New 3-mode bosonic operator realization of SU(2) Lie algebra: From the point of view of squeezing*

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(Received 5 February 2014; revised manuscript received 31 March 2014; published online 16 July 2014)

We consider the quantum mechanical SU(2) transformation $e^{2\lambda J_z}J_\pm e^{-2\lambda J_z}=e^{\pm 2\lambda}J_\pm$ as if the meaning of squeezing with $e^{\pm 2\lambda}$ being squeezing parameter. By studying SU(2) operators $(J_\pm,J_z)$ from the point of view of squeezing we find that $(J_\pm,J_z)$ can also be realized in terms of 3-mode bosonic operators. Employing this realization, we find the natural representation (the eigenvectors of J_+ or J_-) of the 3-mode squeezing operator $e^{2\lambda J_z}$. The idea of considering quantum SU(2) transformation as if squeezing is liable for us to obtain the new bosonic operator realization of SU(2) and new squeezing operators.

Keywords: quantum mechanical SU(2) transformation, 3-mode squeezing operator, new 3-mode bosonic operator realization of SU(2) Lie algebra

PACS: 03.65.−w, 42.50.−p **DOI:** 10.1088/1674-1056/23/9/090302

1. Introduction

In quantum optics theory, squeezing and squeezed states are very important topics. However, how to find multi-mode squeezing mechanism has been a tough task. In this paper we provide a new approach for deriving new multi-mode squeezing operators by trying to relate quantum mechanical SU(2) transformation to squeezing transformation.

In history Schwinger ever pointed out that angular momentum operators (SU(2) generators) could be expressed by virtue of two-mode bose operators[1] with $[a_j,a_i^\dagger]=\delta_{i,j}$

$$J_+=a_1^\dagger a_2,\quad J_-=a_2^\dagger a_1,\quad J_z=\frac{1}{2}\left(a_1^\dagger a_1-a_2^\dagger a_2\right),\tag{1}$$

which obey

$$[J_+,J_-]=2J_z,\quad [J_\pm,J_z]=\mp J_\pm.\tag{2}$$

By noticing

$$e^{2\lambda J_z}J_\pm e^{-2\lambda J_z}=e^{\pm 2\lambda}J_\pm,\tag{3}$$

and comparing with $e^{i\lambda(XP+PX)}Xe^{-i\lambda(XP+PX)}=e^{2\lambda}X$, $e^{i\lambda(XP+PX)}Pe^{-i\lambda(XP+PX)}=e^{-2\lambda}P$, we may consider Eq. (3) as if a squeezing transformation[2] for $J_\pm$ with the squeezing parameter $e^{\pm 2\lambda}$, i.e., $e^{2\lambda J_z}$ is the squeezing operator for $J_\pm$. In this way we have found an alternate form of $J_\pm$, J_z[2]

$$\begin{aligned}J_+&=\frac{1}{2}(a_1-a_2^\dagger)(a_1^\dagger-a_2),\\ J_-&=\frac{1}{2}(a_1+a_2^\dagger)(a_1^\dagger+a_2),\\ J_z&=\frac{1}{2}(a_1^\dagger a_2^\dagger-a_1a_2),\end{aligned}\tag{4}$$

which satisfy both Eqs. (2) and (3), for example,

$$e^{\lambda(a_1^\dagger a_2^\dagger-a_1a_2)}(a_1-a_2^\dagger)e^{-\lambda(a_1^\dagger a_2^\dagger-a_1a_2)}=e^{\lambda}(a_1-a_2^\dagger),$$

$$\begin{aligned}&e^{\lambda(a_1^\dagger a_2^\dagger-a_1a_2)}\frac{1}{2}(a_1-a_2^\dagger)(a_1^\dagger-a_2)e^{-\lambda(a_1^\dagger a_2^\dagger-a_1a_2)}\\ &=e^{2\lambda}\frac{1}{2}(a_1-a_2^\dagger)(a_1^\dagger-a_2),\end{aligned}\tag{5}$$

and

$$S_2=\exp\left\{\lambda\left[a_1^\dagger a_2^\dagger-a_1a_2\right]\right\}=e^{2\lambda J_z}\tag{6}$$

is just the two-mode squeezing operator.[3,4] Noting that in this case J_z is not hermite, and J_+ is not the hermite conjuate of J_-.

