New approach to solving master equations of density operator for the Jaynes Cummings model with cavity damping

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By introducing thermo-entangled state representation $|\eta\rangle$, which can map master equations of density operator in quantum statistics as state-vector evolution equations, and using "dissipative interaction picture" we solve the master equation of Jaynes–Cummings model with cavity damping. In addition we derive the Wigner function for density operator when the atom is initially in the up state $|\uparrow\rangle$ and the cavity mode is in coherent state.

Keywords: Jaynes–Cummings model, entangled state representation, master equations

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1. Introduction

As is well known, decoherence dynamics and the various methods of controlling this process is one of the important topics in quantum optics and the theory of quantum open systems can help to better understand how quantum decoherence is processed. In nature most quantum systems are immersed in reservoir that these systems introduce open system[1,2] and the time evolution of an open quantum system interacting with a memory-less environment can be expressed as a Lindblad master equation as follows:[3,4]

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right] + \sum_n \left[\hat{L}_n\hat{\rho}(t)\hat{L}_n^{\dagger} - \frac{1}{2}\hat{L}_n^{\dagger}\hat{L}_n\hat{\rho}(t) - \frac{1}{2}\hat{\rho}(t)\hat{L}_n^{\dagger}\hat{L}_n\right], \tag{1}$$

where $\hat{\rho}(t)$ is the density of the system, $\hat{H}$ is the Hamiltonian of the system, and $\hat{L}_n$ is the so-called Lindblad operators that are used to model the effect of the environment on the system. Equation (1) describes the general case of a quantum open system. Using some quantum-mechanical models for the reservoir, one can find a differential equation for the time evolution of reduced density operator of the system.

Now, we consider a two-level system interacting with a resonant mode of the electromagnetic field, i.e., Jaynes–Coumming model,[5,6] this model is described without the rotating wave approximation by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \omega\hat{a}^{\dagger}\hat{a} + \lambda(\hat{a}^{\dagger} + \hat{a})(\hat{\sigma}_+ + \hat{\sigma}_-). \tag{2}$$

If damping is contained in J–C model, one needs to solve

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\hat{H},\hat{\rho}\right] + \kappa L_{\mathrm{ir}}\left[\hat{\rho}\right], \tag{3}$$

where κ is the damping constant and $L_{\mathrm{ir}}[\rho]$ takes the form

$$L_{\mathrm{ir}}\left[\hat{\rho}\right] = (2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}). \tag{4}$$

There is a newly developed method, introduced by Fan H Y and Fan Y,[7] from which we can map the density operator by a vector of two-mode Fock space whose first mode is the system mode and the second mode is a fictitious one, and consequently the master equation appears as a Schrödinger-like time evolution equation.

In the present paper, instead of using phase space representation of density operator[8,9] we adopted thermo entangled state (TES) representation to treat time evolution of density operator in the J–C model with cavity damping. The rest of this paper is organized as follows. In Section 2, we review the (TES) representation and in Section 3, we decouple the equation of motion of the density operator. In Section 4, we solve the master equation of J–C model by using "dissipative interaction picture" in entangled representation. In Section 5 Wigner function is derived for a special initial condition. Section 6 is devoted to the calculation. In the Appendix, we derive a relation used in the text.

2. Brief review of the thermo-entangled state representation

In this section, we first briefly review the TES and construct the TES representation in the doubled Fock space in the following form:[10–13]

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta\hat{a}^{\dagger} - \eta^*\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}^{\dagger}\right)|0,\tilde{0}\rangle, \eta \in \mathrm{C}, \tag{5}$$

where $(\hat{a},\hat{a}^{\dagger})$ and $(\hat{b},\hat{b}^{\dagger})$ are the pairs of canonical annihilation and creation operators which are the first (physical) and the second (fictitious) mode respectively. Defining $\hat{\eta} \equiv \hat{a} - \hat{b}^{\dagger}$ and $\hat{\xi} = \hat{a} + \hat{b}^{\dagger}$, one can recognize that equation (5) is eigenvector of

$$\hat{\eta}\,|\eta\rangle = \eta\,|\eta\rangle,$$

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$$\hat{\eta}^{\dagger}|\eta\rangle = \eta^{*}|\eta\rangle. \tag{6}$$

Obviously, when $\eta = 0$

$$|\eta = 0\rangle = \exp\left(-\frac{1}{2}|\eta|^{2}\right)|0,0\rangle = \sum_{n=0}^{\infty}|n,n\rangle. \tag{7}$$

It is obtained that

$$\begin{aligned}\hat{a}|\eta = 0\rangle &= \hat{b}^{\dagger}|\eta = 0\rangle,\\ \hat{a}^{\dagger}|\eta = 0\rangle &= \hat{b}|\eta = 0\rangle.\end{aligned} \tag{8}$$

By the technique (IWOP), one can prove that equation (8) makes up a complete and orthonormal set

$$\int \frac{\mathrm{d}^{2}\eta}{\pi}|\eta\rangle\langle\eta| = 1,\ \ \langle\eta'|\eta\rangle = \pi\delta^{(2)}(\eta'-\eta). \tag{9}$$

It is convenient to introduce the state $|\rho(t)\rangle$[14] for density operator

$$|\rho(t)\rangle \equiv \hat{\rho}(t)|\eta = 0\rangle. \tag{10}$$

This means that we can uniquely represent any density operator $\hat{\rho}(t)$ of physical mode by vector $|\rho(t)\rangle$ from the two-mode Hilbert space. This vector contains all information about the quantum state as the density operator.

