Stability analysis of multi-group deterministic and stochastic epidemic models with vaccination rate*

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We discuss in this paper a deterministic multi-group MSIR epidemic model with a vaccination rate, the basic reproduction number $\mathscr{R}_0$, a key parameter in epidemiology, is a threshold which determines the persistence or extinction of the disease. By using Lyapunov function techniques, we show if $\mathscr{R}_0$ is greater than 1 and the deterministic model obeys some conditions, then the disease will prevail, the infective persists and the endemic state is asymptotically stable in a feasible region. If $\mathscr{R}_0$ is less than or equal to 1, then the infective disappear so the disease dies out. In addition, stochastic noises around the endemic equilibrium will be added to the deterministic MSIR model in order that the deterministic model is extended to a system of stochastic ordinary differential equations. In the stochastic version, we carry out a detailed analysis on the asymptotic behavior of the stochastic model. In addition, regarding the value of $\mathscr{R}_0$, when the stochastic system obeys some conditions and $\mathscr{R}_0$ is greater than 1, we deduce the stochastic system is stochastically asymptotically stable. Finally, the deterministic and stochastic model dynamics are illustrated through computer simulations.

Keywords: MSIR epidemic model, equilibrium, graph theory, Brownian motion

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1. Introduction

The SIR model, proposed firstly by Kermack and McKendrick in 1927, has played a major role in mathematical epidemiology.[1] In the model, a population is divided into three groups: the susceptibles S, the infectives I, and the recovered R; they proved the existence of the threshold and obtained the epidemic threshold result that the density of susceptibles must exceed a critical value in order for an epidemic outbreak to occur. Thereafter, the studies of epidemiology modeling experienced a rapid growth and a tremendous variety of models have now been formulated, mathematically analyzed, and applied to infectious diseases. These researches are applied to specific diseases and have provided some useful and valid references for the characteristics of disease transmission. Based on these results of theoretical analyses, one can predict the future course of an outbreak and evaluate strategies to control an epidemic.[2–12] These models have involved many aspects such as stages of nonhomogeneous mixing, vertical transmission, spatial spread, acquired immunity through vaccinations, quarantine, and so on. Compartments with labels M, E, S, I, and R (passive immune, susceptible, infectious, and recovered) are often used for the epidemiological classes. Hethcote H W discussed the MSEIR model in detail. Threshold theorems involving the basic reproduction number R_0, the contact number σ, and the replacement number R are surveyed for the classic MSEIR epidemic.[13] The choice of which compartments to include in a model depends on the characteristics of the particular disease being modeled and the purpose of the model. If we do not consider the influence of the latent period, then the class E should be omitted; we label this model MSIR. In this MSIR model, if a mother has been infected, then some IgG antibodies are transferred across the placenta, so that her newborn infant is not born into the susceptible compartment but is immune to the disease for the first few months of life due to protection from maternal antibodies. The class M contains these infants with passive immunity. After the maternal antibodies disappear from the body, the infant moves to the susceptible class S. Infants who do not have any passive immunity, because their mothers were never infected, also enter the class S of susceptible individuals, and then the susceptible enters the I class while they are infectious, and then move to the recovered class R with permanent immunity. The MSIR model confers permanent immunity (detailed introduction, see Ref. [13]).

It is reasonable to propose multi-group models to describe the transmission dynamics of diseases in heterogeneous host populations. In multi-group epidemic models, a heterogeneous host population is divided into several homogeneous groups according to a distinct number of sexual partners, con-

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tact patterns, or different geography, etc., so that within-group and inter-group interactions can be modeled separately. In fact, there are already many scholars focusing their study on various forms of multi-group epidemic models.[14–17] Gong *et al.* investigated the global stability of a multi-group SIQR epidemic model with quarantine for computer worms.[14] We have also proved the globally stochastic stability of the multi-group SEIQR and SEIR epidemic models by extending Lyapunov function techniques and theories of stochastic stability, which is one of the main mathematical challenges in analyzing multi-group models.[15,16] Recently, Li *et al.* have paid more attention to multi-group models.[18–20] They first proposed a graph-theoretic approach to the method of global Lyapunov functions, and used it to establish the global stability of the interior equilibrium for more general models.[18] In the current paper, we divide the population of size $N(t)$ into n distinct groups and formulate the n-group ($n \geqslant 2$) MSIR epidemic models as follows:

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{M}_k}{\mathrm{d}t} = b_k(N_k - \widetilde{S}_k) - (\delta_k + d_k^{\mathrm{M}})\widetilde{M}_k - \theta_k \widetilde{M}_k, \\ \dfrac{\mathrm{d}\widetilde{S}_k}{\mathrm{d}t} = b_k \widetilde{S}_k - \varepsilon_k \widetilde{S}_k + \delta_k \widetilde{M}_k - \displaystyle\sum_{j=1}^{n} \frac{\beta_{kj}\widetilde{S}_k(t)\widetilde{I}_j(t)}{N_j} - d_k^{\mathrm{S}}\widetilde{S}_k, \\ \dfrac{\mathrm{d}\widetilde{I}_k}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{n} \frac{\beta_{kj}\widetilde{S}_k(t)\bar{I}_j(t)}{N_j} - (d_k^{\mathrm{I}} + \gamma_k)\widetilde{I}_k, \\ \dfrac{\mathrm{d}\widetilde{R}_k}{\mathrm{d}t} = \gamma_k \widetilde{I}_k(t) + \theta_k \widetilde{M}_k + \varepsilon_k \widetilde{S}_k - d_k^{\mathrm{R}}\widetilde{R}_k, \\ \dfrac{\mathrm{d}N_k}{\mathrm{d}t} = (b_k - d_k)N_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{1}$$

Because the natural births and deaths are not balanced, that is, $b_k \neq d_k$, so that the total population of the model is in an exponentially changing size. It is more difficult to analyze mathematically because the population size N_k is an additional variable, which is governed by a differential equation. In accordance with Guo *et al.*,[6] we investigate the asymptotic behavior of system (1). When studying epidemic systems, we are interested in two problems: one is when the disease will die out, and the other is when the disease will prevail and persist in a population. For a deterministic system, we solve the problems by showing the stability of the two equilibria under different conditions. On the other hand, note that because of environmental noises, the deterministic approach has some limitations in the mathematical modeling transmission of an infectious disease, several authors began to consider the effect of white noise in epidemic models, the parameter perturbation and the perturbations around the positive endemic equilibrium of epidemic models were considered, respectively.[21–24] Beretta *et al.* proved the stability of the epidemic model with stochastic time delays influenced by probability under certain conditions.[6] Yu *et al.* investigated epidemic models with fluctuations around the positive equilibrium and proved globally stochastically asymptotic stability of the positive equilibrium.[25] Imhof and Walcher considered a stochastic chemostat model and proved that the stochastic model led to extinction even though the deterministic counterpart predicts persistence.[26] In the current paper, to examine the influence of white noise on the system (1), we also consider a stochastic version of the MSIR model by perturbing the deterministic system (1) by a white noise and assume that the perturbations are around the positive endemic equilibrium of epidemic models. It is important to consider the effects of uncertainty around the positive endemic equilibrium of the MSIR model. However, to the best of our knowledge, no literature exists regarding this issue on any epidemic models. Thus, the current study is an attempt to fill this gap.

This rest of this paper is organized as follows. We begin in Section 2 with the necessary background with respect to the deterministic multi-group MSIR model and introduce some results of graph theory used by Guo *et al.* in epidemic models. We analyze the global dynamics of the disease-free state by the basic reproduction number $\mathscr{R}_0 \leqslant 1$ (Theorem 1) and derive asymptotic stability of a unique epidemic state in Section 2 (see Theorem 2). In Section 3, we derive the stochastic version from the deterministic model (1) and perform an analysis of the asymptotic behavior of the stochastic model by means of the method of Lyapunov functions and graph theory in Theorem 3. Numerical methods are employed to simulate the dynamic behavior of the deterministic models and the corresponding stochastic models in Section 4. Finally, we give the conclusion of our article in Section 5.

2. Deterministic multi-group MSIR models with varying population size and vaccination rate

The disease transmission diagram is depicted in Fig. 1.

We summarize the parameters in the model (1) by the following list.

$\widetilde{M}_k$: passively immune infants in k-th group;

$\widetilde{S}_k$: susceptibles in k-th group ;

$\widetilde{I}_k$: infectives in k-th group;

$\widetilde{R}_k$: recovered people with immunity in k-th group;

N_k: fraction at which newborns of group k have the passive immunity;

β_{kj}: rate of disease transmission between susceptible individuals in k-th group and infectious individuals in j-th group;

$d_k^{\mathrm{M}}, d_k^{\mathrm{S}}, d_k^{\mathrm{I}}, d_k^{\mathrm{R}}$: mortality rates of susceptible, infectious and recovered individuals in k-th group, respectively;

θ_k: vaccination rates of passively immune infants in k-th group;

ε_k: vaccination rates of susceptibles in k-th group;

δ_k: rate of the transfer out of the passively immune class in group k;

γ_k: recovery rate of infectious individuals in k-th group.

We assume $d_k^{\mathrm{M}} = d_k^{\mathrm{S}} = d_k^{\mathrm{I}} = d_k^{\mathrm{R}} = d_k > 0$, $b_k > 0$, $q_k = b_k - d_k > 0$ and the rest of the parameters be nonnegative for all k. It is clear that the population size be changed in an exponentially increasing manner. In particular, $\beta_{kj} = 0$ if there is no transmission of the disease between compartments S_k and I_j.

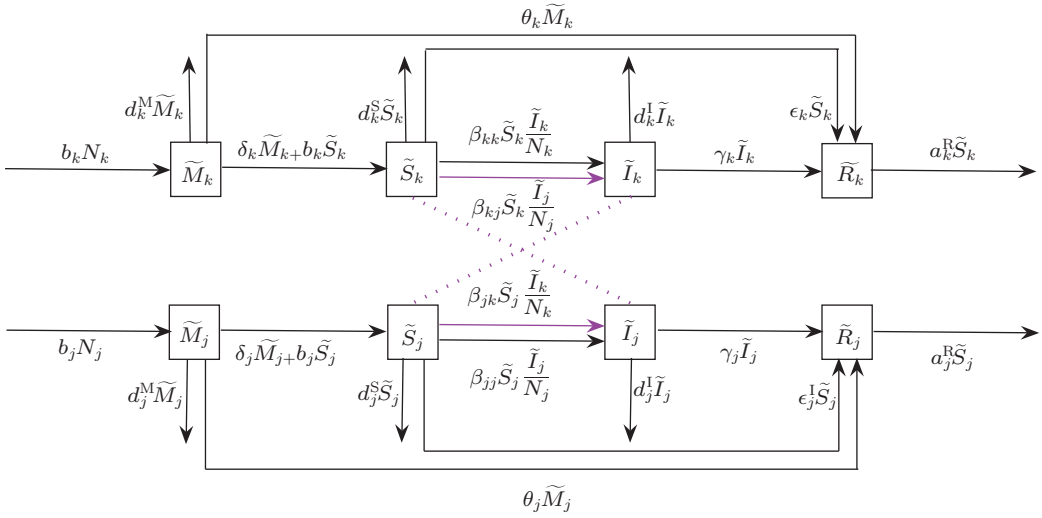


Fig. 1. (color online) The transmission diagram of MSIR endemic model.