In this work, we demonstrate that the idea of studying the quantum mechanical SU(2) transformation from the point of view of squeezing is liable for us to obtain the new 3-mode squeezing operator, and we shall derive new SU(2) realization in terms of 3-mode bosonic operators and then employ it to obtain the natural representation (the eigenvectors of J_+ or J_-) of the 3-mode squeezing operator $e^{2\lambda J_z}$.

2. New 3-mode bosonic operator realization of SU(2) algebra and 3-mode squeezing operator

Recalling the single-mode squeezing operator[4,5] is

$$S_1=\exp\left[\lambda\left(a^2-a^{\dagger 2}\right)\right].\tag{7}$$

Combining Eq. (6) and Eq. (7), we propose the 3-mode squeezing operator as

$$\begin{aligned}S_3=\exp\Bigg\{\frac{\lambda}{3}\Bigg[&\sum_{i=1}^{3}\left(a_i^2-a_i^{\dagger 2}\right)+a_1^\dagger a_2^\dagger+a_1^\dagger a_3^\dagger+a_2^\dagger a_3^\dagger\\ &-a_1a_2-a_1a_3-a_2a_3\Bigg]\Bigg\}.\end{aligned}\tag{8}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11175113 and 11275123) and the Key Project of Natural Science Fund of Anhui Province, China (Grant No. KJ2013A261).

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Obviously, $S_3^{-1}=S_3^{\dagger}$. Using the Baker–Hausdorff formula, we can deduce

$$\begin{aligned}
S_3a_1S_3^{-1} &= \frac{(a_1+a_2+a_3)}{3}+\frac{\left(2a_1^{\dagger}-a_2^{\dagger}-a_3^{\dagger}\right)}{3}\sinh\lambda\\
&\quad+\frac{(2a_1-a_2-a_3)}{3}\cosh\lambda,\\
S_3a_2S_3^{-1} &= \frac{(a_1+a_2+a_3)}{3}+\frac{\left(2a_2^{\dagger}-a_1^{\dagger}-a_3^{\dagger}\right)}{3}\sinh\lambda\\
&\quad+\frac{(2a_2-a_1-a_3)}{3}\cosh\lambda,\\
S_3a_3S_3^{-1} &= \frac{(a_1+a_2+a_3)}{3}+\frac{\left(2a_3^{\dagger}-a_1^{\dagger}-a_2^{\dagger}\right)}{3}\sinh\lambda\\
&\quad+\frac{(2a_3-a_1-a_2)}{3}\cosh\lambda, \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
S_3a_1^{\dagger}S_3^{-1} &= \frac{\left(a_1^{\dagger}+a_2^{\dagger}+a_3^{\dagger}\right)}{3}+\frac{(2a_1-a_2-a_3)}{3}\sinh\lambda\\
&\quad+\frac{\left(2a_1^{\dagger}-a_2^{\dagger}-a_3^{\dagger}\right)}{3}\cosh\lambda,\\
S_3a_2^{\dagger}S_3^{-1} &= \frac{\left(a_1^{\dagger}+a_2^{\dagger}+a_3^{\dagger}\right)}{3}+\frac{(2a_2-a_1-a_3)}{3}\sinh\lambda\\
&\quad+\frac{\left(2a_2^{\dagger}-a_1^{\dagger}-a_3^{\dagger}\right)}{3}\cosh\lambda,\\
S_3a_3^{\dagger}S_3^{-1} &= \frac{\left(a_1^{\dagger}+a_2^{\dagger}+a_3^{\dagger}\right)}{3}+\frac{(2a_3-a_1-a_2)}{3}\sinh\lambda\\
&\quad+\frac{\left(2a_3^{\dagger}-a_1^{\dagger}-a_2^{\dagger}\right)}{3}\cosh\lambda. \qquad (10)
\end{aligned}$$

It then follows

$$\begin{aligned}
&S_3\left(a_3+a_3^{\dagger}-a_2-a_2^{\dagger}\right)S_3^{-1}=\left(a_3+a_3^{\dagger}-a_2-a_2^{\dagger}\right)\mathrm{e}^{\lambda},\\
&S_3\left(a_2+a_2^{\dagger}+a_3+a_3^{\dagger}-2a_1-2a_1^{\dagger}\right)S_3^{-1}\\
&=\left(a_2+a_2^{\dagger}+a_3+a_3^{\dagger}-2a_1-2a_1^{\dagger}\right)\mathrm{e}^{\lambda}. \qquad (11)
\end{aligned}$$