3. Time evolution of density operator in the J–C model

In order to describe Eq. (3) in the interaction picture, which may be called rotational interaction picture, we use the unitary operator

$$U(t) = \exp(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}). \tag{11}$$

Therefore we have

$$U^{-1}\hat{\rho}_{\mathrm{s}}U \Rightarrow \hat{\rho},\ \ U^{-1}\hat{H}U \Rightarrow \hat{H}, \tag{12}$$

Inserting Eq. (12) into Eq. (3) we derive the master equation in the rotational interaction picture as follows:

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} &= -\mathrm{i}\left[\lambda(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})(\hat{\sigma}_{+} + \hat{\sigma}_{-}), \hat{\rho}\right]\\ &\quad + \kappa L_{\mathrm{ir}}[\hat{\rho}].\end{aligned} \tag{13}$$

We write the density operator as

$$\begin{aligned}\hat{\rho} &= \frac{1}{2}\sum_{i=0}^{3}\hat{\rho}_{i}\sigma_{i}\\ &= \begin{pmatrix}\hat{\rho}_{11} & \hat{\rho}_{12}\\ \hat{\rho}_{21} & \hat{\rho}_{22}\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\hat{\rho}_{0}+\hat{\rho}_{3} & \hat{\rho}_{1}-i\hat{\rho}_{2}\\ \hat{\rho}_{1}+\mathrm{i}\hat{\rho}_{2} & \hat{\rho}_{0}-\hat{\rho}_{3}\end{pmatrix}.\end{aligned} \tag{14}$$

Using the Pauli spin matrices $\hat{\sigma}_{+} + \hat{\sigma}_{-} = \hat{\sigma}_{1}, \hat{\sigma}_{y} = \hat{\sigma}_{2}, \hat{\sigma}_{z} = \hat{\sigma}_{3}$ and the unit matrix $\hat{\sigma}_{0} = \hat{1}$, noting that

$$\begin{aligned}\hat{\rho}_{11} &= \langle\uparrow|\hat{\rho}|\uparrow\rangle,\ \ \hat{\rho}_{12} = \langle\uparrow|\hat{\rho}|\downarrow\rangle,\\ \hat{\rho}_{21} &= \langle\downarrow|\hat{\rho}|\uparrow\rangle,\ \ \hat{\rho}_{22} = \langle\downarrow|\hat{\rho}|\downarrow\rangle,\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\hat{\rho}_{0} = \hat{\rho}_{11} + \hat{\rho}_{22},\ \ \hat{\rho}_{1} = \hat{\rho}_{12} + \hat{\rho}_{21},\\ &\hat{\rho}_{2} = \mathrm{i}(\hat{\rho}_{12} - \hat{\rho}_{21}),\ \ \hat{\rho}_{3} = \hat{\rho}_{11} - \hat{\rho}_{22},\end{aligned} \tag{16}$$

according to the above equations, equation (13) can be split into the following four equations:

$$\begin{aligned}\dot{\hat{\rho}}_{0} &= -\mathrm{i}\lambda\left[(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t}), \hat{\rho}_{1}\right] + \kappa L_{\mathrm{ir}}[\hat{\rho}_{0}],\\ \dot{\hat{\rho}}_{1} &= -\mathrm{i}\lambda\left[(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t}), \hat{\rho}_{0}\right] + \kappa L_{\mathrm{ir}}[\hat{\rho}_{1}],\\ \dot{\hat{\rho}}_{2} &= -\lambda\{(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\hat{\rho}_{3} + \hat{\rho}_{3}(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\}\\ &\quad + \kappa L_{\mathrm{ir}}[\hat{\rho}_{2}],\\ \dot{\hat{\rho}}_{3} &= \lambda\{(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\hat{\rho}_{2} + \hat{\rho}_{2}(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\}\\ &\quad + \kappa L_{\mathrm{ir}}[\hat{\rho}_{3}].\end{aligned} \tag{17}$$

One can find that the first two equations and the last equation are coupled, respectively. The first two equations can be decoupled as

$$\dot{\hat{\rho}}_{\pm} = \mp\mathrm{i}\lambda\left[\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t}, \hat{\rho}_{\pm}\right] + \kappa L_{\mathrm{ir}}[\hat{\rho}_{\pm}], \tag{18}$$

where we have introduced

$$\hat{\rho}_{\pm} = \hat{\rho}_{0} \pm \hat{\rho}_{1}. \tag{19}$$

The other two equations can be decoupled. By taking

$$\hat{\rho}_{\mathrm{c}} = \hat{\rho}_{3} + \mathrm{i}\hat{\rho}_{2}, \tag{20}$$

obviously

$$\begin{aligned}\dot{\rho}_{\mathrm{c}} &= -\mathrm{i}\lambda\{(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\hat{\rho}_{\mathrm{c}} + \hat{\rho}_{\mathrm{c}}(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}\mathrm{e}^{-\mathrm{i}\omega t})\}\\ &\quad + \kappa L_{\mathrm{ir}}[\hat{\rho}_{\mathrm{c}}].\end{aligned} \tag{21}$$

It is seen that solving the equation of motion in Eq. (13) is equivalent to solving Eqs. (18) and (21).