Denote the fractions of the population in these classes $\widetilde{M}_k$, $\widetilde{S}_k$, $\widetilde{I}_k$, $\widetilde{R}_k$ are

$$M_k = \frac{\widetilde{M}_k}{N_k}, \quad S_k = \frac{\widetilde{S}_k}{N_k}, \quad I_k = \frac{\widetilde{I}_k}{N_k}, \quad R_k = \frac{\widetilde{R}_k}{N_k}.$$

Note that the number of infectives $\widetilde{I}_k$ could go to infinity even though the fraction I_k goes to zero if the population size N_k grows faster than $\widetilde{I}_k$. To avoid this ambiguity, we focus on the behavior of the fractions in the epidemiological classes. It is convenient to convert to differential equations for the fractions in the epidemiological classes with simplifications by using the differential equation for N_k, we can calculate

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\widetilde{M}_k}{\mathrm{d}t}}{N_k} - q_kM_k, \quad \frac{\mathrm{d}S_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\widetilde{S}_k}{\mathrm{d}t}}{N_k} - q_kS_k,$$
$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\widetilde{I}_k}{\mathrm{d}t}}{N_k} - q_kI_k, \quad \frac{\mathrm{d}R_k}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\widetilde{R}_k}{\mathrm{d}t}}{N_k} - q_kR_k,$$

furthermore, eliminating the differential equation for S_k by using $S_k = 1 - M_k - I_k - R_k$ and $b_k - d_k = q_k > 0$, the ordinary differential equations for the MSIR model becomes

$$\begin{cases} \dfrac{\mathrm{d}M_k}{\mathrm{d}t} = -(\delta_k+\theta_k)M_k + (d_k+q_k)I_k + (d_k+q_k)R_k, \\ \dfrac{\mathrm{d}I_k}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{n}\beta_{kj}(1-M_k-I_k-R_k)I_j - (d_k+\gamma_k+q_k)I_k, \\ \dfrac{\mathrm{d}R_k}{\mathrm{d}t} = \varepsilon_k + (\theta_k-\varepsilon_k)M_k + (\gamma_k-\varepsilon_k)I_k \\ \qquad -(d_k+q_k+\varepsilon_k)R_k, \quad k=1,2,\ldots,n. \end{cases} \tag{2}$$

Equilibrium solutions $(M_1^*, I_1^*, R_1^*, \ldots, M_n^*, I_n^*, R_n^*) \in R^{3n}$ of system (2) obey the following equation:

$$\begin{cases} 0 = -(\delta_k+\theta_k)M_k^* + (d_k+q_k)I_k^* + (d_k+q_k)R_k^*, \\ 0 = \displaystyle\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_j^* - (d_k+\gamma_k+q_k)I_k^*, \\ 0 = \varepsilon_k + (\theta_k-\varepsilon_k)M_k^* + (\gamma_k-\varepsilon_k)I_k^* \\ \qquad -(d_k+q_k+\varepsilon_k)R_k^*, \quad k=1,2,\ldots,n. \end{cases} \tag{3}$$

Then it is easy to verify that the trivial solution of system (3) is given by

$$P_0 = (M_1^0, I_1^0, R_1^0, \ldots, M_n^0, I_n^0, R_n^0) \in R_+^{3n},$$

where

$$M_k^0 = \frac{(d_k+q_k)\varepsilon_k}{d_k\delta_k + q_k\delta_k + \varepsilon_k\delta_k + \varepsilon_k\theta_k + \varepsilon_kd_k + \varepsilon_kq_k}, \quad I_k^0 = 0,$$
$$R_k^0 = \frac{(\delta_k+\theta_k)\varepsilon_k}{d_k\delta_k + q_k\delta_k + \varepsilon_k\delta_k + \varepsilon_k\theta_k + \varepsilon_kd_k + \varepsilon_kq_k},$$
$$k \in \{1,\ldots,n\}.$$

In epidemiology, it is called the disease-free equilibrium, at which the population remains in the absence of disease. Nontrivial solutions of system (3) with $I_k^* > 0$ for some $k \in \{1,\ldots,n\}$ are called endemic equilibria, at which the disease persists. Our main task is that we will find some conditions that determine whether the disease dies out (i.e., the fraction I_k goes to zero) or remains endemic (i.e., the fraction I_k remains positive) for system (2).

We can check that the right sides of Eq. (2) are smooth, so that solution of system (2) with initial condition

$$(M_1(0), I_1(0), R_1(0), \ldots, M_n(0), I_n(0), R_n(0)) \in R_+^{3n}$$

has a unique solution and remains nonnegative. Therefore, in what follows, we consider system (2) in R_+^{3n}.

A suitable bounded region in the non-negative cone of R_+^{3n} for system (2) is

$$\Gamma = \left\{(M_1,I_1,R_1,\ldots,M_n,I_n,R_n)\in R_+^{3n}\middle|0\leqslant M_k,\right.$$
$$\left. I_k,R_k,M_k+I_k+R_k\leqslant 1,\quad 1\leqslant k\leqslant n\right\}.$$

Because no solution paths leave through any boundary, therefore, it can be verified that region Γ is positively invariant for system (2). Let

$$\overset{\circ}{\Gamma} = \left\{(M_1,I_1,R_1,\ldots,M_n,I_n,R_n)\in R_+^{3n}\middle|0< M_k,\right.$$
$$\left. I_k,R_k,M_k+I_k+R_k< 1,1\leqslant k\leqslant n\right\}.$$

It is clear that $\overset{\circ}{\Gamma}$ is the interior of Γ. Set

$$\mathscr{F} = \begin{pmatrix} \beta_{11}(1-M_1^0-I_1^0-R_1^0) & \cdots & \beta_{1n}(1-M_1^0-I_1^0-R_1^0)\\ \vdots & \ddots & \vdots\\ \beta_{n1}(1-M_n^0-I_n^0-R_nn^0) & \cdots & \beta_{nn}(1-M_n^0-I_n^0-R_n^0)\end{pmatrix},$$
$$\mathscr{V} = \operatorname{diag}(d_k+\gamma_k+q_k) = \begin{pmatrix} d_1+\gamma_1+q_1 & 0 & \cdots & 0\\ 0 & d_2+\gamma_2+q_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & d_n+\gamma_n+q_n\end{pmatrix}.$$

Then the next generation matrix is

$$\mathscr{F}\mathscr{V}^{-1} = \begin{pmatrix} \dfrac{\beta_{11}(1-M_1^0-I_1^0-R_1^0)}{d_1+\gamma_1+q_1} & \cdots & \dfrac{\beta_{1n}(1-M_1^0-I_1^0-R_1^0)}{d_n+\gamma_n+q_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\beta_{n1}(1-M_n^0-I_n^0-R_n^0)}{d_1+\gamma_1+q_1} & \cdots & \dfrac{\beta_{nn}(1-M_n^0-I_n^0-R_n^0)}{d_n+\gamma_n+q_n}\end{pmatrix},$$

and hence, the basic reproduction number $\mathscr{R}_0$ is defined as follows:

$$\mathscr{R}_0 = \rho(\mathscr{F}\mathscr{V}^{-1}) = \max\left\{|\lambda|;\lambda\in\sigma(\mathscr{F}\mathscr{V}^{-1})\right\}, \qquad (4)$$

where $\rho(\cdot)$ and $\sigma(\cdot)$ denote the spectral radius and the set of eigenvalues of a matrix, respectively. The parameter $\mathscr{R}_0$ is referred to as the basic reproduction number. Its biological significance is that if $\mathscr{R}_0<1$, the disease dies out, while if $\mathscr{R}_0>1$, the disease becomes endemic.

2.1. Asymptotical stability of the disease-free equilibrium

In the study of population systems, extinction and persistence are two of the most important issues. We will discuss the extinction of the deterministic MSIR model (2) in this section, that is, we will find some conditions that determine when the disease dies out (i.e., the fraction I_k goes to zero) for system (2). First, following, Ref. [20], we prepare a matrix whose spectral radius has a similar threshold property to that of $\mathscr{R}_0$. Let

$$\mathscr{M}^0 := \mathscr{V}^{-1}\mathscr{F} = \begin{pmatrix} \dfrac{\beta_{11}(1-M_1^0-I_1^0-R_1^0)}{d_1+\gamma_1+q_1} & \cdots & \dfrac{\beta_{1n}(1-M_1^0-I_1^0-R_1^0)}{d_1+\gamma_1+q_1}\\ \vdots & \ddots & \vdots\\ \dfrac{\beta_{n1}(1-M_n^0-I_n^0-R_n^0)}{d_n+\gamma_n+q_n} & \cdots & \dfrac{\beta_{nn}(1-M_n^0-I_n^0-R_n^0)}{d_n+\gamma_n+q_n}\end{pmatrix}.$$

Then the following lemma immediately follows.

Lemma 1 $\rho(\mathscr{M}^0)\leqslant 1$ if and only if $\mathscr{R}_0\leqslant 1$.

Using Lemma 1, we give the following theorem, which is one of the main results of this paper. This proof is similar to that of Ref. [20]. When $\mathscr{R}_0\leqslant 1$, the asymptotic dynamics of model (2) are summarized in the following theorem.

Theorem 1 Assume $B=(\beta_{kj})$ is irreducible. If $\mathscr{R}_0\leqslant 1$, then the disease-free equilibrium P_0 of system (2) is globally asymptotically stable in Γ and there does not exist any endemic equilibrium P^*.

Proof From Lemma 1, we have $\rho(\mathscr{M}^0)\leqslant 1$. Following Ref. [20], we define a matrix-valued function

$$\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})=\begin{pmatrix}\dfrac{\beta_{11}(1-M_1-I_1-R_1)}{d_1+\gamma_1+q_1} & \cdots & \dfrac{\beta_{1n}(1-M_1-I_1-R_1)}{d_1+\gamma_1+q_1}\\ \vdots & \ddots & \vdots\\ \dfrac{\beta_{n1}(1-M_n-I_n-R_n)}{d_n+\gamma_n+q_n} & \cdots & \dfrac{\beta_{n1}(1-M_n-I_n-R_n)}{d_n+\gamma_n+q_n}\end{pmatrix}$$

on R_+^{3n}, where $\boldsymbol{M}=(M_1,\ldots,M_n)^{\mathrm{T}}$, $\boldsymbol{I}=(I_1,\ldots,I_n)^{\mathrm{T}}$ and $\boldsymbol{R}=(R_1,\ldots,R_n)^{\mathrm{T}}$. Note that $\mathscr{M}(\boldsymbol{M}^0,\boldsymbol{I}^0,\boldsymbol{R}^0)=\mathscr{M}^0$, where $\boldsymbol{M}^0=(M_1^0,\ldots,M_n^0)^{\mathrm{T}}$, $\boldsymbol{I}^0=(I_1^0,\ldots,I_n^0)^{\mathrm{T}}$, $\boldsymbol{R}^0=(R_1^0,\ldots,R_n^0)^{\mathrm{T}}$, and first we claim that there does not exist any endemic equilibrium P^* in Γ. Suppose that $(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})\neq(\boldsymbol{M}^0,\boldsymbol{I}^0,\boldsymbol{R}^0)$. Then we have $0<\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})<\mathscr{M}^0$. Since nonnegative matrix $\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})+\mathscr{M}^0$ is irreducible, it follows from the Perron–Frobenius theorem (see Corollary 2.1.5 of Ref. [27]) that $\rho(\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R}))<\rho(\mathscr{M}^0)\leqslant 1$. This implies that equation $\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})\boldsymbol{I}=\boldsymbol{I}$ has only the trivial solution $\boldsymbol{I}=0$, where $\boldsymbol{I}=(I_1,\ldots,I_n)^{\mathrm{T}}$. Hence, the claim is true.