Thus

$$\begin{aligned}
&S_3\left(a_3+a_3^{\dagger}-a_2-a_2^{\dagger}\right)\left(a_2+a_2^{\dagger}+a_3+a_3^{\dagger}-2a_1-2a_1^{\dagger}\right)S_3^{-1}\\
&=\mathrm{e}^{2\lambda}\left(a_3+a_3^{\dagger}-a_2-a_2^{\dagger}\right)\left(a_2+a_2^{\dagger}+a_3+a_3^{\dagger}-2a_1-2a_1^{\dagger}\right), \qquad (12)\\
&S_3\left(a_2-a_2^{\dagger}-a_3+a_3^{\dagger}\right)\left(a_2-a_2^{\dagger}+a_3-a_3^{\dagger}-2a_1+2a_1^{\dagger}\right)S_3^{-1}\\
&=\left(a_2-a_2^{\dagger}-a_3+a_3^{\dagger}\right)\left(a_2-a_2^{\dagger}+a_3-a_3^{\dagger}-2a_1+2a_1^{\dagger}\right)\mathrm{e}^{-2\lambda}. \qquad (13)
\end{aligned}$$

Comparing Eqs. (12) and (13) with Eq. (3), we consider Eqs. (12) and (13) as the 3-mode squeezing transformation, and we find that the new 3-mode bosonic operator realization of SU(2) generators are

$$\mathrm{e}^{2\lambda J_z}=S_3, \qquad (14)$$

$$\begin{aligned}
J_+ &= \frac{1}{4\sqrt{3}}\left(a_3+a_3^{\dagger}-a_2-a_2^{\dagger}\right)\left(a_2+a_2^{\dagger}+a_3+a_3^{\dagger}-2a_1-2a_1^{\dagger}\right)\\
&= \frac{1}{2\sqrt{3}}(x_3-x_2)(x_2+x_3-2x_1), \qquad (15)
\end{aligned}$$

where we have used[6]

$$x_j=\frac{a_j+a_j^{\dagger}}{\sqrt{2}},\quad (j=1,2,3), \qquad (16)$$

and

$$\begin{aligned}
J_- &= \frac{1}{4\sqrt{3}}\left(a_2-a_2^{\dagger}-a_3+a_3^{\dagger}\right)\left(a_2-a_2^{\dagger}+a_3-a_3^{\dagger}-2a_1+2a_1^{\dagger}\right)\\
&= \frac{1}{2\sqrt{3}}(p_3-p_2)(p_2+p_3-2p_1), \qquad (17)
\end{aligned}$$

where we have used[6]

$$p_j=\frac{a_j-a_j^{\dagger}}{\sqrt{2}\mathrm{i}},\quad (j=1,2,3), \qquad (18)$$

as well as

$$\begin{aligned}
J_z &= \frac{1}{6}\left[\sum_{i=1}^{3}\left(a_i^2-a_i^{\dagger 2}\right)+a_1^{\dagger}a_2^{\dagger}+a_1^{\dagger}a_3^{\dagger}+a_2^{\dagger}a_3^{\dagger}-a_1a_2-a_1a_3-a_2a_3\right]\\
&= \frac{\mathrm{i}}{12}\left[(p_2+p_3-2p_1)(x_2+x_3-2x_1)\right.\\
&\quad\left.+3(x_3-x_2)(p_3-p_2)\right]. \qquad (19)
\end{aligned}$$

While $J_{\pm}$ are Hermite operators, J_z is anti-hermite

$$J_z^{\dagger}=-J_z. \qquad (20)$$

3. The eigenvector of $J_{\pm}$ and the natural representation of the 3-mode squeezing operator $\mathrm{e}^{2\lambda J_z}$

Since $J_{\pm}$ both are Hermite operators, from $\mathrm{e}^{2\lambda J_z}J_{\pm}\mathrm{e}^{-2\lambda J_z}=\mathrm{e}^{\pm 2\lambda}J_{\pm}$, we conclude that J_- possesses its eigenvector $|\varphi\rangle$

$$J_-|\varphi\rangle=\varphi|\varphi\rangle, \qquad (21)$$

where φ is real, $|\varphi\rangle$ is orthnormal and complete, then

$$\begin{aligned}
\mathrm{e}^{2\lambda J_z}J_-|\varphi\rangle &= \mathrm{e}^{2\lambda J_z}J_-\mathrm{e}^{-2\lambda J_z}\mathrm{e}^{2\lambda J_z}|\varphi\rangle\\
&= \mathrm{e}^{-2\lambda}J_-\mathrm{e}^{2\lambda J_z}|\varphi\rangle\\
&= \varphi\mathrm{e}^{2\lambda J_z}|\varphi\rangle, \qquad (22)
\end{aligned}$$

so

$$J_-\mathrm{e}^{2\lambda J_z}|\varphi\rangle=\varphi\mathrm{e}^{2\lambda}\mathrm{e}^{2\lambda J_z}|\varphi\rangle. \qquad (23)$$