4. Unraveling master equations through entangled state representation

In this section, we unravel the master equations (18) and (19) through (TES) representation. Therefore acting on both sides of Eq. (18) by $|\eta = 0\rangle$ and using Eq. (8), we can find a Schrödinger-like time evolution equation

$$\begin{aligned}\frac{\mathrm{d}|\rho_{\pm}(t)\rangle}{\mathrm{d}t} &= \left\{\mp\mathrm{i}\lambda\left[\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} + \hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t} - \hat{b}\mathrm{e}^{\mathrm{i}\omega t} - \hat{b}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega t}\right]\right.\\ &\quad \left. + \frac{\gamma}{2}\left(2\hat{a}\hat{b} - \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\right)\right\}|\rho_{\pm}(t)\rangle\\ &= \left\{\mp\mathrm{i}\lambda(\hat{\eta}\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{\eta}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t}) + \frac{\gamma}{2}\hat{L}\right\}\\ &= \left\{\hat{g}_{\pm}(t) + \frac{\gamma}{2}\hat{L}\right\}|\rho_{\pm}(t)\rangle,\end{aligned} \tag{22}$$

where

$$\hat{g}_{\pm}(t) = \mp \mathrm{i}\lambda(\hat{\eta}\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{\eta}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t}),$$
$$\hat{L} = (2\hat{a}\hat{b} - \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}). \tag{23}$$

By using commutation relations $\left[\hat{L},\hat{\eta}\right] = -\hat{\eta}$ and $\left[\hat{L},\hat{\eta}^{\dagger}\right] = -\hat{\eta}^{\dagger}$ one can find that

$$\left[\hat{g}_{\pm}(t) + \frac{\gamma}{2}\hat{L}, \hat{g}_{\pm}(t') + \frac{\gamma}{2}\hat{L}\right] \neq 0. \tag{24}$$

It is difficult to solve Eq. (22) therefore we define "dissipative interaction picture" in entangled representation[14,15] for $\rho_{\pm}(t)$ as follows:

$$|\rho_{\pm}(t)\rangle_{\mathrm{I}} = \mathrm{e}^{-\gamma t\hat{L}/2}|\rho_{\pm}(t)\rangle\,. \tag{25}$$

For new variable $|\rho_{\pm}(t)\rangle_{\mathrm{I}}$ we have

$$\frac{\mathrm{d}\,|\rho_{\pm}(t)\rangle_{\mathrm{I}}}{\mathrm{d}t} = \hat{g}^{\mathrm{I}}_{\pm}(t)\,|\rho_{\pm}(t)\rangle_{\mathrm{I}},$$
$$\hat{g}^{\mathrm{I}}_{\pm}(t) \triangleq \mathrm{e}^{-\gamma t\hat{L}/2}\hat{g}_{\pm}(t)\,\mathrm{e}^{\gamma t\hat{L}/2}, \tag{26}$$

where

$$\begin{aligned}\hat{g}^{\mathrm{I}}_{\pm}(t) &= \mathrm{e}^{-\gamma t\hat{L}/2}\hat{g}_{\pm}(t)\,\mathrm{e}^{\gamma t\hat{L}/2}\\ &= \mp\mathrm{i}\lambda\,\mathrm{e}^{\gamma t/2}\left\{\hat{\eta}\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{\eta}^{\dagger}\,\mathrm{e}^{\mathrm{i}\omega t}\right\}.\end{aligned} \tag{27}$$

Now we have

$$\left[\hat{g}^{\mathrm{I}}_{\pm}(t), \hat{g}^{\mathrm{I}}_{\pm}(t')\right] = 0. \tag{28}$$

Therefore we can integrate Eq. (26) and find

$$|\rho_{\pm}(t)\rangle_{\mathrm{I}} = \exp\left\{\int_0^t \mathrm{d}t'\hat{g}^{\mathrm{I}}_{\pm}(t')\right\}|\rho_{\pm}(0)\rangle_{\mathrm{s}}\,. \tag{29}$$

To find more simplified form of Eq. (29) we write

$$\left\{\int_0^t \mathrm{d}t'\hat{g}_{\mathrm{I}}(t')\right\} = \{\lambda_{\pm}(t)\hat{\eta}^{\dagger} - \lambda^{*}_{\pm}(t)\hat{\eta}\}, \tag{30}$$

where the time dependent parameter $\lambda_{\pm}(t)$ is defined as

$$\begin{aligned}\lambda_{\pm}(t) &\triangleq \mp\mathrm{i}\lambda\int_0^t \mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\omega t' + \gamma t'/2}\\ &= \mp\frac{2\mathrm{i}\lambda}{2\mathrm{i}\omega + \gamma}\{\mathrm{e}^{\gamma t/2 + \mathrm{i}\omega t} - 1\}.\end{aligned} \tag{31}$$

Then substituting Eq. (30) into Eq. (29) yields

$$|\rho_{\pm}(t)\rangle_{\mathrm{I}} = \exp\{\lambda_{\pm}(t)\hat{\eta}^{\dagger} - \lambda^{*}_{\pm}(t)\hat{\eta}\}\,|\rho_{\pm}(0)\rangle_{\mathrm{s}}\,. \tag{32}$$