Next we claim that the disease-free equilibrium P_0 is globally asymptotically stable in Γ. From the Perron–Frobenius theorem (see Theorem 2.1.4 of Ref. [27]), it follows that the nonnegative irreducible matrix $\mathscr{M}^0$ has a strictly positive left eigenvector $\ell:=(\ell_1,\ldots,\ell_n)\geqslant 0$ associated with the eigenvalue $\rho(\mathscr{M}^0)$. Let us define a Lyapunov function

$$L(\boldsymbol{I})=\sum_{i=1}^{n}\ell_i\frac{I_i}{d_i+\gamma_i+q_i} \tag{5}$$

on R_+^n, whose derivative along the trajectories of system (2) is

$$\begin{aligned}L'(\boldsymbol{I}) &= \sum_{i=1}^{n}\ell_i\left(\frac{(1-M_i-I_i-R_i)\sum_{j=1}^{n}\beta_{ij}I_j}{d_i+\gamma_i+q_i}-I_i\right)\\ &= \ell\cdot\Big(\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})-\boldsymbol{E}_n\Big)\boldsymbol{I}\\ &\leqslant \ell\cdot\Big(\mathscr{M}^0-\boldsymbol{E}_n\Big)\boldsymbol{I}\\ &= \ell\cdot(\rho(\mathscr{M}^0)-1)\boldsymbol{I}\\ &\leqslant 0,\end{aligned} \tag{6}$$

where $\boldsymbol{E}_n$ and $\cdot$ denote the $n\times n$ identity matrix and the inner product of vectors, respectively. Suppose that $\rho(\mathscr{M}^0)<1$. Then $L'(\boldsymbol{I})=0$ if and only if $\boldsymbol{I}=0$. Suppose that $\rho(\mathscr{M}^0)=1$. Then it follows from Eq. (6) that $L'(\boldsymbol{I})=0$ implies

$$\ell\cdot\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})=\ell\cdot\boldsymbol{I}, \tag{7}$$

hence, if $(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})\neq(\boldsymbol{M}^0,\boldsymbol{I}^0,\boldsymbol{R}^0)$ then $\ell\cdot\mathscr{M}(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})<\ell\cdot\mathscr{M}^0=\rho(\mathscr{M}^0)\ell=\ell$ and thus $\boldsymbol{I}=0$ is the only solution of Eq. (7). Summarizing the statements, we see that $L'(\boldsymbol{I})=0$ if and only if $\boldsymbol{I}=0$ or $(\boldsymbol{M},\boldsymbol{I},\boldsymbol{R})=(\boldsymbol{M}^0,\boldsymbol{I}^0,\boldsymbol{R}^0)$, which implies that the compact invariant subset of the set where $L'(\boldsymbol{I})=0$ is only the singleton $\{P_0\}\subset\Gamma$. Thus, from the La Salle invariance principle, it follows that the disease free equilibrium P_0 is globally asymptotically stable in Γ.

2.2. Asymptotical stability of an endemic equilibrium in the deterministic MSIR model

We will discuss the persistence of the deterministic MSIR model (2) in this section, that is, we will find some conditions that determine when the disease remains endemic (i.e., the fraction I_k remains positive) for system (2). Firstly, we introduce some knowledge on the graph theory, which is useful to the proofs of asymptotic stability of an endemic equilibrium in this section and the next section.

The matrix $\boldsymbol{B}=(\beta_{kj})$ denotes the contact matrix. Associated with $\boldsymbol{B}$, one can construct a directed graph $\mathfrak{L}=G(\boldsymbol{B})$ whose vertex k represents the k-th group, $k=1,\ldots,n$. A directed edge exists from vertex k to vertex j if and only if $\beta_{kj}>0$. Throughout the paper, we assume that $\boldsymbol{B}$ is irreducible. This is equivalent to $G(\boldsymbol{B})$ being strongly connected. Biologically, this is the same as assuming that any two groups k and j have a direct or indirect route of transmission. More specifically, individuals in I_j can infect ones in S_k directly or indirectly. Now consider the linear system

$$\overline{\mathfrak{B}}\varsigma=0, \tag{8}$$

where

$$\overline{\mathfrak{B}}=\begin{bmatrix}\sum_{i\neq 1}\overline{\beta}_{1i} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1}\\ -\overline{\beta}_{12} & \sum_{i\neq 2}\overline{\beta}_{2i} & \cdots & -\overline{\beta}_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ -\overline{\beta}_{1n} & \overline{\beta}_{2n} & \cdots & \sum_{i\neq n}\overline{\beta}_{ni}\end{bmatrix},$$

and $\overline{\beta}_{kj}=\beta_{kj}S_k^*I_j^*=\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_j^*$, $\overline{\beta}_{kj}>0, 1\leqslant k,j\leqslant n$. Let $\mathfrak{L}=G(\boldsymbol{B})$ denote the directed graph associated with matrix $\boldsymbol{B}$ (and $(\overline{\beta}_{kj})_{n\times n}$), and C_{jk} denotes the cofactor of the (j,k) entry of $\overline{\mathfrak{B}}$.

We have the following fundamental lemma.[18]

Lemma 2 (Kirchhoff's matrix-tree theorem) Assume $(\overline{\beta}_{kj})_{n\times n}$ is irreducible and $n\geqslant 2$. Then the following results hold:

(i) The solution space of system (8) has dimension one, with a basis

$$(\varsigma_1,\varsigma_2,\ldots,\varsigma_n)=(C_{11},C_{22},\ldots,C_{nn}).$$

(ii) For $1\leqslant k\leqslant n$,

$$C_{kk}=\sum_{T\in\mathbb{T}_k}W(T)=\sum_{T\in\mathbb{T}_k}\prod_{(r,m)\in E(T)}\overline{\beta}_{rm}>0,$$

where $\mathbb{T}_k$ is the set of all directed spanning subtrees of $\mathfrak{L}$ that are rooted at vertex k, $W(T)$ is the weight of a directed tree T, and $E(T)$ denotes the set of directed arcs in a directed tree T.

Let $\mathscr{R}_0$ be defined in Eq. (4). If $\mathscr{R}_0 > 1$, the following proposition for system (2) is known in the literature and its proof is standard (see Refs. [17] and [28]).

Proposition 1 Assume that $\boldsymbol{B} = (\beta_{kj})$ is irreducible. Then the following statements If $\mathscr{R}_0 > 1$, then P_0 is unstable and system (2) is uniformly persistent in Γ.

Uniform persistence of Eq. (2), together with uniform boundedness of solutions in $\overset{\circ}{\Gamma}$, implies the existence of an equilibrium of Eq. (2) in $\overset{\circ}{\Gamma}$. Summarizing the statements, we have the following corollary.

Corollary 1 Assume that $\boldsymbol{B} = (\beta_{kj})$ is irreducible. If $\mathscr{R}_0 > 1$, then equation (2) has at least one endemic equilibrium.

Moreover, based on the result of the existence of endemic equilibrium, we prove the following theorem, which is one of the main results of this paper.

Theorem 2 Assume that $\boldsymbol{B} = (\beta_{kj})$ is irreducible and $\gamma_k > \varepsilon_k, \gamma_k + \theta_k > 2\varepsilon_k$. If $\mathscr{R}_0 > 1$, then system (2) has a unique endemic equilibrium P^*, and P^* is asymptotically stable.

Proof Here, we will consider the case of $n \geqslant 2$. First change the variables $\boldsymbol{y}_k = M_k - M_k^*, \boldsymbol{z}_k = I_k - I_k^*, \boldsymbol{p}_k = R_k - R_k^*$ and system (2) can be written as

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{y}_k}{\mathrm{d}t} = -(\delta_k + \theta_k)\boldsymbol{y}_k + (d_k + q_k)\boldsymbol{z}_k + (d_k + q_k)\boldsymbol{p}_k, \\ \dfrac{\mathrm{d}\boldsymbol{z}_k}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{n} \beta_{kj}(1 - M_k^* - I_k^* - R_k^*)\boldsymbol{z}_j \\ \qquad\qquad - \displaystyle\sum_{j=1}^{n} \beta_{kj}(\boldsymbol{y}_k + \boldsymbol{z}_k + \boldsymbol{p}_k)(I_j^* + \boldsymbol{z}_j) \\ \qquad\qquad - (d_k + \gamma_k + q_k)\boldsymbol{z}_k, \\ \dfrac{\mathrm{d}\boldsymbol{p}_k}{\mathrm{d}t} = (\theta_k - \varepsilon_k)\boldsymbol{y}_k + (\gamma_k - \varepsilon_k)\boldsymbol{z}_k \\ \qquad\qquad - (d_k + q_k + \varepsilon_k)\boldsymbol{p}_k, \quad k = 1, 2, \ldots, n. \end{cases} \tag{9}$$

Note that $\overline{\mathfrak{B}}$ is the Laplacian matrix of the matrix $(\overline{\beta}_{kj})_{n\times n}$. Since $(\beta_{kj})_{n\times n}$ is irreducible, the matrices $(\overline{\beta}_{kj})_{n\times n}$ and $\overline{\mathfrak{B}}$ are also irreducible. The column sums of the Laplacian matrix $\overline{\mathfrak{B}}$ are zero. Therefore, it follows from Lemma 2.1 of Ref. [20] that the solution space of linear system $\overline{\mathfrak{B}}\varsigma = 0$ has dimension one with a basis

$$\varsigma := (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)^{\mathrm{T}} = (C_{11}, C_{22}, \ldots, C_{nn})^{\mathrm{T}},$$

where C_{kk} denotes the cofactor of the k-th diagonal entry of $\overline{\mathfrak{B}}$. For such $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)$, we define a Lyapunov function

$$\begin{aligned} W(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{p}) :&= \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \Bigg\{ \frac{(\gamma_k - \varepsilon_k)(\delta_k + 2\varepsilon_k)\boldsymbol{y}_k^2}{2(d_k + q_k)\Big[(d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)\Big]} \\ &\quad + \frac{(\gamma_k + \theta_k - 2\varepsilon_k)(\boldsymbol{y}_k + \boldsymbol{p}_k)^2 + (\delta_k + 2\varepsilon_k)\boldsymbol{p}_k^2}{2\Big((d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)\Big)} \Bigg\} + \sum_{k=1}^{n} \frac{\varsigma_k}{2I_k^*} \boldsymbol{z}_k^2, \\ W'(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{p}) :&= \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \Bigg\{ \frac{(\gamma_k - \varepsilon_k)(\delta_k + 2\varepsilon_k)\boldsymbol{y}_k \dot{\boldsymbol{y}}_k}{(d_k + q_k)\Big[(d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)\Big]} \\ &\quad + \frac{(\gamma_k + \theta_k - 2\varepsilon_k)(\boldsymbol{y}_k + \boldsymbol{p}_k)(\dot{\boldsymbol{y}}_k + \dot{\boldsymbol{p}}_k) + (\delta_k + 2\varepsilon_k)\boldsymbol{p}_k \dot{\boldsymbol{p}}_k}{(d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)} \Bigg\} + \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \boldsymbol{z}_k \dot{\boldsymbol{z}}_k \\ &= W_1 + W_2, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{y} &= (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n), \quad \boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n), \quad \boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n), \\ W_1 &= \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \sum_{j=1}^{n} \beta_{kj} I_j^* \Bigg\{ \frac{(\gamma_k - \varepsilon_k)(\delta_k + 2\varepsilon_k)\boldsymbol{y}_k \dot{\boldsymbol{y}}_k}{(d_k + q_k)\Big[(d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)\Big]} \\ &\quad + \frac{(\gamma_k + \theta_k - 2\varepsilon_k)(\boldsymbol{y}_k + \boldsymbol{p}_k)(\dot{\boldsymbol{y}}_k + \dot{\boldsymbol{p}}_k) + (\delta_k + 2\varepsilon_k)\boldsymbol{p}_k \dot{\boldsymbol{p}}_k}{(d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k)} \Bigg\}, \\ W_2 &= \sum_{k=1}^{n} \frac{\varsigma_k}{I_k^*} \boldsymbol{z}_k \dot{\boldsymbol{z}}_k. \end{aligned}$$