Thus, $\mathrm{e}^{2\lambda J_z}|\varphi\rangle$ is the eigenvector of J_- with the eigenvalue being $\varphi\mathrm{e}^{2\lambda}$, which indicates

$$\mathrm{e}^{2\lambda J_z}|\varphi\rangle=A(\lambda)\left|\varphi\mathrm{e}^{2\lambda}\right\rangle, \qquad (24)$$

where $A(\lambda)$ is real and can be determined by the unitarity of $\mathrm{e}^{2\lambda J_z}$, i.e., by using

$$\int|\varphi\rangle\langle\varphi|\,\mathrm{d}\varphi=1,\langle\varphi|\,\varphi'\rangle=\delta(\varphi-\varphi'), \qquad (25)$$

we have

$$\mathrm{e}^{2\lambda J_z} = A(\lambda)\int \left|\varphi \mathrm{e}^{2\lambda}\right\rangle \langle\varphi|\,\mathrm{d}\varphi, \tag{26}$$

then

$$\begin{aligned}\left(\mathrm{e}^{2\lambda J_z}\right)^{\dagger} &= A(\lambda)\int |\varphi\rangle \left\langle \varphi \mathrm{e}^{2\lambda}\right| \mathrm{d}\varphi \\ &= \mathrm{e}^{-2\lambda J_z} \\ &= A(-\lambda)\int \left|\varphi \mathrm{e}^{-2\lambda}\right\rangle \langle\varphi|\,\mathrm{d}\varphi \\ &= A(-\lambda)\int |\varphi\rangle \left\langle \varphi \mathrm{e}^{2\lambda}\right| \mathrm{e}^{2\lambda}\,\mathrm{d}\varphi. \end{aligned} \tag{27}$$

Hence, $A(\lambda) = A(-\lambda)\,\mathrm{e}^{2\lambda}$, and

$$A(\lambda) = \mathrm{e}^{\lambda}, \tag{28}$$

then we have the natural representation of the 3-mode squeezing operator $\mathrm{e}^{2\lambda J_z}$

$$\mathrm{e}^{2\lambda J_z} = \mathrm{e}^{\lambda}\int \mathrm{d}\varphi \left|\varphi \mathrm{e}^{2\lambda}\right\rangle \langle\varphi|. \tag{29}$$

Similarly, J_+ possesses its eigenvector $|\psi\rangle$,

$$J_+|\psi\rangle = \psi|\psi\rangle, \tag{30}$$

$\mathrm{e}^{2\lambda J_z}$, as a squeezing operator, has a natural representation

$$\mathrm{e}^{2\lambda J_z} = \mathrm{e}^{-\lambda}\int \mathrm{d}\psi \left|\psi \mathrm{e}^{-2\lambda}\right\rangle \langle\psi|. \tag{31}$$

Finally, according to the method in Ref. [7] we may directly write down the explicit form of $|\varphi\rangle$

$$\begin{aligned}&|\varphi(x,r_1,r_2)\rangle \\ &= \frac{1}{\sqrt{6}}\pi^{-3/4}\exp\left\{\frac{1}{6}\sum_{j=1}^{3}\left(a_j^{\dagger 2}\right) - \frac{1}{12}\left(2x^2+r_1^2+3r_2^2\right)\right. \\ &\quad -\frac{2}{3}\left(a_1^{\dagger}a_2^{\dagger}+a_1^{\dagger}a_3^{\dagger}+a_2^{\dagger}a_3^{\dagger}\right)+\frac{\sqrt{2}}{3}x\left(a_1^{\dagger}+a_2^{\dagger}+a_3^{\dagger}\right) \\ &\quad \left.+\frac{\sqrt{2}}{6}\mathrm{i}r_1\left(a_2^{\dagger}+a_3^{\dagger}-2a_1^{\dagger}\right)+\frac{\sqrt{2}}{2}\mathrm{i}r_2\left(a_3^{\dagger}-a_2^{\dagger}\right)\right\}|000\rangle, \end{aligned} \tag{32}$$