Using Eqs. (8) and (10) one can decompose Eq. (32) as follows:

$$\begin{aligned}|\rho_{\pm}(t)\rangle_{\mathrm{I}} &= \exp\{\lambda_{\pm}(t)\hat{\eta}^{\dagger} - \lambda^{*}_{\pm}(t)\hat{\eta}\}\rho_{\pm}(0)\,|\eta = 0\rangle\\ &= \mathrm{e}^{\lambda_{\pm}\hat{a}^{\dagger} - \lambda^{*}_{\pm}\hat{a}}\,\mathrm{e}^{-\lambda_{\pm}\hat{b} + \lambda^{*}_{\pm}\hat{b}^{\dagger}}\rho_{\pm}(0)\,|\eta = 0\rangle\\ &= \mathrm{e}^{\lambda_{\pm}\hat{a}^{\dagger} - \lambda^{*}_{\pm}\hat{a}}\rho_{\pm}(0)\,\mathrm{e}^{-(\lambda_{\pm}\hat{a}^{\dagger} - \lambda^{*}_{\pm}\hat{a})}\,|\eta = 0\rangle\\ &= \hat{D}(\lambda_{\pm})\rho_{\pm}(0)\hat{D}^{\dagger}(\lambda_{\pm})\,|\eta = 0\rangle\,,\end{aligned} \tag{33}$$

where $\hat{D}(\lambda) \equiv \mathrm{e}^{\lambda_{\pm}\hat{a}^{\dagger} - \lambda^{*}_{\pm}\hat{a}}$ is defined as displaced operator of first mode. To find density operator $\rho_{\pm}(t)$ in Schrödinger picture we should return rotational interaction picture

$$\begin{aligned}&|\rho_{\pm}(t)\rangle\\ &= \mathrm{e}^{\gamma t2\hat{L}/2}\,|\rho_{\pm}(t)\rangle_{\mathrm{I}}\\ &= \mathrm{e}^{\gamma t\{2\hat{a}\hat{b} - \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\}/2}\,|\rho_{\pm}(t)\rangle_{\mathrm{I}}\\ &= \mathrm{e}^{\gamma t\{2\hat{a}\hat{b} - \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\}/2}\hat{D}(\lambda_{\pm})\rho_{\pm}(0)\hat{D}^{\dagger}(\lambda_{\pm})\,|\eta = 0\rangle\,.\end{aligned} \tag{34}$$

To solve the above equation one can immediately understand that this equation is similar in form to the master equation of lossy channel (or cavity at zero temperature),[16,17] therefore we have

$$\begin{aligned}\rho_{\pm}(t) &= \mathrm{e}^{-\gamma t\hat{a}^{\dagger}\hat{a}/2}\left\{\sum_{n=0}^{\infty}\frac{T^n}{n!}\hat{a}^n\hat{D}(\lambda_{\pm})\rho_{\pm}(0)\hat{D}^{\dagger}(\lambda_{\pm})\hat{a}^{\dagger n}\right\}\\ &\quad\times\mathrm{e}^{-\gamma t\hat{a}^{\dagger}\hat{a}/2}.\end{aligned} \tag{35}$$

Consequently, density operator in Schrödinger picture is as follows:

$$\begin{aligned}\hat{\rho}^{\mathrm{s}}_{\pm}(t) &= \sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-(\mathrm{i}\omega + \gamma/2)t\hat{a}^{\dagger}\hat{a}}\hat{a}^n\hat{D}(\lambda_{\pm})\hat{\rho}^{\mathrm{s}}_{\pm}(0)\\ &\quad\times\hat{D}^{\dagger}(\lambda_{\pm})\hat{a}^{\dagger n}\mathrm{e}^{(\mathrm{i}\omega - \gamma/2)t\hat{a}^{\dagger}\hat{a}}.\end{aligned} \tag{36}$$

Similarly, the explicit form of evolution of density operator $\hat{\rho}_{\mathrm{c}}$ in Eq. (21) can be derived. Acting on both sides of Eq. (21) with $|\eta = 0\rangle$ we have

$$\begin{aligned}\frac{\mathrm{d}\,|\rho_{\mathrm{c}}(t)\rangle}{\mathrm{d}t} &= \left\{\frac{\gamma}{2}(2\hat{a}\hat{b} - \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}) - \mathrm{i}\lambda(\hat{a}\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{a}^{\dagger}\,\mathrm{e}^{\mathrm{i}\omega t}\right.\\ &\quad\left.+\,\hat{b}\,\mathrm{e}^{\mathrm{i}\omega t} + \hat{b}^{\dagger}\,\mathrm{e}^{-\mathrm{i}\omega t})\right\}|\rho_{\mathrm{c}}(t)\rangle\\ &= \left\{\frac{\gamma}{2}\hat{L} + \hat{g}_{\mathrm{c}}(t)\right\}|\rho_{\mathrm{c}}\rangle\,,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\hat{g}_{\mathrm{c}}(t) &= -\mathrm{i}\lambda\{(\hat{a} + \hat{b}^{\dagger})\,\mathrm{e}^{-\mathrm{i}\omega t} + (\hat{a}^{\dagger} + \hat{b})\,\mathrm{e}^{\mathrm{i}\omega t}\}\\ &= -\mathrm{i}\lambda\{\hat{\xi}\,\mathrm{e}^{-\mathrm{i}\omega t} + \hat{\xi}^{\dagger}\,\mathrm{e}^{\mathrm{i}\omega t}\}.\end{aligned} \tag{38}$$