For the sake of simplicity, we denote

$$K = (d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k).$$

We calculate

$$\begin{aligned}
W_1 &= \sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\Bigg\{\frac{(\gamma_k-\varepsilon_k)(\delta_k+2\varepsilon_k)}{(d_k+q_k)K}\Big(-(\delta_k+\theta_k)\boldsymbol{y}_k^2+(d_k+q_k)\boldsymbol{y}_k\boldsymbol{z}_k+(d_k+q_k)\boldsymbol{y}_k\boldsymbol{p}_k\Big)\\
&\quad+\frac{\gamma_k+\theta_k-2\varepsilon_k}{K}(\boldsymbol{y}_k+\boldsymbol{p}_k)\big[-\delta_k\boldsymbol{y}_k+(d_k+q_k+\gamma_k)\boldsymbol{z}_k-\varepsilon_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\big]\\
&\quad+\frac{\delta_k+2\varepsilon_k}{K}\big[(\theta_k-\varepsilon_k)\boldsymbol{y}_k\boldsymbol{p}_k+(\gamma_k-\varepsilon_k)\boldsymbol{p}_k\boldsymbol{z}_k-(d_k+q_k+\varepsilon_k)\boldsymbol{p}_k^2\big]\Bigg\}\\
&=\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\Bigg[-\frac{(\gamma_k-\varepsilon_k)(\delta_k+2\varepsilon_k)(\delta_k+\theta_k)+(d_k+q_k)(\gamma_k+p_k-2\varepsilon_k)(\delta_k+\varepsilon_k)}{(d_k+q_k)K}\boldsymbol{y}_k^2\\
&\quad+\boldsymbol{y}_k\boldsymbol{z}_k+\boldsymbol{p}_k\boldsymbol{z}_k-\frac{(d_k+q_k+\varepsilon_k)(\delta_k+2\varepsilon_k)+\varepsilon_k(\gamma_k+\theta_k-2\varepsilon_k)}{K}\boldsymbol{p}_k^2\Bigg],\\
W_2 &= \sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\boldsymbol{z}_k\Bigg[\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)\boldsymbol{z}_j-\sum_{j=1}^{n}\beta_{kj}(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)(I_j^*+\boldsymbol{z}_j)-(d_k+\gamma_k+q_k)\boldsymbol{z}_k\Bigg]\\
&=\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[-\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2+\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)\boldsymbol{z}_k\boldsymbol{z}_j-\sum_{j=1}^{n}\beta_{kj}\frac{(1-M_k^*-I_k^*-R_k^*)I_j^*}{I_k^*}\boldsymbol{z}_k^2\\
&\quad-\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k-\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\\
&=\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[\Bigg(\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_k^*I_j^*\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*}-\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_k^*I_j^*\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\Bigg)\\
&\quad-\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2-\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k-\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Bigg]\\
&=\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[\Bigg(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*}-\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\Bigg)-\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2-\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k\\
&\quad-\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Bigg].
\end{aligned}$$

It follows from the arithmetic-geometric mean that

$$\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*}\leqslant\frac{1}{2}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2+\frac{1}{2}\Big(\frac{\boldsymbol{z}_j}{I_j^*}\Big)^2,$$

therefore,

$$\begin{aligned}
W_2 &\leqslant \sum_{k=1}^{n}\varsigma_k\Bigg[\sum_{j=1}^{n}\overline{\beta}_{kj}\Big(\frac{1}{2}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2+\frac{1}{2}\Big(\frac{\boldsymbol{z}_j}{I_j^*}\Big)^2-\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\Big)\Bigg]\\
&\quad-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2\\
&\quad+\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k+\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Bigg]\\
&=\frac{1}{2}\sum_{k=1}^{n}\varsigma_k\Bigg[\sum_{j=1}^{n}\overline{\beta}_{kj}\Big(\Big(\frac{\boldsymbol{z}_j}{I_j^*}\Big)^2-\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\Big)\Bigg]\\
&\quad-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2+\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k\\
&\quad+\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Bigg]. \qquad (10)
\end{aligned}$$

It follows from $\overline{\mathfrak{B}}\varsigma=0$ and $\overline{\beta}_{jk}=\beta_{jk}(1-M_j^*-I_j^*-R_j^*)I_k^*$ that

$$\sum_{j=1}^{n}\bar{\beta}_{jk}\varsigma_j=\sum_{k=1}^{n}\bar{\beta}_{ki}\varsigma_k=\sum_{j=1}^{n}\bar{\beta}_{kj}\varsigma_k,$$

thus

$$\begin{aligned}
&\sum_{k=1}^{n}\varsigma_k\sum_{j=1}^{n}\bar{\beta}_{kj}\Big(\frac{\boldsymbol{z}_j}{I_j^*}\Big)^2\\
&=\sum_{j=1}^{n}\varsigma_j\sum_{k=1}^{n}\bar{\beta}_{jk}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2=\sum_{k=1}^{n}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\sum_{j=1}^{n}\bar{\beta}_{jk}\varsigma_j\\
&=\sum_{k=1}^{n}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2\sum_{j=1}^{n}\bar{\beta}_{kj}\varsigma_k=\sum_{k=1}^{n}\varsigma_k\sum_{j=1}^{n}\bar{\beta}_{kj}\Big(\frac{\boldsymbol{z}_k}{I_k^*}\Big)^2. \qquad (11)
\end{aligned}$$

Substituting this into Eq. (10), we obtain

$$\begin{aligned}
W_2 &\leqslant -\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\Bigg[\sum_{j=1}^{n}\beta_{kj}I_j^*\boldsymbol{z}_k^2+\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k\\
&\quad+\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Bigg], \qquad (12)
\end{aligned}$$

thus,

$$
\begin{aligned}
W'(\boldsymbol{y},\boldsymbol{z},\boldsymbol{p}) &= W_1+W_2\\
&\leqslant -\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{(\gamma_k-\varepsilon_k)(\delta_k+2\varepsilon_k)(\delta_k+\theta_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)(\delta_k+\varepsilon_k)}{(d_k+q_k)K}\boldsymbol{y}_k^2\right.\\
&\quad\left.+\frac{(d_k+q_k+\varepsilon_k)(\delta_k+2\varepsilon_k)+\varepsilon_k(\gamma_k+\theta_k-2\varepsilon_k)}{K}\boldsymbol{p}_k^2+\boldsymbol{z}_k^2\right)-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\\
&\leqslant W_0+\left(-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{k=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\right)
\end{aligned}
$$

where

$$
\begin{aligned}
W_0 :&= -\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{(\gamma_k-\varepsilon_k)(\delta_k+2\varepsilon_k)(\delta_k+\theta_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)(\delta_k+\varepsilon_k)}{(d_k+q_k)K}\boldsymbol{y}_k^2\right.\\
&\quad\left.+\frac{(d_k+q_k+\varepsilon_k)(\delta_k+2\varepsilon_k)+\varepsilon_k(\gamma_k+\theta_k-2\varepsilon_k)}{K}\boldsymbol{p}_k^2+\boldsymbol{z}_k^2\right).
\end{aligned}
$$

We denote $Y_k=(\boldsymbol{y}_k,\boldsymbol{z}_k,\boldsymbol{p}_k)$ and $\boldsymbol{Y}=(Y_1,\ldots,Y_n)$, then

$$
|Y_k|=\sqrt{\boldsymbol{y}_k^2(t)+\boldsymbol{z}_k^2+\boldsymbol{p}_k^2},\quad |\boldsymbol{Y}|=\left(\sum_{k=1}^{n}|Y_k|^2\right)^{1/2}.
$$

It is clear that $\left(-\sum_{k=1}^{n}\frac{\varsigma_k}{I_k^*}\sum_{k=1}^{n}\beta_{kj}\widetilde{I}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\right)$ is an infinitesimal of higher order of $|\boldsymbol{Y}|^2$ for $|\boldsymbol{Y}|\to 0$. Let

$$
\begin{aligned}
\omega = \min_{k\in\{1,\ldots,n\}}&\left\{\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\frac{(\gamma_k-\varepsilon_k)(\delta_k+2\varepsilon_k)(\delta_k+\theta_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)(\delta_k+\varepsilon_k)}{(d_k+q_k)K},\frac{\varsigma_k}{I_k^*},\right.\\
&\left.\frac{\varsigma_k}{I_k^*}\sum_{j=1}^{n}\beta_{kj}I_j^*\frac{(d_k+q_k+\varepsilon_k)(\delta_k+2\varepsilon_k)+\varepsilon_k(\gamma_k+\theta_k-2\varepsilon_k)}{K}\right\},
\end{aligned}
$$

thus

$$
W'(\boldsymbol{y},\boldsymbol{z},\boldsymbol{p})\leqslant -\omega\sum_{k=1}^{n}|Y_k|^2+O(|\boldsymbol{Y}|^2). \tag{13}
$$

Hence, $W'(\boldsymbol{y},\boldsymbol{z},\boldsymbol{p})$ is negative-definite in a sufficiently small neighborhood of $\boldsymbol{Y}=\boldsymbol{0}$ for $t\geqslant 0$. On the other hand, it is easy to see $W(\boldsymbol{y},\boldsymbol{z},\boldsymbol{p})$ is a positive-definite decrescent function, which implies the endemic equilibrium P^* is asymptotically stable.

3. Stochastic stability of the endemic equilibrium of multi-group stochastic MSIR model

In this section, we formulate the stochastic version of the deterministic MSIR model and find some conditions that determine when the disease remains endemic (i.e., the fraction I_k remains positive) for the stochastic system. Under assumption $\mathscr{R}_0>1$ and $\boldsymbol{B}=(\beta_{kj})_{n\times n}$ is irreducible, we know from Section 2 that there exists a unique positive endemic equilibrium P^* in $\overset{\circ}{\Gamma}$. Furthermore, we assume stochastic perturbations are of a white noise type, which are directly proportional to distances $M_k(t)$, $I_k(t)$, and $R_k(t)$ from values of M_k^*, I_k^*, and R_k^*, influence on the $M_k(t)$, $I_k(t)$, and $R_k(t)$ respectively. Such a type of stochastic perturbations was first proposed in Refs. [6] and [7] and later successfully used in many other models described by stochastic differential and stochastic difference equations (see, for instance Refs. [8]–[10], [29], and [30]). Therefore, system (2) results in

$$
\begin{cases}
\dfrac{\mathrm{d}M_k}{\mathrm{d}t} = -(\delta_k+\theta_k)M_k+(d_k+q_k)I_k\\
\qquad\qquad +(d_k+q_k)R_k+\sigma_{1k}(M_k-M_k^*)\dfrac{\mathrm{d}B_{1k}}{\mathrm{d}t},\\
\dfrac{\mathrm{d}I_k}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{n}\beta_{kj}(1-M_k-I_k-R_k)I_j\\
\qquad\qquad -(d_k+\gamma_k+q_k)I_k+\sigma_{2k}(I_k-I_k^*)\dfrac{\mathrm{d}B_{2k}}{\mathrm{d}t},\\
\dfrac{\mathrm{d}R_k}{\mathrm{d}t} = \varepsilon_k+(\theta_k-\varepsilon_k)M_k+(\gamma_k-\varepsilon_k)I_k\\
\qquad\qquad -(d_k+q_k+\varepsilon_k)R_k+\sigma_{3k}(R_k-R_k^*)\dfrac{\mathrm{d}B_{3k}}{\mathrm{d}t},\\
\qquad\qquad k=1,2,\ldots,n,
\end{cases} \tag{14}
$$

where $B_{1k}(t)$, $B_{2k}(t)$, and $B_{3k}(t)$ are independent standard Brownian motions and $\sigma_{ik}^2>0$ represent the intensities of $B_{ik}(t)$ $(i=1,2,3)$, respectively. Obviously, stochastic sys-

tem (14) has the same equilibrium points as system (2). Next, let us now proceed to discuss the asymptotic stability of the system (14). In this paper, unless otherwise specified, let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant t_0},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets). Let $B_{ik}(t)$ be the Brownian motions defined on this probability space. If $\mathscr{R}_0>1$, then the stochastic system (14) can be centered at its endemic equilibrium $P^*=(M_1^*,I_1^*,R_1^*,\ldots,M_n^*,I_n^*,R_n^*)$, by the change of variables

$$y_k=M_k-M_k^*,\quad z_k=I_k-I_k^*,\quad p_k=R_k-R_k^*,$$

we obtain the following system:

$$\begin{cases}\dfrac{\mathrm{d}y_k}{\mathrm{d}t}=-(\delta_k+\theta_k)y_k+(d_k+q_k)z_k\\ \qquad\qquad +(d_k+q_k)p_k+\sigma_{1k}y_k\dfrac{\mathrm{d}B_{1k}}{\mathrm{d}t},\\ \dfrac{\mathrm{d}z_k}{\mathrm{d}t}=\displaystyle\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)z_j\\ \qquad\qquad -\displaystyle\sum_{j=1}^{n}\beta_{kj}(y_k+z_k+p_k)(I_j^*+z_j)\\ \qquad\qquad -(d_k+\gamma_k+q_k)z_k+\sigma_{2k}z_k\dfrac{\mathrm{d}B_{2k}}{\mathrm{d}t},\\ \dfrac{\mathrm{d}p_k}{\mathrm{d}t}=(\theta_k-\varepsilon_k)y_k+(\gamma_k-\varepsilon_k)z_k\\ \qquad\qquad -(d_k+q_k+\varepsilon_k)p_k+\sigma_{3k}p_k\dfrac{\mathrm{d}B_{3k}}{\mathrm{d}t},\\ \qquad\qquad k=1,2,\ldots,n.\end{cases}\tag{15}$$