and

$$\begin{aligned}&|\psi(p,s_1,s_2)\rangle \\ &= \frac{1}{\sqrt{6}}\pi^{-3/4}\exp\left\{-\frac{1}{6}\sum_{j=1}^{3}\left(a_j^{\dagger 2}\right) - \frac{1}{12}\left(2p^2+s_1^2+3s_2^2\right)\right. \\ &\quad +\frac{2}{3}\left(a_1^{\dagger}a_2^{\dagger}+a_1^{\dagger}a_3^{\dagger}+a_2^{\dagger}a_3^{\dagger}\right)+\frac{\sqrt{2}}{3}\mathrm{i}p\left(a_1^{\dagger}+a_2^{\dagger}+a_3^{\dagger}\right) \\ &\quad \left.+\frac{\sqrt{2}}{6}s_1\left(a_2^{\dagger}+a_3^{\dagger}-2a_1^{\dagger}\right)+\frac{\sqrt{2}}{2}s_2\left(a_3^{\dagger}-a_2^{\dagger}\right)\right\}|000\rangle, \end{aligned} \tag{33}$$

where x, r_1, r_2, and p, s_1, s_2 are real parameters. Denoting $|\varphi(x,r_1,r_2)\rangle$ as $|\varphi\rangle$ and using $\left[a_i, :f\left(a_i,a_i^{\dagger}\right):\right] =: \frac{\partial}{\partial a_i^{\dagger}} f\left(a_i,a_i^{\dagger}\right):$, we derive

$$\begin{aligned} a_1|\varphi\rangle &= \left[\frac{\sqrt{2}}{6}(2x-2r_1\mathrm{i})+\frac{1}{3}\left(a_1^{\dagger}-2a_2^{\dagger}-2a_3^{\dagger}\right)\right]|\varphi\rangle, \\ a_2|\varphi\rangle &= \left[\frac{\sqrt{2}}{6}(2x+r_1\mathrm{i})-\frac{\sqrt{2}}{2}\mathrm{i}r_2+\frac{1}{3}\left(a_2^{\dagger}-2a_1^{\dagger}-2a_3^{\dagger}\right)\right]|\varphi\rangle, \\ a_3|\varphi\rangle &= \left[\frac{\sqrt{2}}{6}(2x+r_1\mathrm{i})+\frac{\sqrt{2}}{2}\mathrm{i}r_2+\frac{1}{3}\left(a_3^{\dagger}-2a_1^{\dagger}-2a_2^{\dagger}\right)\right]|\varphi\rangle, \end{aligned} \tag{34}$$

so we have

$$\begin{aligned}(x_1+x_2+x_3)|\varphi\rangle &= x|\varphi\rangle, \\ (p_2+p_3-2p_1)|\varphi\rangle &= r_1|\varphi\rangle, \\ (p_3-p_2)|\varphi\rangle &= r_2|\varphi\rangle. \end{aligned} \tag{35}$$

Using Eqs. (17) and (35), we can straightforward check

$$J_-|\varphi\rangle = \frac{1}{2\sqrt{3}}r_1r_2|\varphi\rangle. \tag{36}$$

Using the similar method, we can also check

$$J_+|\psi\rangle = \frac{1}{2\sqrt{3}}s_1s_2|\psi\rangle. \tag{37}$$

4. Conclusion

By considering the quantum mechanical transformation $\mathrm{e}^{2\lambda J_z}J_{\pm}\mathrm{e}^{-2\lambda J_z} = \mathrm{e}^{\pm 2\lambda}J_{\pm}$ as if the meaning of squeezing with $\mathrm{e}^{\pm 2\lambda}$ being squeezing parameter. We have found that $(J_{\pm},J_z)$ can also be realized in terms of 3-mode bosonic operators which leads to a 3-mode squeezing operator $\mathrm{e}^{2\lambda J_z}$. Employing this realization we find the natural representation (the eigenvectors of J_+ or J_-) of $\mathrm{e}^{2\lambda J_z}$. The idea of studying quantum SU(2) transformation from the point of view of squeezing is liable for us to obtain the new multi-mode squeezing operators and corresponding multimode squeezed states.

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