It is a simple task to check the following commutation relations:

$$\begin{aligned}&\left[\hat{L},\hat{\xi}\right] = 3\hat{a} - \hat{b}^{\dagger} = 2\hat{a} + \hat{\eta} = \hat{M}, \quad \left[\hat{L},\hat{M}\right] = \hat{\xi},\\ &\left[\hat{L},\hat{\xi}^{\dagger}\right] = 3\hat{b} - \hat{a}^{\dagger} = 2\hat{b} - \hat{\eta}^{\dagger} = \hat{N}, \ \left[\hat{L},\hat{N}\right] = \hat{\xi}^{\dagger}.\end{aligned} \tag{39}$$

To solve Eq. (37) we define "dissipative interaction picture" for $\rho_{\mathrm{c}}(t)$ as follows:

$$|\rho_{\mathrm{c}}(t)\rangle_{\mathrm{I}} = \mathrm{e}^{-\gamma t\hat{L}/2}\,|\rho_{\mathrm{c}}(t)\rangle\,. \tag{40}$$

Therefore, we have

$$\frac{\mathrm{d}\,|\rho_{\mathrm{c}}(t)\rangle_{\mathrm{I}}}{\mathrm{d}t} = \hat{g}^{\mathrm{I}}_{\mathrm{c}}(t)\,|\rho_{\mathrm{c}}(t)\rangle_{\mathrm{I}},$$
$$\hat{g}^{\mathrm{I}}_{\mathrm{c}}(t) = \mathrm{e}^{-\gamma t\hat{L}/2}\hat{g}_{\mathrm{c}}(t)\,\mathrm{e}^{\gamma t\hat{L}/2}. \tag{41}$$

Making use of commutation relations (39) one can find

$$\begin{aligned}\hat{g}_{\rm c}^{\rm I}(t) &= -{\rm i}\lambda\,{\rm e}^{-\gamma[\hat{L},\cdots]/2}\{\hat{\xi}\,{\rm e}^{-{\rm i}\omega t}+\hat{\xi}^{\dagger}\,{\rm e}^{{\rm i}\omega t}\}\\ &= -{\rm i}\lambda\left\{{\rm e}^{-{\rm i}\omega t}\left(\hat{\xi}\cosh\frac{\gamma t}{2}-\hat{M}\sinh\frac{\gamma t}{2}\right)\right.\\ &\quad\left.+\,{\rm e}^{{\rm i}\omega t}\left(\hat{\xi}^{\dagger}\cosh\frac{\gamma t}{2}-\hat{N}\sinh\frac{\gamma t}{2}\right)\right\}. \qquad (42)\end{aligned}$$

Because of $\left[\hat{g}_{\rm c}^{\rm I}(t),\hat{g}_{\rm c}^{\rm I}(t')\right]\neq 0$ we cannot integrate Eq. (40), thus we define new operators $\hat{A}(t),\hat{B}(t)$ as follows:

$$\begin{aligned}\hat{A}(t) &= -{\rm i}\lambda\cosh\frac{\gamma t}{2}(\hat{\xi}\,{\rm e}^{-{\rm i}\omega t}+\hat{\xi}^{\dagger}\,{\rm e}^{{\rm i}\omega t}),\\ \hat{B}(t) &= {\rm i}\lambda\sinh\frac{\gamma t}{2}(\hat{M}\,{\rm e}^{-{\rm i}\omega t}+\hat{N}\,{\rm e}^{{\rm i}\omega t}). \qquad (43)\end{aligned}$$

Now by using the formula in appendix we can decompose Eq. (41) and find

$$\begin{aligned}|\rho_{\rm c}(t)\rangle_{\rm I} &= \exp\left\{\int_0^t\hat{B}(s)\,{\rm d}s\right\}\exp\left\{\int_0^t\hat{A}(s)\,{\rm d}s\right\}\\ &\quad\times\exp\left\{\int_0^t{\rm d}s\int_0^{\rm s}{\rm d}s' f(s,s')\right\}|\rho_{\rm c}(0)\rangle_{\rm I}, \qquad (44)\end{aligned}$$

where

$$\begin{aligned}&f(s,s')\\ &= \left[\hat{A}(s),\hat{B}(s')\right]\\ &= -8\lambda^2\left(\cosh\left(\frac{\gamma}{2}s\right)\sinh\left(\frac{\gamma}{2}s'\right)\cos\left(\omega(s-s')\right)\right), \qquad (45)\end{aligned}$$