It is clear that the stability of the equilibrium of the system (14) is equivalent to the stability of zero solution of system (15). Considering the d-dimensional stochastic differential equation[31]

$$\mathrm{d}x(t)=f(x(t),t)\mathrm{d}t+g(x(t),t)\mathrm{d}B(t),\quad t\geqslant t_0.\tag{16}$$

If the assumptions of the existence-and-uniqueness theorem are satisfied, then, for any given initial value $x(t_0)=x_0\in\mathbb{R}^d$, equation (16) has a unique global solution denoted by $x(t;t_0,x_0)$. For the purpose of stability, we assume in this section $f(\mathbf{0},t)=0$ and $g(\mathbf{0},t)=0$ for all $t\geqslant t_0$. Therefore, equation (16) admits a solution $x(t)\equiv\mathbf{0}$ which is called the trivial solution or equilibrium position.

Let κ denote the family of all continuous nondecreasing functions $\mu:\mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(0)=0$ and $\mu(r)>0$ if $r>0$. For $0<h\leqslant\infty$ and $S_h=\{x\in\mathbb{R}^d,|x|<h\}$, denote $C^{2,1}(S_h\times[t_0,\infty);\mathbb{R}_+)$ the family of all non-negative functions $V(x,t)$ on $S_h\times[t_0,\infty)$ which are continuously twice differentiable in x and once in t. Define the differential operator L associated with Eq. (16) by

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{d}f_i(x,t)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d}[g^{\mathrm{T}}(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_ix_j}.$$

If L acts on a function $V\in C^{2,1}(\mathbb{R}^d\times[t_0,\infty);\mathbb{R}_+)$, then

$$\begin{aligned}LV(x,t)&=V_t(x,t)+V_t(x,t)f(x,t)\\&\quad+\frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(x,t)V_{xx}(x,t)g(x,t)].\end{aligned}$$

Definition 1 1) The trivial solution of Eq. (16) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon\in(0,1)$ and $r>0$, there exists a $\delta=\delta(\varepsilon,r,t_0)>0$ such that

$$P\{|x(t;t_0,x_0)|<r,\quad t\geqslant t_0\}\geqslant 1-\varepsilon,$$

whenever $|x_0|<\delta$. Otherwise, it is said to be stochastically unstable.

2) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable and for every $\varepsilon\in(0,1)$, there exists a $\delta_0=\delta_0(\varepsilon,t_0)>0$ such that

$$P\{\lim_{t\to\infty}x(t;t_0,x_0)=0\}\geqslant 1-\varepsilon,$$

whenever $|x_0|<\delta_0$.

3) The trivial solution is said to be stochastically asymptotically stable in the large if it is stochastically stable and for all $x_0\in\mathbb{R}^d$

$$P\{\lim_{t\to\infty}x(t;t_0,x_0)=0\}=1.$$

Definition 2 A continuous non-negative function $V(x,t)$ is said to be decrescent if for some $\mu\in\kappa$:

$$V(x,t)\leqslant\mu(|x(t)|),\quad (x,t)\in S_h\times[t_0,\infty).$$

Before presenting the main theorem we put forward a lemma in Ref. [31].

Lemma 3[31] If there exists a positive-definite decrescent function $V(x,t)\in C^{2,1}(S^h\times[t_0,\infty);\mathbb{R}_+)$ such that $\mathrm{LV}(x,t)$ is negative-definite, then the trivial solution of Eq. (16) is stochastically asymptotically stable.

From the above lemma, we can obtain the stochastically asymptotically stability of equilibrium as follows.

Theorem 3 Assume that $\mathfrak{B}=(\beta_{kj})$ is irreducible and $\gamma_k>\varepsilon_k,\gamma_k+\theta_k>2\varepsilon_k$. If $\mathscr{R}_0>1$, then if the following condition is satisfied:

$$\begin{aligned}&\sigma_{1k}^2<\frac{2(\delta_k+\theta_k)(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+2(\delta_k+\varepsilon_k)(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)}{(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)},\\&\sigma_{2k}^2<2\sum_{j=1}^{n}\beta_{kj}I_j^*,\quad\sigma_{3k}^2<2\varepsilon_k+\frac{2(d_k+q_k)(\delta_k+2\varepsilon_k)}{\theta_k+\delta_k+\gamma_k},\end{aligned}\tag{17}$$

the endemic equilibrium P^* is stochastically asymptotically stable.

Proof It is easy to see that we only need to prove the zero solution of Eq. (14) is stochastically asymptotically stable. Let $\boldsymbol{x}_k(t) = (\boldsymbol{y}_k(t), \boldsymbol{z}_k(t), \boldsymbol{p}_k(t))^{\mathrm{T}} \in R^3, k = 1, \ldots, n$ and $\boldsymbol{x}(t) = (\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t))^{\mathrm{T}} \in R^{3n}$. We define the Lyapunov function $V(\boldsymbol{x}(t))$ as follows:

$$V(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}\left(u_k\boldsymbol{y}_k^2 + v_k\boldsymbol{z}_k^2 + w_k\boldsymbol{p}_k^2 + m_k(\boldsymbol{y}_k+\boldsymbol{p}_k)^2\right), \quad (18)$$

where $u_k > 0$, $v_k > 0$, $w_k > 0$, and $m_k > 0$ are real positive constants to be chosen later. Then it can be described as the quadratic form

$$V(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k,$$

where

$$Q = \begin{pmatrix} u_k+m_k & 0 & m_k \\ 0 & v_k & 0 \\ m_k & 0 & w_k+m_k \end{pmatrix}$$

is a symmetric positive-definite matrix. Therefore, it is obviously that $V(\boldsymbol{x})$ is a positive-definite decrescent. For the sake of simplicity, equation (18) may be divided into four functions: $V(\boldsymbol{x}) = V_1(\boldsymbol{x}) + V_2(\boldsymbol{x}) + V_3(\boldsymbol{x}) + V_4(\boldsymbol{x})$, where

$$V_1(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}u_k\boldsymbol{y}_k^2, \quad V_2(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}v_k\boldsymbol{z}_k^2,$$
$$V_3(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}w_k\boldsymbol{p}_k^2,$$
$$V_4(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n}m_k(\boldsymbol{y}_k+\boldsymbol{p}_k)^2.$$

Using Itô's formula, we compute

$$\begin{aligned}\mathrm{LV}_1 &= \sum_{k=1}^{n}u_k\boldsymbol{y}_k\Big(-(\delta_k+\theta_k)\boldsymbol{y}_k + (d_k+q_k)\boldsymbol{z}_k \\ &\quad + (d_k+q_k)\boldsymbol{p}_k\Big) + \frac{1}{2}\sum_{k=1}^{n}u_k\sigma_{1k}^2\boldsymbol{y}_k^2 \\ &= -\sum_{k=1}^{n}u_k\left(\delta_k+\theta_k-\frac{1}{2}\sigma_{1k}^2\right)\boldsymbol{y}_k^2 \\ &\quad + \sum_{k=1}^{n}u_k\Big((d_k+q_k)\boldsymbol{y}_k\boldsymbol{z}_k + (d_k+q_k)\boldsymbol{y}_k\boldsymbol{p}_k\Big). \quad (19)\end{aligned}$$

Similarly, from Itô's formula, we obtain

$$\begin{aligned}\mathrm{LV}_2 &= \sum_{k=1}^{n}v_k\boldsymbol{z}_k\Big(\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)\boldsymbol{z}_j \\ &\quad - \sum_{j=1}^{n}\beta_{kj}(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)(I_j^*+\boldsymbol{z}_j) \\ &\quad - (d_k+\gamma_k+q_k)\boldsymbol{z}_k\Big) + \frac{1}{2}\sum_{k=1}^{n}v_k\sigma_{2k}^2\boldsymbol{z}_k^2 \\ &= \sum_{k=1}^{n}v_k\Big(-\Big(\sum_{j=1}^{n}\beta_{kj}I_j^* - \frac{1}{2}\sigma_{2k}^2\Big)\boldsymbol{z}_k^2 \\ &\quad + \sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)\boldsymbol{z}_k\boldsymbol{z}_j \\ &\quad - \sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k \\ &\quad - \sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j\Big) \\ &\quad - \sum_{j=1}^{n}\beta_{kj}\frac{(1-M_k^*-I_k^*-R_k^*)I_j^*}{I_k^*}\boldsymbol{z}_k^2 \\ &= \sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_k^*I_j^*\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*} \\ &\quad - \sum_{j=1}^{n}\beta_{kj}(1-M_k^*-I_k^*-R_k^*)I_k^*I_j^*\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2\Big) \\ &\quad - \sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\beta_{kj}I_j^* - \frac{1}{2}\sigma_{2k}^2\Big)\boldsymbol{z}_k^2 \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j \\ &= \sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*} - \sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2\Big) \\ &\quad - \sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\beta_{kj}I_j^* - \frac{1}{2}\sigma_{2k}^2\Big)\boldsymbol{z}_k^2 \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{u}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j. \quad (20)\end{aligned}$$

It follows from the arithmetic-geometric mean that

$$\frac{\boldsymbol{z}_k}{I_k^*}\frac{\boldsymbol{z}_j}{I_j^*} \leqslant \frac{1}{2}\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2 + \frac{1}{2}\left(\frac{\boldsymbol{z}_j}{I_j^*}\right)^2,$$

therefore

$$\begin{aligned}\mathrm{LV}_2 &\leqslant \sum_{k=1}^{n}\frac{1}{2}v_k\Big(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\Big(\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2 + \left(\frac{\boldsymbol{z}_j}{I_j^*}\right)^2\Big) \\ &\quad - \sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2\Big) \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{w}_k)\boldsymbol{z}_j \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k \\ &\quad - \sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\beta_{kj}I_j^* - \frac{1}{2}\sigma_{2k}^2\Big)\boldsymbol{z}_k^2 \\ &= \frac{1}{2}\sum_{k=1}^{n}v_k\Big(\sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\boldsymbol{z}_j}{I_j^*}\right)^2 - \sum_{j=1}^{n}\overline{\beta}_{kj}I_k^*\left(\frac{\boldsymbol{z}_k}{I_k^*}\right)^2\Big) \\ &\quad - \sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}I_j^*(\boldsymbol{y}_k+\boldsymbol{p}_k)\boldsymbol{z}_k\end{aligned}$$

$$- \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} z_k (\boldsymbol{y}_k + \boldsymbol{z}_k + \boldsymbol{p}_k) z_j$$
$$- \sum_{k=1}^{n} v_k \Big(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \Big) z_k^2. \tag{21}$$

