives κ ($G_{0s1/2}$ and $G_{0p3/2}$).

The ground state wave function of upper component of Dirac spinor for pseudospin symmetry is obtained using Eqs. (6) and (60) in the approximation scheme of centrifugal term given as

$$F_{0\kappa}(r) = \left\{\frac{a\left\{\frac{1}{2}\left(\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}-\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}+1-2V_2\right)\cos(ar)\right\}}{\left(M-E_{n\kappa}+C_{\rm ps}\right)}-a\frac{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}+\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}+(\kappa+V_3)}{\left(M-E_{n\kappa}+C_{\rm ps}\right)}\right\} \times (1-\cos(ar))^{-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)-B_{\rm ps}}-\frac{1}{4}}(1+\cos(ar))^{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_{\rm ps}\right)+B_{\rm ps}}-\frac{1}{4}}. \quad (62)$$

For exact pseudospin symmetry which occurs when $C_{\rm ps} = 0$, the upper spinor in Eq. (62) exists if $M \neq E_{n\kappa}$, which means that there is no positive bound state energy for pseudospin symmetry. The graph of the upper component of Dirac spinor expressed in Eq. (62) as a function of κ for fixed $a = 0.05$ fm^{-1} with and without tensor are shown in Figs. 4 and 5, respectively. It can be seen from Figs. 4 and 5 that the waves for negative κ are sensitive to the presence of tensor potential.

Since $\beta_{\rm ps}$ and $\alpha_{\rm ps}$ expressed in Eqs. (50) and (51) are n-dependence thus the orthogonality integral of Romanovski polynomial can not be produced by the orthogonality integral of wave function expressed in Eq. (35) and so the number of polynomials that are orthogonal are not finite.

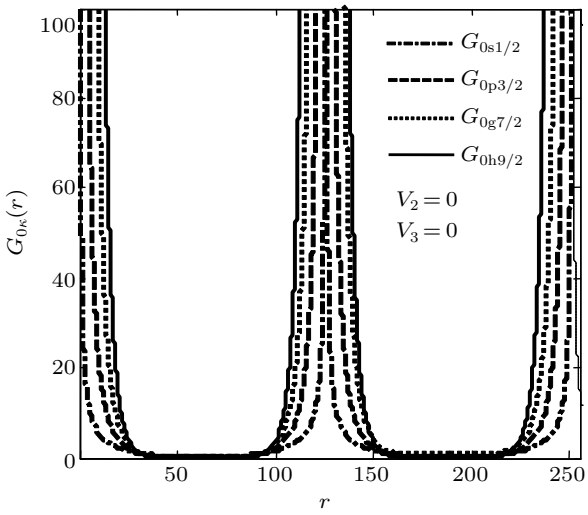


Fig. 2. The graph of the wave function of the lower component of Dirac spinor $G_{0\kappa}(r)$ as a function of κ for the fixed potential width $a = 0.05$ fm^{-1} in the case of pseudospin symmetry without tensor potential.

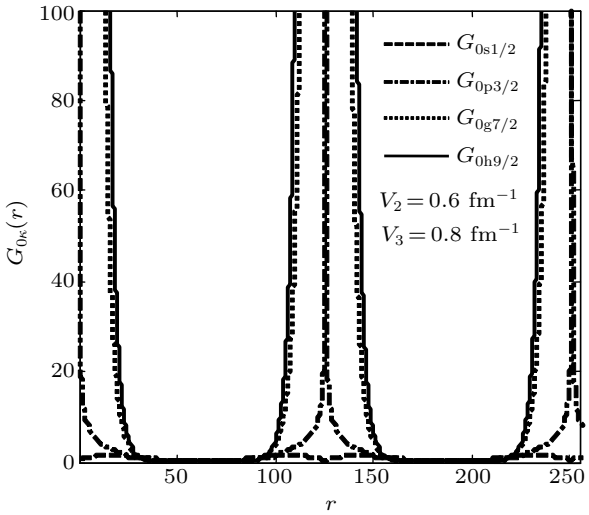


Fig. 3. The graph of the wave function of the lower component of Dirac spinor $G_{0\kappa}(r)$ as a function of κ for the fixed potential width $a = 0.05$ fm^{-1} in the case of pseudospin symmetry with tensor potential.