and

$$\int_0^t{\rm d}s\int_0^{\rm s}f(s,s')\,{\rm d}s'=F(t). \qquad (46)$$

Therefore, we have

$$\begin{aligned}&|\rho_{\rm c}(t)\rangle_{\rm I}\\ &= {\rm e}^{F(t)}\exp\left\{{\rm i}\lambda\int_0^t\sinh\left(\frac{\gamma s}{2}\right)(\hat{M}\,{\rm e}^{-{\rm i}\omega s}+\hat{N}\,{\rm e}^{{\rm i}\omega s})\,{\rm d}s\right\}\\ &\quad\times\exp\left\{-{\rm i}\lambda\int_0^t\cosh\left(\frac{\gamma s}{2}\right)(\hat{\xi}\,{\rm e}^{-{\rm i}\omega s}+\hat{\xi}^{\dagger}\,{\rm e}^{{\rm i}\omega s})\,{\rm d}s\right\}\\ &\quad\times|\rho_{\rm c}(0)\rangle_{\rm I}\\ &= {\rm e}^{F(t)}e^{\mu_2^*(t)\hat{M}-\mu_2(t)\hat{N}}\,{\rm e}^{\mu_1(t)\hat{\xi}^{\dagger}-\mu_1^*(t)\hat{\xi}}\,|\rho_{\rm c}(0)\rangle_{\rm I}, \qquad (47)\end{aligned}$$

where

$$\begin{aligned}\mu_1(t) &\equiv -{\rm i}\lambda\int_0^t\cosh\left(\frac{\gamma s}{2}\right){\rm e}^{{\rm i}\omega s}\,{\rm d}s,\\ \mu_2(t) &\equiv -{\rm i}\lambda\int_0^t\sinh\left(\frac{\gamma s}{2}\right){\rm e}^{{\rm i}\omega s}\,{\rm d}s. \qquad (48)\end{aligned}$$

To find the density operator $\rho_{\rm c}(t)$ we decompose any of the operators ${\rm e}^{\mu_1(t)\hat{\xi}^{\dagger}-\mu_1^*(t)\hat{\xi}}$ and ${\rm e}^{\mu_2^*(t)\hat{M}-\mu_2(t)\hat{N}}$ as a product of two operators: one is related to the physical mode and the other to the fictitious mode, therefore we have

$$\begin{aligned}&{\rm e}^{\mu_1(t)\hat{\xi}^{\dagger}-\mu_1^*(t)\hat{\xi}}\,|\rho_{\rm c}(0)\rangle_{\rm s}\\ &= {\rm e}^{\mu_1(\hat{a}^{\dagger}+\hat{b})-\mu_1^*(\hat{a}+\hat{b}^{\dagger})}\hat{\rho}_{\rm c}(0)\,|\eta=0\rangle\\ &= {\rm e}^{\mu_1\hat{a}^{\dagger}-\mu_1^*\hat{a}}e^{\mu_1\hat{b}-\mu_1^*\hat{b}^{\dagger}}\hat{\rho}_{\rm c}(0)\,|\eta=0\rangle\\ &= {\rm e}^{\mu_1\hat{a}^{\dagger}-\mu_1^*\hat{a}}\hat{\rho}_{\rm c}(0)e^{\mu_1\hat{a}^{\dagger}-\mu_1^*\hat{a}}\,|\eta=0\rangle\\ &= \hat{D}(\mu_1)\hat{\rho}_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\,|\eta=0\rangle, \qquad (49)\end{aligned}$$

and

$$\begin{aligned}&{\rm e}^{\mu_2^*\hat{M}-\mu_2\hat{N}}e^{\mu_1^*(t)\hat{\xi}-\mu_1(t)\hat{\xi}^{\dagger}}\,|\rho_{\rm c}(0)\rangle_{\rm I}\\ &= {\rm e}^{\mu_2^*(\hat{\eta}+2\hat{a})}\,{\rm e}^{\mu_2(\hat{\eta}^{\dagger}-2\hat{b})}\hat{D}(\mu_1)\rho_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\,|\eta=0\rangle\\ &= {\rm e}^{2\mu_2^*\hat{a}}e^{\mu_2^*\hat{\eta}}\,{\rm e}^{\mu_2\hat{\eta}^{\dagger}}\hat{D}(\mu_1)\rho_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)e^{-2\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle\\ &= {\rm e}^{2\mu_2^*\hat{a}}e^{\mu_2^*\hat{a}+\mu_2\hat{a}^{\dagger}}\,{\rm e}^{-\mu_2\hat{b}-\mu_2^*\hat{b}^{\dagger}}\hat{D}(\mu_1)\rho_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\\ &\quad\times{\rm e}^{-2\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle\\ &= {\rm e}^{2\mu_2^*\hat{a}}\,{\rm e}^{\mu_2^*\hat{a}+\mu_2\hat{a}^{\dagger}}\hat{D}(\mu_1)\rho_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)e^{-2\mu_2\hat{a}^{\dagger}}\\ &\quad\times{\rm e}^{-\mu_2\hat{a}^{\dagger}-\mu_2^*\hat{a}}\,|\eta=0\rangle. \qquad (50)\end{aligned}$$