Let $v_k = C_{kk}/I_k^*$, so $\varsigma_k = v_k I_k^*$. It follows from $\overline{\mathfrak{B}}\varsigma = 0$ and $\overline{\beta}_{jk} = \beta_{jk}(1 - M_j^* - I_j^* - R_j^*) I_k^*$ that

$$\sum_{j=1}^{n} \bar{\beta}_{jk} \varsigma_j = \sum_{k=1}^{n} \bar{\beta}_{kj} \varsigma_k,$$

which implies

$$\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \bar{\beta}_{kj} I_k^* \Big(\frac{z_j}{I_j^*} \Big)^2 = \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \bar{\beta}_{kj} I_k^* \Big(\frac{z_k}{I_k^*} \Big)^2.$$

Hence, inequality (21) becomes

$$\begin{aligned}
\mathrm{LV}_2 \leqslant & - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{y}_k + \boldsymbol{p}_k) z_k \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} z_k (\boldsymbol{y}_k + \boldsymbol{z}_k + \boldsymbol{p}_k) z_j \\
& - \sum_{k=1}^{n} v_k \Big(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \Big) z_k^2, \\
\mathrm{LV}_3 = & \sum_{k=1}^{n} w_k \boldsymbol{p}_k \Big((\theta_k - \varepsilon_k) \boldsymbol{y}_k + (\gamma_k - \varepsilon_k) z_k \\
& - (d_k + q_k + \varepsilon_k) \boldsymbol{p}_k \Big) + \frac{1}{2} \sum_{k=1}^{n} w_k \sigma_{3k}^2 \boldsymbol{p}_k^2 \\
= & - \sum_{k=1}^{n} w_k \big(d_k + q_k + \varepsilon_k - \frac{1}{2}\sigma_{3k}^2 \big) \boldsymbol{p}_k^2 \\
& + \sum_{k=1}^{n} w_k (\theta_k - \varepsilon_k) \boldsymbol{y}_k \boldsymbol{p}_k + \sum_{k=1}^{n} w_k (\gamma_k - \varepsilon_k) z_k \boldsymbol{p}_k, \\
\mathrm{LV}_4 = & \sum_{k=1}^{n} m_k (\boldsymbol{y}_k + \boldsymbol{p}_k) \Big(- (\delta_k + \varepsilon_k) \boldsymbol{y}_k \\
& + (d_k + q_k + \gamma_k - \varepsilon_k) z_k - \varepsilon_k \boldsymbol{p}_k \Big) \\
& + \frac{1}{2} \sum_{k=1}^{n} m_k (\sigma_{1k}^2 \boldsymbol{y}_k^2 + \sigma_{3k}^2 \boldsymbol{p}_k^2) \\
= & - \sum_{k=1}^{n} m_k (\delta_k + \varepsilon_k) \boldsymbol{y}_k^2 - \sum_{k=1}^{n} m_k \varepsilon_k \boldsymbol{p}_k^2 \\
& + \frac{1}{2} \sum_{k=1}^{n} m_k (\sigma_{1k}^2 \boldsymbol{y}_k^2 + \sigma_{3k}^2 \boldsymbol{p}_k^2) \\
& - \sum_{k=1}^{n} m_k (\delta_k + 2\varepsilon_k) \boldsymbol{y}_k \boldsymbol{p}_k \\
& + \sum_{k=1}^{n} m_k (d_k + q_k + \gamma_k - \varepsilon_k) (\boldsymbol{y}_k + \boldsymbol{p}_k) z_k.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\mathrm{LV} = & \ \mathrm{LV}_1 + \mathrm{LV}_2 + \mathrm{LV}_3 + \mathrm{LV}_4 \\
\leqslant & - \sum_{k=1}^{n} \Big[u_k (\delta_k + \theta_k) + m_k (\delta_k + \varepsilon_k) - \frac{1}{2}(u_k + m_k) \sigma_{1k}^2 \Big] \boldsymbol{y}_k^2 \\
& - \sum_{k=1}^{n} v_k \big(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \big) z_k^2 \\
& - \sum_{k=1}^{n} \Big[w_k \big(d_k + q_k + \varepsilon_k \big) + m_k \varepsilon_k - \frac{1}{2}(w_k + m_k) \sigma_{3k}^2 \Big] \boldsymbol{p}_k^2 \\
& + \sum_{k=1}^{n} w_k (\theta_k - \varepsilon_k) \boldsymbol{y}_k \boldsymbol{p}_k + \sum_{k=1}^{n} w_k (\gamma_k - \varepsilon_k) z_k \boldsymbol{p}_k \\
& + \sum_{k=1}^{n} u_k (d_k + q_k) (\boldsymbol{y}_k z_k + \boldsymbol{y}_k \boldsymbol{p}_k) \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} I_j^* (\boldsymbol{y}_k + \boldsymbol{p}_k) z_k - \sum_{k=1}^{n} m_k (\delta_k + 2\varepsilon_k) \boldsymbol{y}_k \boldsymbol{p}_k \\
& + \sum_{k=1}^{n} m_k (d_k + q_k + \gamma_k - \varepsilon_k) (\boldsymbol{y}_k + \boldsymbol{p}_k) z_k \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} z_k (\boldsymbol{y}_k + \boldsymbol{z}_k + \boldsymbol{p}_k) z_j \\
= & - \sum_{k=1}^{n} \Big[u_k (\delta_k + \theta_k) + m_k (\delta_k + \varepsilon_k) - \frac{1}{2}(u_k + m_k) \sigma_{1k}^2 \Big] \boldsymbol{y}_k^2 \\
& - \sum_{k=1}^{n} v_k \big(\sum_{j=1}^{n} \beta_{kj} I_j^* - \frac{1}{2}\sigma_{2k}^2 \big) z_k^2 \\
& - \sum_{k=1}^{n} \Big[w_k \big(d_k + q_k + \varepsilon_k \big) + m_k \varepsilon_k - \frac{1}{2}(w_k + m_k) \sigma_{3k}^2 \Big] \boldsymbol{p}_k^2 \\
& + \sum_{k=1}^{n} \Big[w_k (\gamma_k - \varepsilon_k) - v_k \sum_{j=1}^{n} \beta_{kj} I_j^* \\
& + m_k (d_k + q_k + \gamma_k - \varepsilon_k) \Big] z_k \boldsymbol{p}_k \\
& + \sum_{k=1}^{n} \Big[u_k (d_k + q_k) - v_k \sum_{j=1}^{n} \beta_{kj} I_j^* \\
& + m_k (d_k + q_k + \gamma_k - \varepsilon_k) \Big] \boldsymbol{y}_k z_k \\
& + \sum_{k=1}^{n} \Big[u_k (d_k + q_k) + w_k (\theta_k - \varepsilon_k) - m_k (\delta_k + 2\varepsilon_k) \Big] \boldsymbol{y}_k \boldsymbol{p}_k \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} z_k (\boldsymbol{y}_k + \boldsymbol{z}_k + \boldsymbol{p}_k) z_j.
\end{aligned}$$

We can choose u_k, w_k, and m_k such that

$$u_k (d_k + q_k) + w_k (\theta_k - \varepsilon_k) - m_k (\delta_k + 2\varepsilon_k) = 0,$$
$$u_k (d_k + q_k) - v_k \sum_{j=1}^{n} \beta_{kj} I_j^* + m_k (d_k + q_k + \gamma_k - \varepsilon_k) = 0,$$
$$w_k (\gamma_k - \varepsilon_k) - v_k \sum_{j=1}^{n} \beta_{kj} I_j^* + m_k (d_k + q_k + \gamma_k - \varepsilon_k) = 0,$$

i.e.,

$$u_k = \frac{\sum_{j=1}^{n} \beta_{kj} I_j^* (\delta_k + 2\varepsilon_k)(\gamma_k - \varepsilon_k)}{(d_k + q_k) K} v_k,$$
$$m_k = \frac{(\gamma_k + \theta_k - 2\varepsilon_k) \sum_{j=1}^{n} \beta_{kj} I_j^*}{K} v_k,$$
$$w_k = \frac{\sum_{j=1}^{n} \beta_{kj} I_j^* (\delta_k + 2\varepsilon_k)}{K} v_k,$$

where

$$K = (d_k + q_k)(\gamma_k + \theta_k - 2\varepsilon_k) + (\gamma_k - \varepsilon_k)(\delta_k + \gamma_k + \theta_k),$$

then

$$
\begin{aligned}
\mathrm{LV} \leqslant & -\sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2(\delta_k+\theta_k)(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+2(\delta_k+\varepsilon_k)(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)}{2(d_k+q_k)K}\right.\\
& \left.-\frac{[(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)]\sigma_{1k}^2}{2(d_k+q_k)K}\right)v_k\boldsymbol{y}_k^2\\
& -\sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2\varepsilon_k(\theta_k+\delta_k+\gamma_k)+2(d_k+q_k)(\delta_k+2\varepsilon_k)-(\theta_k+\delta_k+\gamma_k)\sigma_{3k}^2}{2K}\right)v_k\boldsymbol{p}_k^2\\
& -\sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j-\sum_{k=1}^{n}v_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right)\boldsymbol{z}_k^2\\
= & \ \mathrm{LV}_0-\sum_{k=1}^{n}v_k\sum_{j=1}^{n}\beta_{kj}\boldsymbol{z}_k(\boldsymbol{y}_k+\boldsymbol{z}_k+\boldsymbol{p}_k)\boldsymbol{z}_j, \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{LV}_0 := & -\sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2(\delta_k+\theta_k)(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+2(\delta_k+\varepsilon_k)(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)}{2(d_k+q_k)K}\right.\\
& \left.-\frac{\big((\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)\big)\sigma_{1k}^2}{2(d_k+q_k)K}\right)v_k\boldsymbol{y}_k^2\\
& -\sum_{k=1}^{n}\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2\varepsilon_k(\theta_k+\delta_k+\gamma_k)+2(d_k+q_k)(\delta_k+2\varepsilon_k)-(\theta_k+\delta_k+\gamma_k)\sigma_{3k}^2}{2K}\right)v_k\boldsymbol{p}_k^2\\
& -\sum_{k=1}^{n}v_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right)\boldsymbol{z}_k^2.
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathscr{A}_k = & \sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2(\delta_k+\theta_k)(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+2(\delta_k+\varepsilon_k)(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)}{2(d_k+q_k)K}\right.\\
& \left.-\frac{[(\delta_k+2\varepsilon_k)(\gamma_k-\varepsilon_k)+(d_k+q_k)(\gamma_k+\theta_k-2\varepsilon_k)]\sigma_{1k}^2}{2(d_k+q_k)K}\right)v_k,\\
\mathscr{B}_k = & \ v_k\left(\sum_{j=1}^{n}\beta_{kj}I_j^*-\frac{1}{2}\sigma_{2k}^2\right),\\
\mathscr{C}_k = & -\sum_{j=1}^{n}\beta_{kj}I_j^*\left(\frac{2\varepsilon_k(\theta_k+\delta_k+\gamma_k)+2(d_k+q_k)(\delta_k+2\varepsilon_k)-(\theta_k+\delta_k+\gamma_k)\sigma_{3k}^2}{2K}\right)v_k,
\end{aligned}
$$

then

$$
\mathrm{LV}_0=-\sum_{k=1}^{n}(\mathscr{A}_k\boldsymbol{y}_k^2+\mathscr{B}_k\boldsymbol{z}_k^2+\mathscr{C}_k\boldsymbol{p}_k^2). \qquad (23)
$$

The proofs above show that if the condition (17) is satisfied, then $\mathscr{A}_k$, $\mathscr{B}_k$, and $\mathscr{C}_k$ are positive constants. Let

$$
\lambda=\min_{k\in\{1,\ldots,n\}}\{\mathscr{A}_k,\mathscr{B}_k,\mathscr{C}_k\},
$$

then $\lambda>0$. From Eqs. (22) and (23), one sees that

$$
\begin{aligned}
\mathrm{LV} \leqslant & -\lambda\sum_{k=1}^{n}\left(|\boldsymbol{x}_k(t)|^2+O(|\boldsymbol{x}_k(t)|^2)\right)\\
= & -\lambda|\boldsymbol{x}(t)|^2+O(|\boldsymbol{x}(t)|^2), \qquad (24)
\end{aligned}
$$

where

$$
\begin{aligned}
|\boldsymbol{x}_k(t)| &= \sqrt{\boldsymbol{y}_k^2(t)+\boldsymbol{z}_k^2(t)+\boldsymbol{p}_k^2(t)},\\
|\boldsymbol{x}(t)| &= \left(\sum_{k=1}^{n}|\boldsymbol{x}_k(t)|^2\right)^{1/2},
\end{aligned}
$$

and $O(|\boldsymbol{x}(t)|^2)$ is an infinitesimal of the higher order of $|\boldsymbol{x}(t)|^2$ for $t\to\infty$. Hence $\mathrm{LV}(\boldsymbol{x},t)$ is negative-definite in a sufficiently small neighborhood of $\boldsymbol{x}=\boldsymbol{0}$ for $t\geqslant 0$. According to Lemma 3, we therefore conclude that the zero solution of Eq. (15) is stochastically asymptotically stable. The proof is complete.