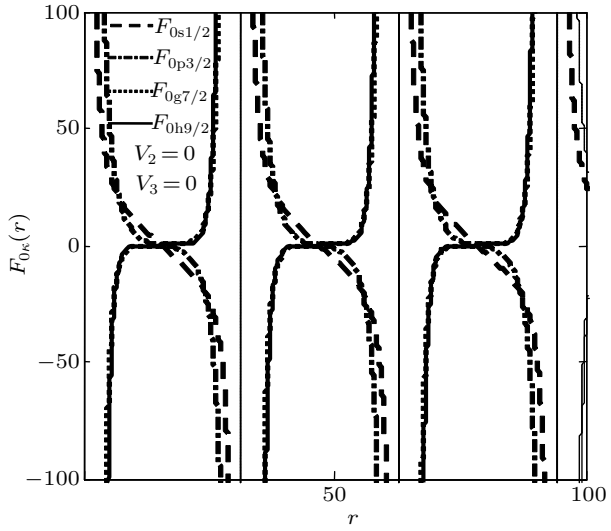


Fig. 4. The graph of the ground state wave function of the upper component of Dirac spinor, $F_{0\kappa}(r)$ as a function of κ for the fixed potential width $a = 0.05$ fm^{-1} in the case of pseudospin symmetry without tensor potential.

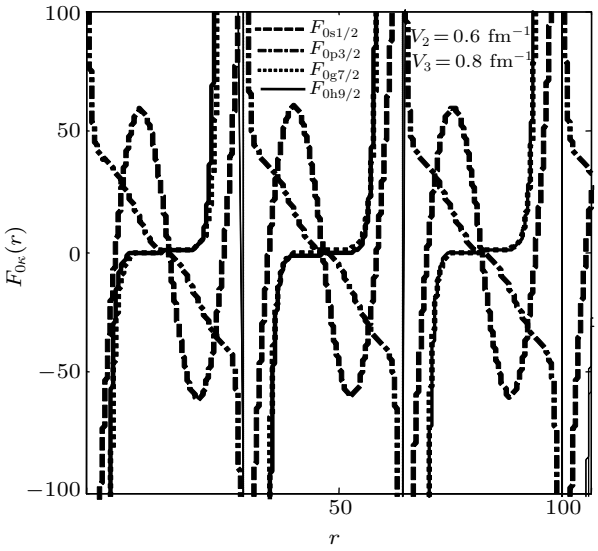


Fig. 5. The graph of the ground state wave function of the upper component of Dirac spinor, $F_{0\kappa}(r)$ as a function of κ for the fixed potential width $a = 0.05$ fm^{-1} in the case of pseudospin symmetry with tensor potential.

4.2. Solution of spin symmetry

The Dirac equation for spin symmetry is obtained by inserting Eqs. (1) and (36) into Eq. (8) given as

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa}{r}a\left(V_2\cot(ar)+V_3\csc(ar)\right)\right.\\&\left.-a^2V_2\csc^2(ar)\right\}F_{n\kappa}(r) - \left[a^2\left(V_2\cot(ar)+V_3\csc(ar)\right)^2\right.\\&\left.+a^2V_3\csc(ar)\cot(ar)\right]F_{n\kappa}(r)\\&-a^2\left(\frac{V_0}{\sin^2(ar)} - \frac{V_1\cos(ar)}{\sin^2(ar)}\right)(M+E_{n\kappa}-C_\mathrm{s})F_{n\kappa}(r)\\&=(M-E_{n\kappa})(M+E_{n\kappa}-C_\mathrm{s})F_{n\kappa}(r). \qquad (63)\end{aligned}$$

By inserting the approximation of the centrifugal term in Eq. (38) into Eq. (63), we obtain

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{a^2\left[\kappa(\kappa+1)+2\kappa V_3+V_3^2+V_2^2-V_2+V_0\gamma_\mathrm{s}\right]}{\sin^2(ar)}\right\}F_{n\kappa}(r)\\&-\frac{a^2(2\kappa V_2+2V_2V_3-V_3-V_1\gamma_\mathrm{s})\cos(ar)}{\sin^2(ar)}F_{n\kappa}(r)\\&=\left\{(M-E_{n\kappa})(M+E_{n\kappa}-C_\mathrm{s})-a^2V_2^2\right\}F_{n\kappa}(r). \qquad (64)\end{aligned}$$