By means of ${\rm e}^{\hat{A}+\hat{B}}={\rm e}^{\hat{A}}\,{\rm e}^{\hat{B}}\,{\rm e}^{-[\hat{A},\hat{B}]/2}$, one can recast Eq. (50) into

$$\begin{aligned}|\rho_{\rm c}(t)\rangle_{\rm I} &= {\rm e}^{F(t)}e^{2\mu_2^*\hat{a}}e^{\mu_2^*\hat{a}+\mu_2\hat{a}^{\dagger}}\hat{D}(\mu_1)\hat{\rho}_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\\ &\quad\times{\rm e}^{-2\mu_2\hat{a}^{\dagger}}\,{\rm e}^{-\mu_2\hat{a}^{\dagger}-\mu_2^*\hat{a}}\,|\eta=0\rangle\\ &= {\rm e}^{F(t)-2|\mu_2|^2}\,{\rm e}^{2\mu_2^*\hat{a}}e^{\mu_2^*\hat{a}+\mu_2\hat{a}^{\dagger}}\hat{D}(\mu_1)\hat{\rho}_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\\ &\quad\times{\rm e}^{-\mu_2\hat{a}^{\dagger}-\mu_2^*\hat{a}}e^{-2\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle\\ &= {\rm e}^{F(t)-2|\mu_2|^2}\,{\rm e}^{2\mu_2^*\hat{a}}\,{\rm e}^{2\mu_2^*\hat{a}-\mu_2^*\hat{a}+\mu_2\hat{a}^{\dagger}}\hat{D}(\mu_1)\hat{\rho}_{\rm c}(0)\\ &\quad\times\hat{D}^{\dagger}(-\mu_1)\,{\rm e}^{-2\mu_2\hat{a}^{\dagger}+\mu_2\hat{a}^{\dagger}-\mu_2^*\hat{a}}\,{\rm e}^{-2\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle\\ &= {\rm e}^{F(t)-4|\mu_2|^2}\,{\rm e}^{4\mu_2^*\hat{a}}\hat{D}(\mu_2)\hat{D}(\mu_1)\hat{\rho}_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1)\\ &\quad\times\hat{D}^{\dagger}(-\mu_2)\,{\rm e}^{-4\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle\\ &= {\rm e}^{F(t)-4|\mu_2|^2+\mu_1\mu_2^*+\mu_1^*\mu_2}\,{\rm e}^{4\mu_2^*\hat{a}}\hat{D}(\mu_1+\mu_2)\hat{\rho}_{\rm c}(0)\\ &\quad\times\hat{D}^{\dagger}(-\mu_1-\mu_2)\,{\rm e}^{-4\mu_2\hat{a}^{\dagger}}\,|\eta=0\rangle, \qquad (51)\end{aligned}$$

where we have employed the following relations:

$$\begin{aligned}\hat{D}(\alpha)\hat{D}(\beta) &= \exp\left\{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)\right\}\hat{D}(\alpha+\beta),\\ \hat{D}^{\dagger}(\alpha)\hat{D}^{\dagger}(\beta) &= \exp\left\{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)\right\}\hat{D}^{\dagger}(\alpha+\beta). \qquad (52)\end{aligned}$$

Now using the result of the master equation lossy channel,[17] we can find the density operator $\hat{\rho}_{\rm c}(t)$ in the Schrödinger picture

$$\begin{aligned}\hat{\rho}_{\rm c}^{\rm s}(t) &= {\rm e}^{F(s,s')+\mu_1\mu_2^*+\mu_1^*\mu_2-4|\mu_2|^2}\sum_{n=0}^{\infty}\frac{T^n}{n!}\,{\rm e}^{-\left({\rm i}\omega+\frac{\gamma}{2}\right)t\hat{a}^{\dagger}\hat{a}}\hat{a}^n\\ &\quad\times{\rm e}^{4\mu_2^*\hat{a}}\hat{D}(\mu_1+\mu_2)\hat{\rho}_{\rm c}(0)\hat{D}^{\dagger}(-\mu_1-\mu_2)\\ &\quad\times{\rm e}^{-4\mu_2\hat{a}^{\dagger}}\hat{a}^{\dagger n}\,{\rm e}^{-\left(\frac{\gamma}{2}-{\rm i}\omega\right)t\hat{a}^{\dagger}\hat{a}}. \qquad (53)\end{aligned}$$

5. Wigner function for density operator $\hat{\rho}_{\pm}$

In this section, we find the Wigner function of density operator (see Eq. (36)) for a given initial state. Suppose that the atom is initially in the up state $|\uparrow\rangle$ and the cavity mode is in a coherent state $|\alpha_0\rangle$, then the initial condition for density operator will be

$$\rho_0=|\alpha_0\rangle\langle\alpha_0|\otimes|\uparrow\rangle\langle\uparrow|. \qquad (54)$$

Then substituting Eq. (54) into Eqs. (19) and (20) yields

$$\rho_{\pm}(0)=\rho_{\rm c}(0)=|\alpha_0\rangle\langle\alpha_0|. \tag{55}$$

Noticing

$$\begin{aligned}&{\rm e}^{-\mu\hat{a}^{\dagger}\hat{a}}|\alpha_0\rangle\langle\alpha_0|{\rm e}^{-\mu^*\hat{a}^{\dagger}\hat{a}}\\ &=\exp\{|\alpha_0|^2({\rm e}^{-(\mu+\mu^*)}-1)\}\left|{\rm e}^{-\mu}\alpha_0\right\rangle\left\langle{\rm e}^{-\mu}\alpha_0\right|,\end{aligned} \tag{56}$$

equation (36) becomes

$$\begin{aligned}\hat{\rho}^{\rm s}_{\pm}(t)=&\left|{\rm e}^{-({\rm i}\omega+\gamma/2)t}(\alpha_0+\lambda_{\pm})\right\rangle\\ &\times\left\langle{\rm e}^{-({\rm i}\omega+\gamma/2)t}(\alpha_0+\lambda_{\pm})\right|.\end{aligned} \tag{57}$$