4. Numerical simulation

To illustrate the various theoretical results presented above, the systems (2) and (14) are simulated for various sets of parameters. Numerical methods are employed to solve and to depict the behavior of the passively immune, infectious, and recovered with respect to time. We numerically simulate the solution of systems (2) and (14) when $n=2$. In this case, we have

$$\mathscr{M}_0 = \begin{bmatrix} \dfrac{\beta_{11}(1-M_1^0-I_1^0-R_1^0)}{d_1^{\mathrm{I}}+q_1+\gamma_1} & \dfrac{\beta_{12}(1-M_1^0-I_1^0-R_1^0)}{d_1^{\mathrm{I}}+q_1+\gamma_1} \\ \dfrac{\beta_{21}(1-M_2^0-I_2^0-R_2^0)}{d_2^{\mathrm{I}}+q_2+\gamma_2} & \dfrac{\beta_{22}(1-M_2^0-I_2^0-R_2^0)}{d_2^{\mathrm{I}}+q_2+\gamma_2} \end{bmatrix} := \begin{bmatrix} \beta_{11}K_1 & \beta_{12}K_1 \\ \beta_{21}K_2 & \beta_{22}K_2 \end{bmatrix},$$

$$\mathscr{R}_0 = \rho(\mathscr{M}_0) = \frac{\beta_{11}K_1+\beta_{22}K_2+\sqrt{(\beta_{11}K_1-\beta_{22}K_2)^2+4\beta_{12}\beta_{21}K_1K_2}}{2},$$

where

$$K_i = \frac{1-M_i^0-I_i^0-R_i^0}{d_i^{\mathrm{I}}+q_i+\gamma_i}, \quad i=1,2,$$

$$M_i^0 = \frac{(d_i+q_i)\varepsilon_i}{d_i\delta_i+q_i\delta_i+\varepsilon_i\delta_i+\varepsilon_i\theta_i+\varepsilon_i d_i+\varepsilon_i q_i}, \quad I_i^0 = 0,$$

$$R_i^0 = \frac{(\delta_i+\theta_i)\varepsilon_i}{d_i\delta_i+q_i\delta_i+\varepsilon_i\delta_i+\varepsilon_i\theta_i+\varepsilon_i d_i+\varepsilon_i q_i}, \quad i=1,2.$$

Let us give the first set of parameter values

$$\beta_{11}=0.25,\ \beta_{12}=0.4,\ d_1=0.011,\ q_1=0.027,$$
$$\delta_1=0.06,\ \theta_1=0.1,\ \varepsilon_1=0.05,\ \gamma_1=0.65,$$
$$\beta_{21}=0.37,\ \beta_{22}=0.5,\ d_2=0.013,\ q_2=0.029,$$
$$\delta_2=0.09,\ \theta_2=0.25,\ \varepsilon_2=0.1,\ \gamma_2=0.6.$$

It is easy to compute

$$\mathscr{R}_0 = 0.14426 < 1.$$

We can obtain the disease-free equilibrium $M_1^0=0.1560$, $I_1^0=0$, $R_1^0=0.6568$, $M_2^0=0.1000$, $I_2^0=0$, and $R_2^0=0.8099$. Moreover, we choose $M_1(0)=0.03$, $I_1(0)=0.37$, $R_1(0)=0.22$, $M_2(0)=0.71$, $I_2(0)=0.63$, and $R_2(0)=0.05$ as the initial values, we keep all these initial values unchanged in each example unless otherwise stated. In the absence of noise, we simulate the asymptotic stability of the disease-free equilibrium of deterministic system (2) in Fig. 2.

Figure 2 illustrates the case where the parameters of deterministic MSIR model (2) verified the conditions of Theorem 1. It is observed that disease-free equilibrium state P_0 is globally asymptotically stable when $\mathscr{R}_0 \leqslant 1$. The computer simulations shown in Fig. 2 clearly support this result.

Given the second set of parameter values

$$\beta_{11}=0.6,\ \beta_{12}=0.5,\ d_1=0.011,\ q_1=0.017,$$
$$\delta_1=0.04,\ \theta_1=0.018,\ \varepsilon_1=0.0015,\ \gamma_1=0.002,$$
$$\beta_{21}=0.5,\ \beta_{22}=0.46,\ d_2=0.013,\ q_2=0.01,$$
$$\delta_2=0.03,\ \theta_2=0.015,\ \varepsilon_2=0.01,\ \gamma_2=0.012.$$

We calculate

$$\mathscr{R}_0 = 1.002 > 1.$$

We can obtain the endemic equilibrium $M_1^*=0.3150$, $I_1^*=0.4184$, $R_1^*=0.2341$, $M_2^*=0.3286$, $I_2^*=0.2735$, and $R_2^*=0.3694$. In the absence of noise, we simulate the asymptotic stability of the endemic equilibrium of deterministic system (2) in Fig. 3.

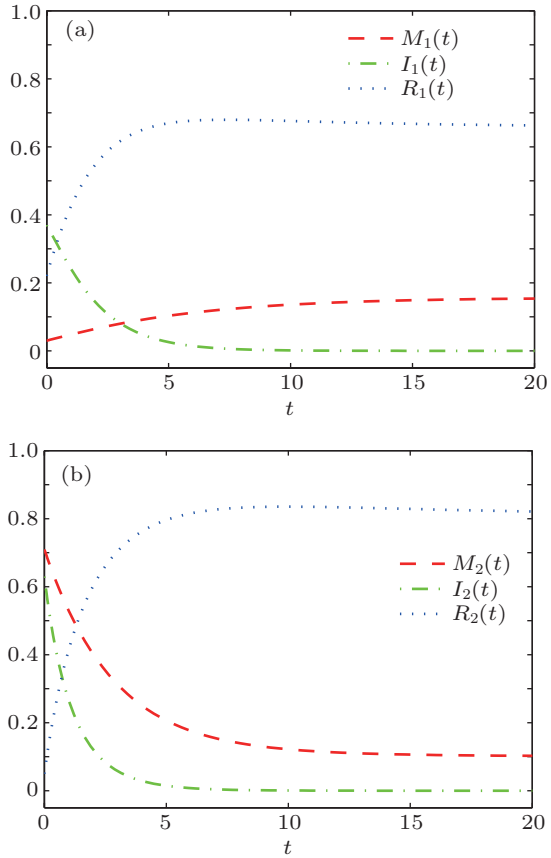


Fig. 2. (color online) Computer simulation of the asymptotic stability of the disease-free equilibrium of MSIR model (2) for initial conditions $M_1(0)=0.03$, $I_1(0)=0.37$, $R_1(0)=0.22$, $M_2(0)=0.71$, $I_2(0)=0.63$, and $R_2(0)=0.05$. (a) $M_1(t)$, $I_1(t)$, and $R_1(t)$ and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

Figure 3 supports the Theorem 2 which shows that the endemic equilibrium P^* of the system (2) is asymptotically stable with the conditions of Theorem 2.

On the other hand, we show the numerical simulation of the stochastic system (14). Given the discretization of system (14) for $t=0, \Delta t, 2\Delta t, \ldots, n\Delta t$, and $k=1,2$.

$$\begin{cases} M_{k,i+1} = M_{k,i} + \Big(-(\delta_k+\theta_k)M_{k,i} + (d_k+q_k)I_{k,i} \\ \qquad + (d_k+q_k)R_{k,i}\Big)\Delta t + \sigma_{1k}(M_{k,i}-M_k^*)\sqrt{\Delta t}\varepsilon_{1k,i}, \\ I_{k,i+1} = I_{k,i} + \Big(\beta_{k1}(1-M_{k,i}-I_{k,i}-R_{k,i})I_{1,i} \\ \qquad + \beta_{k2}(1-M_{k,i}-I_{k,i}-R_{k,i})I_{2,i} \\ \qquad - (d_k+\gamma_k+q_k)I_{k,i}\Big)\Delta t \\ \qquad + \sigma_{2k}(I_{k,i}-I_k^*)\sqrt{\Delta t}\varepsilon_{2k,i}, \\ R_{k,i+1} = R_{k,i} + \Big((\theta_k-\varepsilon_k)M_{k,i} + (\gamma_k-\varepsilon_k)I_{k,i} \\ \qquad - (d_k+q_k+\varepsilon_k)R_{k,i}\Big)\Delta t \\ \qquad + \sigma_{3k}(R_{k,i}-R_k^*)\sqrt{\Delta t}\varepsilon_{3k,i}, \end{cases} \tag{25}$$

where time increment $\Delta t > 0$, and $\varepsilon_{1k,i}$, $\varepsilon_{2k,i}$, and $\varepsilon_{3k,i}$ are $N(0,1)$-distributed independent random variables which can be generated numerically by pseudo-random number generators.