By setting

$$A_\mathrm{s} = \left[\kappa(\kappa+1)+2\kappa V_3+V_3^2+V_2^2-V_2+V_0\gamma_\mathrm{s}\right], \qquad (65)$$

$$B_\mathrm{s} = (V_3-2\kappa V_2-2V_2V_3+V_1\gamma_\mathrm{s}), \qquad (66)$$

$$E_\mathrm{s}' = \left\{\frac{(M-E_{n\kappa})(M+E_{n\kappa}-C_\mathrm{s})}{a^2} - V_2^2\right\}, \qquad (67)$$

$$\gamma_\mathrm{s} = (M+E_{n\kappa}-C_\mathrm{s}), \qquad (68)$$

in Eq. (64), then equation (64) reduces to a one-dimensional Schrödinger-type equation given as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{a^2A_\mathrm{s}}{\sin^2(ar)} + \frac{a^2B_\mathrm{s}\cos(ar)}{\sin^2(ar)}\right\}F_{n\kappa}(r) = a^2E_\mathrm{s}'F_{n\kappa}(r). \qquad (69)$$

By introducing new variable $\cos(ar) = \mathrm{i}x$ in Eq. (69), we have

$$\left\{(1+x^2)\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + \frac{A_\mathrm{s}}{(1+x^2)} - \frac{B_\mathrm{s}\mathrm{i}x}{(1+x^2)} + E_\mathrm{s}'\right\}F_{n\kappa}(x) = 0, \qquad (70)$$

and setting a new wave function

$$F_{n\kappa}(x) = g_n(x) = \left(1+x^2\right)^{\frac{\beta_s}{2}} \mathrm{e}^{-\frac{\alpha_s}{2}\tan^{-1}x} D_n^{(\beta_s,\alpha_s)}(x), \quad (71)$$

in Eq. (70), we obtain

$$\left(1+x^2\right)\frac{\partial^2 D_n^{(\beta_s,\alpha_s)}(x)}{\partial x^2} + \left[x(2\beta_s+1)-\alpha_s\right]\frac{\partial D_n^{(\beta_s,\alpha_s)}(x)}{\partial x} - \left\{\frac{\beta_s\alpha_s x - \frac{1}{2}\alpha_s x - \frac{\alpha_s^2}{4} + \beta_s^2 - \beta_s - A_s + B_s \mathrm{i}x}{(1+x^2)} - E_s' - \beta_s^2\right\} \times D_n^{(\beta_s,\alpha_s)}(x) = 0. \quad (72)$$

Setting

$$\beta_s\alpha_s - \frac{1}{2}\alpha_s + B_s\mathrm{i} = 0, \quad -\frac{\alpha_s^2}{4} + \beta_s^2 - \beta_s - A_s = 0, \quad (73)$$

in Eq. (72), we get the differential equation for Romanovski polynomials given as

$$\left(1+x^2\right)\frac{\partial^2 D_n^{(\beta_s,\alpha_s)}(x)}{\partial x^2} + \left[x(2\beta_s+1)-\alpha_s\right]\frac{\partial D_n^{(\beta_s,\alpha_s)}(x)}{\partial x} - \left(-E_s' - \beta_s^2\right) D_n^{(\beta_s,\alpha_s)}(x) = 0. \quad (74)$$

From Eq. (73), we obtain the values of α_s and β_s that have physical meanings given as

$$\alpha_s = -\mathrm{i}\sqrt{\frac{\left(\frac{1}{4}+A_s\right) + \sqrt{\left(\frac{1}{4}+A_s\right)^2 - B_s^2}}{2}}, \quad (75)$$

$$\beta_s - 1/2 = \sqrt{\frac{\left(\frac{1}{4}+A_s\right) - \sqrt{\left(\frac{1}{4}+A_s\right)^2 - B_s^2}}{2}}. \quad (76)$$

Comparing Eqs. (26) and (74), we obtain

$$(2\beta_s+1) = 2(1-p), \quad \alpha_s = -q', \quad (77)$$

$$\beta_s - 1/2 = \pm\sqrt{-E_s'} - n - 1/2. \quad (78)$$

The relativistic energy equation is obtained from Eqs. (67), (76), and (78) given as

$$\left\{\frac{(E_{n\kappa}-M)(E_{n\kappa}+M-C_s)}{a^2} + V_2^2\right\} = \left\{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)+B_s} - \frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)-B_s} + n + 1/2\right\}^2. \quad (79)$$