Equation (57) shows that if the initial condition for density operator $\rho_{\pm}(0)$ is the coherent state $|\alpha_0\rangle$, then at later times it continues to retain the coherent state with a time dependent complex parameter

$$\alpha_0(t)={\rm e}^{-({\rm i}\omega+\gamma/2)t}(\alpha_0+\lambda_{\pm}). \tag{58}$$

Now, by inserting Eq. (31) into Eq. (58), we arrive at

$$\alpha_0(t)=\mp\frac{2{\rm i}\lambda}{2{\rm i}\omega+\gamma}+{\rm e}^{-({\rm i}\omega+\gamma/2)t}\left(\alpha_0\pm\frac{2{\rm i}\lambda}{2{\rm i}\omega+\gamma}\right). \tag{59}$$

The Wigner function for density operator is defined as

$$W(\alpha,t)={\rm Tr}\{\hat{U}_0(\alpha)\hat{\rho}(t)\}, \tag{60}$$

where the Wigner operator $\hat{U}_0(\alpha)$ has the form

$$\hat{U}_0(\alpha)=2:\exp\{-2(\alpha-\hat{a})(\alpha^*-\hat{a}^{\dagger})\}:. \tag{61}$$

Inserting Eqs. (57) and (60) into Eq. (59) yields

$$W_{\pm}(\alpha,t)=2\exp\{-2|\alpha-\alpha_0(t)|^2\}. \tag{62}$$

Obviously, for $\gamma t\gg 1$ one can find

$$\alpha_0(t)=\mp\frac{2{\rm i}\lambda}{2{\rm i}\omega+\gamma},$$

which is independent of the initial state, and this means that the system loses its memory. By the same method one can find $W_{\rm c}(\alpha,t)$ for $\hat{\rho}^{\rm s}_{\rm c}(t)$.

6. Conclusions

In this paper, we for the first time introduce a new type of the interaction picture related to dissipative processes, dissipative interaction picture in entangled state representation, and employ this picture to treat the time evolution of density operators in the J–C model with cavity damping. This technical method also provides us with a fresh view of solving master equation in another problem. Moreover we derive the Wigner function $W_{\pm}(\alpha,t)$ of density operator for special initial condition and we show that the system loses its memory when $\gamma t\gg 1$.

Appendix A

Here, we reproduce the proof of Eq. (44) given in Ref. [1]. Suppose that

$$\frac{\rm d}{{\rm d}t}\hat{V}(t)=\{\hat{A}(t)+\hat{B}(t)\}\hat{V}(t),\ \ \hat{V}(0)=1, \tag{A1}$$

where

$$\begin{aligned}&\left[\hat{A}(t),\hat{A}(t')\right]=\left[\hat{B}(t),\hat{B}(t')\right]=0,\\ &\left[\hat{A}(t),\hat{B}(t')\right]=f(t,t').\end{aligned} \tag{A2}$$

It is difficult to solve Eq. (A1) because of commutation relation $\left[\hat{A}(t)+\hat{B}(t),\hat{A}(t')+\hat{B}(t')\right]\neq 0$. In order to solve this problem, we defined new operator $\hat{U}(t)$ as follows:

$$\begin{aligned}\hat{U}(t)&=\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{V}(t),\\ \hat{V}(t)&=\exp\left\{\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{U}(t).\end{aligned} \tag{A3}$$

Therefore we have

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\hat{U}(t)=&\frac{\rm d}{{\rm d}t}\left(\exp\{-\int_0^t\hat{B}(t'){\rm d}t'\}\right)\hat{V}(t)\\ &+\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\frac{\rm d}{{\rm d}t}\hat{V}(t)\\ =&-\hat{B}(t)\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{V}(t)\\ &+\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\{\hat{A}(t)+\hat{B}(t)\}\hat{V}(t)\\ =&\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{A}(t)\hat{V}(t).\end{aligned} \tag{A4}$$

Substituting Eq. (A3) into Eq. (A4) yields

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\hat{U}(t)=&\exp\left\{-\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{A}(t)\exp\left\{\int_0^t\hat{B}(t'){\rm d}t'\right\}\hat{U}(t)\\ =&\left\{\hat{A}(t)-\int_0^t\left[\hat{B}(t'),\hat{A}(t)\right]{\rm d}t'\right.\\ &\left.+\frac{1}{2}\int_0^t\left[\hat{B}(t'),\left[\hat{B}(t'),\hat{A}(t)\right]\right]+\ldots\right\}\hat{U}(t)\\ =&\left\{\hat{A}(t)+\int_0^t f(t,t'){\rm d}t'\right\}\hat{U}(t).\end{aligned} \tag{A5}$$

Obviously, because of commutation relation

$$\left[\hat{A}(t)+\int_0^t f(t,s){\rm d}s,\hat{A}(t')+\int_0^{t'}f(t',s'){\rm d}s'\right]=0, \tag{A6}$$

we can easily integrate Eq. (A5) and find

$$\begin{aligned}\hat{V}(t)=&\exp\left\{\int_0^t{\rm d}s\hat{B}(s)\right\}\exp\left\{\int_0^t{\rm d}s\hat{A}(s)\right\}\\ &\times\exp\left\{\int_0^t{\rm d}s\int_0^s f(s,s'){\rm d}s'\right\}.\end{aligned} \tag{A7}$$

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