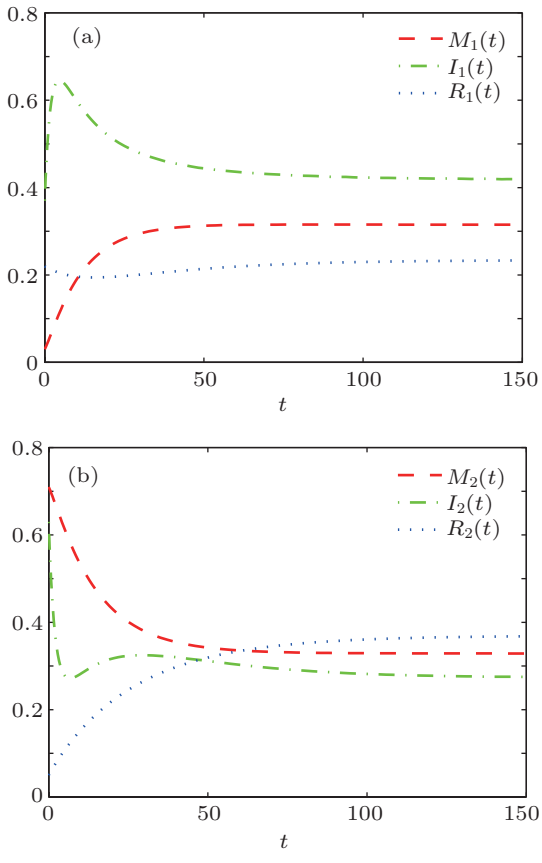


Fig. 3. (color online) Computer simulation of the asymptotic stability of the endemic equilibrium of deterministic MSIR model (2) with initial condition $M_1(0) = 0.03$, $I_1(0) = 0.37$, $R_1(0) = 0.22$, $M_2(0) = 0.71$, $I_2(0) = 0.63$, $R_2(0) = 0.05$, and step size $\Delta t = 0.001$. (a) $M_1(t)$, $I_1(t)$, and $R_1(t)$ and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

To see the effect of the noise intensity, we consider three series of different values. Figure 4 corresponds to $\sigma_{11} = 0.25$, $\sigma_{21} = 0.81$, $\sigma_{31} = 0.20$, $\sigma_{12} = 0.26$, $\sigma_{22} = 0.75$, $\sigma_{32} = 0.23$, it is easy to verify the noise intensity and system parameters obey condition (17). We can therefore conclude, by Theorem 3, that the endemic equilibrium P^* of stochastic model (14) is asymptotically stable. The computer simulations shown in Fig. 4 agree well with the mathematical result. Note that the condition (17) is just a sufficient condition. When this condition is not satisfied, the stochastic system (14) may be or may not be stable. If we choose $\sigma_{11} = 12$, $\sigma_{21} = 15$, $\sigma_{31} = 25$, $\sigma_{12} = 28$, $\sigma_{22} = 15$, $\sigma_{32} = 16$, then the solution of the stochastic system (14) is not asymptotically stable but explodes to infinity at the finite time (see Fig. 5).

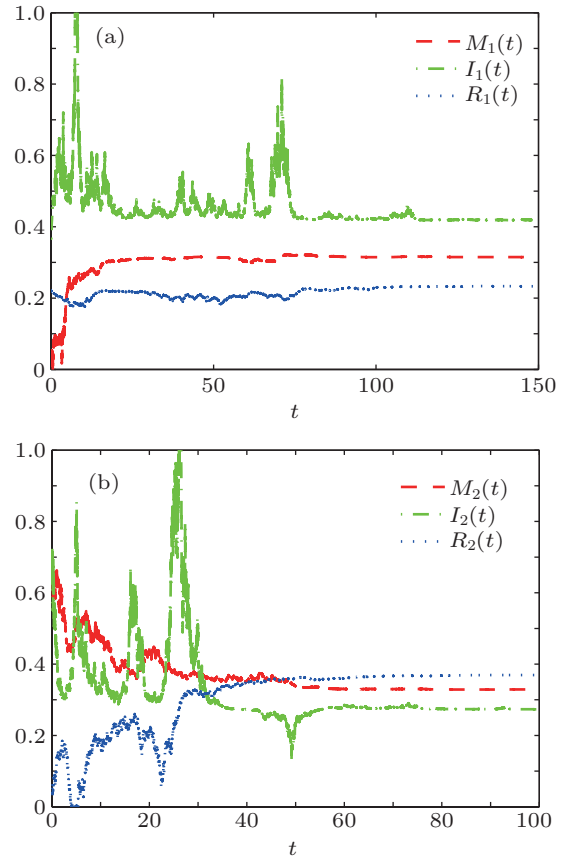


Fig. 4. (color online) Computer simulation of the asymptotic stability of stochastic MSIR model (14) for $\sigma_{11} = 0.25$, $\sigma_{21} = 0.81$, $\sigma_{31} = 0.20$, $\sigma_{12} = 0.26$, $\sigma_{22} = 0.75$, $\sigma_{32} = 0.23$, and $\Delta t = 0.001$. (a) $M_1(t)$, $I_1(t)$, and $R_1(t)$ and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

We show the phase space portrait for the deterministic MSIR endemic model in Fig. 6. For the stochastic case, figure 6 corresponds to $\sigma_{11} = 0.25$, $\sigma_{21} = 0.81$, $\sigma_{31} = 0.20$, $\sigma_{12} = 0.26$, $\sigma_{22} = 0.75$, $\sigma_{32} = 0.23$, the noise intensity and system parameters obey the condition (17), therefore, by Theorem 10, the endemic equilibrium P^* of stochastic model (14) is asymptotically stable. We can see from Fig. 7 that oscillation appears under environmental driving forces, which actually affect the deterministic curves shown in Fig. 6. These two trajectories in Fig. 7 still maintain the same total trend as those of Fig. 6 and reach the same equilibrium points as the deterministic version. Simulations agree with our result.

Let us analyze the impact of vaccination rates of the passively immune infants and the susceptibles on the MSIR model

by assigning different values to them, as provided in Table 1 by calculating the equilibrium of system (2), which is, of course, an equilibrium of system (14).

Analyzing the data in Table 1, it shows that the higher the values of the vaccination rates are, the higher the value of the endemic equilibrium is.

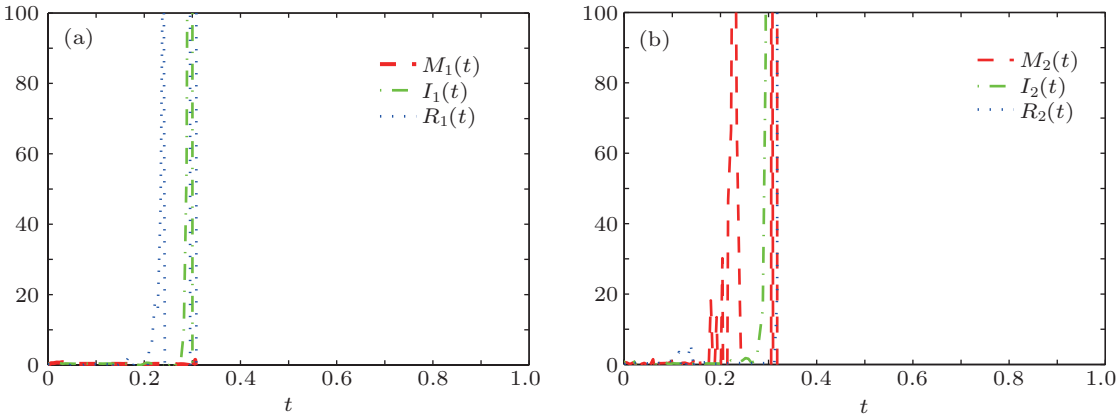


Fig. 5. (color online) Stochastic trajectories of MSIR model (14) for $\sigma_{11} = 12$, $\sigma_{21} = 15$, $\sigma_{31} = 25$, $\sigma_{12} = 28$, $\sigma_{22} = 15$, $\sigma_{32} = 16$. (a) $M_1(t)$, $I_1(t)$, and $R_1(t)$ and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

Table 1. The influences of the different vaccination rates on equilibrium.

	$\theta_1 = 0.018,\ \varepsilon_1 = 0.0015,$ $\theta_2 = 0.015,\ \varepsilon_2 = 0.0100$	$\theta_1 = 0.020,\ \varepsilon_1 = 0.0017,$ $\theta_2 = 0.017,\ \varepsilon_2 = 0.0105$	$\theta_1 = 0.022,\ \varepsilon_1 = 0.0019,$ $\theta_2 = 0.019,\ \varepsilon_2 = 0.011$
M_1^*	0.3150	0.3078	0.3010
I_1^*	0.4184	0.4086	0.3993
R_1^*	0.2341	0.2511	0.2672
M_2^*	0.3286	0.3192	0.3104
I_2^*	0.2735	0.2651	0.2572
R_2^*	0.3694	0.3872	0.4041

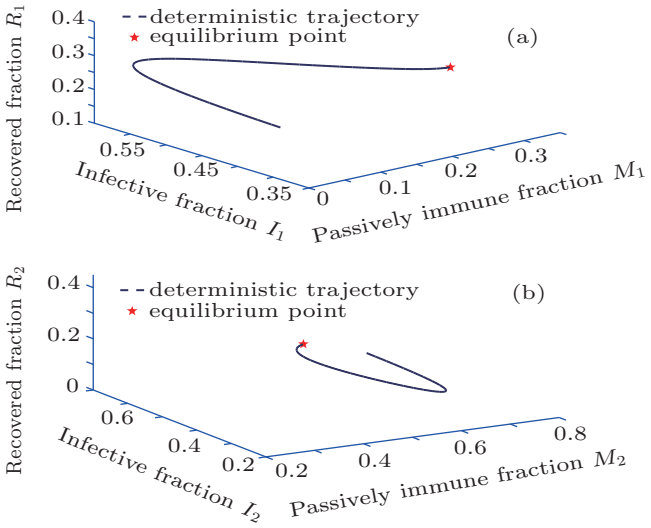


Fig. 6. (color online) Phase space portrait of the deterministic MSIR model (2) for initial condition $M_1(0) = 0.03$, $I_1(0) = 0.37$, $R_1(0) = 0.22$, $M_2(0) = 0.71$, $I_2(0) = 0.63$, and $R_2(0) = 0.05$. (a) M_1(t), I_1(t), and R_1(t) and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

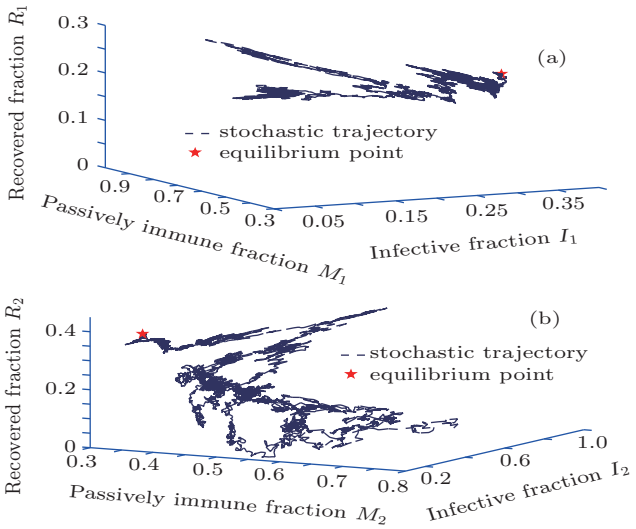


Fig. 7. (color online) Phase space portrait of the MSIR model (14) with stochastic perturbations for $\sigma_{11} = 0.25$, $\sigma_{21} = 0.8$, $\sigma_{31} = 0.20$, $\sigma_{12} = 0.26$, $\sigma_{22} = 0.75$, $\sigma_{32} = 0.23$, and $\Delta t = 0.001$. (a) M_1(t), I_1(t), and R_1(t) and (b) $M_2(t)$, $I_2(t)$, and $R_2(t)$.

5. Conclusion

This paper presented a mathematical study describing the dynamical behavior of an MSIR epidemic model. Our purpose was based on analyzing this behavior using both a deterministic model and a stochastic one. The basic reproduction number $\mathscr{R}_0$ is still an important characteristic, which determines the stability of disease-free equilibrium and endemic equilibrium.

If the basic reproduction number $\mathscr{R}_0 \leqslant 1$, then the disease-free equilibrium is globally asymptotically stable and the disease cannot "invade" the population. The deterministic model has unique endemic equilibrium if the reproduction number $\mathscr{R}_0$ is greater than one; however, stability of the endemic equilibrium is determined not only by the threshold quantity $\mathscr{R}_0$, but also by the other conditions $\gamma_k > \varepsilon_k, \gamma_k + \theta_k > 2\varepsilon_k$. In this case, the disease will persist at the endemic equilibrium level if it is initially present. Moreover, the investigation of this stochastic model revealed that the stochastic stability of P^* depends on the magnitude of the intensity of noise as well as the parameters involved within the model system. The higher vaccination rates of the passively immune infants and the susceptibles have important influences on diminishing the prevalence rate of the disease. Understanding these transmission characteristics of the MSIR model in communities and regions are useful in making general forecasts and estimating the uncertainty in forecasts, and formulating the control programs in order to decrease the transmission of some epidemics.

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