The relativistic energies for spin symmetry calculated by using Eq. (79) are presented in Table 3. From Table 3, we see that the degeneracy occurs for a pair of states $(n,l,j=l+1/2)$ with $(n,l,j=l-1/2)$. The degeneracy is removed by the presence of trigonometric cotangent and cosecant tensor potential as shown in the 5th and 9th columns. The relativistic energy spectra for spin symmetry as a function of the width of the potential, a, calculated from Eq. (79) without the tensor potential are presented in Table 4. The graphs of relativistic energies without tensor potential as a function of the width of potential are shown in Fig. 6(a) for $n=0$ and in Fig. 6(b) for $n=1$. From Figs. 6(a) and 6(b), we see that the energy increases with the increase of the width of potential a. From Eqs. (28), (33), (76), (77), and (78), we obtain the weight function and the Romanovski polynomials for upper wave function given as

$$w^{(p,q')}(x) = (1+x^2)^{\beta_s-1/2}\,\mathrm{e}^{\frac{B_s\mathrm{i}}{(\beta_s-1/2)}\tan^{-1}(x)}, \quad (80)$$

and

$$R_n^{(p,q')}(x) = \frac{1}{(1+x^2)^{\beta_s-1/2}\,\mathrm{e}^{\frac{B_s\mathrm{i}}{(\beta_s-1/2)}\tan^{-1}(x)}} \times \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{(1+x^2)^{\beta_s-1/2+n}\,\mathrm{e}^{\frac{B_s\mathrm{i}}{(\beta_s-1/2)}\tan^{-1}(x)}\right\}. \quad (81)$$

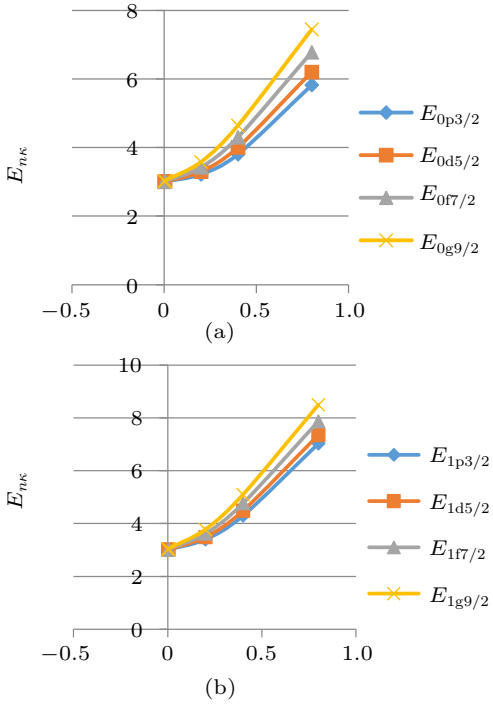


Fig. 6. (color online) The graphs of the energy spectra in the spin symmetric case as a function of the width of the potential for $n=0$ (a) and $n=1$ (b).

Table 3. Energy spectra for trigonometric Scarf potential with/without new tensor potential for $V_0 = 4$ fm^{-1}, $V_1 = 3$ fm^{-1}, $a = 0.05$ fm^{-1}, $M = 3$ fm^{-1}, and $C_s = 5$ fm^{-1}.

1	$n,\kappa<0$	$(l,j=l+1/2)$	$E_{n\kappa}<0$ $V_2\&V_3=0$	$E_{n\kappa}<0$ $V_2=0.6, V_3=0.8$	$n,\kappa>0$	$(l,j=l-1/2)$	$E_{n\kappa}<0$ $V_2\ \&\ V_3=0$	$E_{n\kappa}<0$ $V_2=0.6, V_3=0.8$
1	0, −2	$0p_{3/2}$	3.02112	3.01396	0, 1	$0p_{1/2}$	3.02112	3.03046
2	0, −3	$0d_{5/2}$	3.03287	3.02089	0, 2	$0d_{3/2}$	3.03287	3.04545
3	0, −4	$0f_{7/2}$	3.04905	3.03354	0, 3	$0f_{5/2}$	3.04905	3.06464
4	0, −5	$0g_{9/2}$	3.06924	3.05070	0, 4	$0g_{7/2}$	3.06924	3.06464
1	1, −2	$1p_{3/2}$	3.03757	3.02833	1, 1	$1p_{1/2}$	3.03757	3.04967
2	1, −3	$2d_{5/2}$	3.05239	3.03754	1, 2	$2d_{3/2}$	3.05239	3.06774
3	1, −4	$3f_{7/2}$	3.07178	3.05342	1, 3	$3f_{5/2}$	3.07178	3.08997
4	1, −5	$4g_{9/2}$	3.09503	3.07386	1, 4	$4g_{7/2}$	3.09503	3.11576

Table 4. Energy spectra for trigonometric Scarf potential without new tensor potential for $V_0 = 4$ fm^{-1}, $V_1 = 3$ fm^{-1}, $a = 0.05$ fm^{-1}, $M = 3$ fm^{-1}, $C_s = -5$ fm^{-1}, $V_2 = V_3 = 0$ and variation of a.

l	n, κ	$(l, j = l+1/2)$	$a = 0.8$	$a = 0.4$	$a = 0.2$	$a = 0.004$	$a = 0.002$
1	0, −2, 1	$0p_{3/2}, 0p_{1/2}$	5.82453	3.80403	3.21972	3.00923	3.00231
2	0, −3, 2	$0d_{5/2}, 0d_{3/2}$	6.20022	4.00516	3.30071	3.01357	3.00341
3	0, −4, 3	$0f_{7/2}, 0f_{5/2}$	6.77779	4.30135	3.42650	3.02120	3.00536
4	0, −5, 4	$0g_{9/2}, 0g_{7/2}$	7.44723	4.63963	3.57751	3.03180	3.00810
5	1, −2, 1	$1p_{3/2}, 1p_{1/2}$	7.02816	4.29865	3.39729	3.01839	3.00463
6	1, −3, 2	$2d_{5/2}, 2d_{3/2}$	7.34843	4.48579	3.48519	3.02424	3.00613
7	1, −4, 3	$3f_{7/2}, 3f_{5/2}$	7.86737	4.76899	3.61778	3.03398	3.00865
8	1, −5, 4	$4g_{9/2}, 4g_{7/2}$	8.49050	5.09707	3.77337	3.04683	3.01204

The upper component of Dirac spinor obtained from Eqs. (71) and (81) is given

$$F_{n\kappa}(x) = (1+x^2)^{-\frac{\beta_s}{2}+\frac{1}{2}} \mathrm{e}^{-\frac{B_s \mathrm{i}}{2(\beta_s-1/2)}\tan^{-1}x} \times \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{(1+x^2)^{\beta_s-1/2+n}\mathrm{e}^{\frac{B_s \mathrm{i}}{(\beta_s-1/2)}\tan^{-1}(x)}\right\}. \quad (82)$$

Using Eqs. (58), (77), and (78), we obtain the weight function as a function of radial variable given as

$$w^{(p,q')}(x) = (1-\cos(ar))^{-\sqrt{\left(\frac{1}{4}+A_s\right)-B_s}} \times (1+\cos(ar))^{\sqrt{\left(\frac{1}{4}+A_s\right)+B_s}}. \quad (83)$$

The ground state of upper component of Dirac spinor for any κ state, in the case of spin symmetry is obtained from Eq. (82) when $n = 0$,

$$F_{0\kappa}(x) = (1-\cos(ar))^{-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)-B_s}+\frac{1}{4}} \times (1+\cos(ar))^{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s.\right)+B_s}+\frac{1}{4}}. \quad (84)$$

The lower component of Dirac spinor for spin symmetry is obtained using Eqs. (5) and (84), given as

$$G_{0\kappa}(r) = \left\{ a\frac{\left\{\frac{1}{2}\left(\sqrt{\left(\frac{1}{4}+A_s\right)+B_s}-\sqrt{\left(\frac{1}{4}+A_s\right)-B_s}+1+2V_2\right)\cos(ar)\right\}}{(M+E_{n\kappa}-C_s)} - a\frac{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)+B_s}+\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)-B_s}-(\kappa+V_3)}{(M+E_{n\kappa}-C_s)} \right\} \times (1-\cos(ar))^{-\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)-B_s}-\frac{1}{4}} (1+\cos(ar))^{\frac{1}{2}\sqrt{\left(\frac{1}{4}+A_s\right)+B_s}-\frac{1}{4}}. \quad (85)$$

For exact spin symmetry which occurs when $C_s = 0$, the lower spinor exists if $M \neq -E_{n\kappa}$, therefore, the system has no negative bound state relativistic energies.

The equations of ground state wave functions for spin symmetry, both for upper and lower components of Dirac spinors are basically the same as that in the pseudospin symmetry limit, the only difference is the values of A, B, and C therefore, it can be shown that the graphs of the ground state wave functions in spin symmetric case are similar to that in pseudospin symmetric case.

5. Conclusion

The Dirac equation for trigonometric Scarf potential with a trigonometric cotangent and cosecant tensor potential in the approximation scheme of centrifugal term is exactly solved using Romanovski polynomials both in pseudospin and spin symmetric cases. The bound state relativistic energy spectra are calculated using Matlab 2011 and are positive in spin symmetric case and negative in pseudospin symmetric case. The trigonometric cotangent and cosecant tensor potential removes the degeneracy energies both in pseudospin and spin symmetries. The lower and upper components of Dirac spinors are obtained exactly in the approximation scheme of centrifugal term by using Romanovski polynomials both for pseudospin and spin symmetries. The tensor potential has significant effects on the wave function for negative values of κ.

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