The mass limit of white dwarfs with strong magnetic fields in general relativity*

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Recently, U. Das and B. Mukhopadhyay proposed that the Chandrasekhar limit of a white dwarf could reach a new high level ($2.58M_\odot$) if a superstrong magnetic field were considered (Das U and Mukhopadhyay B 2013 *Phys. Rev. Lett.* **110** 071102), where the structure of the strongly magnetized white dwarf (SMWD) is calculated in the framework of Newtonian theory (NT). As the SMWD has a far smaller size, in contrast with the usual expectation, we found that there is an obvious general relativistic effect (GRE) in the SMWD. For example, for the SMWD with a one Landau level system, the super-Chandrasekhar mass limit in general relativity (GR) is approximately 16.5% lower than that in NT. More interestingly, the maximal mass of the white dwarf will be first increased when the magnetic field strength keeps on increasing and reaches the maximal value $M = 2.48M_\odot$ with $B_{\rm D} = 391.5$. Then if we further increase the magnetic fields, surprisingly, the maximal mass of the white dwarf will decrease when one takes the GRE into account.

Keywords: strongly magnetize field, white dwarf, general relativity effect

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1. Introduction

About 80 years ago, Chandrasekhar set a famous mass limit for the white dwarf, that is, the mass of a white dwarf cannot exceed $1.44M_\odot$.[1] This limit is obtained from the fact that in a white dwarf the inward gravitational force is balanced by the outward pressure produced by the degenerate electrons. A white dwarf is not only an interesting theoretical issue, but also has important application in astrophysics. If an accreting white dwarf in a binary system obtains mass continually from its companion star, its mass may exceed the Chandrasekhar limit and will explode, named a type-Ia supernova explosion. Type-Ia supernovae play a key role in understanding the expansion of the universe, as all of the supernovae come from the same mechanism and can be used as standard candles in measuring cosmic distances.[2,3] The most famous result using the Ia supernovae is the discovery that the universe expansion is accelerating.[4,5]

Recently, observations have shown that there exist some peculiar type-Ia supernovae, such as SN2003fg, SN2006gz, SN2007if, and SN2009dc.[6,7] They have extraordinarily higher luminosities but lower kinetic energies. These type-Ia supernovae suggest that their progenitors, white dwarfs in binary systems, have masses up to $2.4M_\odot$–$2.8M_\odot$, which is far higher than the classical Chandrasekhar limit.[8] If this is true, some fundamental questions are raised: what is the new super-Chandrasekhar mass limit and what new mechanism can support the new limit? Moreover, does this mean that a new understanding of standard candles is needed? In fact, for another compact star—the neutron star—it has also been a hot topic to seek the mass limit in both observations and theoretical studies in recent years.[9–11]

Some efforts have been made in seeking the new mechanism and new mass limit of white dwarfs. For example, an accreting white dwarf accompanied by a differential rotation can support a super-Chandrasekhar limit.[12] Das and Mukhopadhyay also proposed a new mechanism recently, that is, a superstrong magnetic field in a white dwarf can support a higher mass limit.[8,13,14] They established a new mass limit up to $2.58M_\odot$ by considering a central magnetic field as strong as 8.8×10^{17} Gs (1 Gs = 10^{-4} T).[13] This result is very interesting and may provide a new standard candle in understanding the expansion history of the universe. It is worth noting that in the works of Das and Mukhopadhyay all of the structures of SMWDs are calculated in the framework of Newtonian gravity theory. As we have known, usually we use the ratio $R/R_{\rm g}$ to reflect the strength of the general relativistic effect (GRE), where R and $R_{\rm g}$ ($=2GM/c^2$) are the radius of the white dwarf and the Schwarzschild radius of the same star, respectively. The larger value of ratio means a weaker GRE, while in the white dwarf typically this ratio is several thousands; hence the GRE in a classical white dwarf is very weak and in most cases it is enough to deal with white dwarfs in the framework of Newtonian theory (NT).[15] However, for the SMWD, as it has a far smaller radius and has a figure similar to the neutron star, hence the ratio $R/R_{\rm g}$ will be much smaller than the usual case, which indicates that the GRE may not be neglected

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as in the usual white dwarf situation. Thus the white dwarfs with strong magnetic field should be treated similar to the neutron stars, that is, by using the TOV equation as the hydrostatic equation.[16–20]

The paper is organized as follows. After the introduction, in Section 2 we briefly recall the modified equation of state (EOS) of white dwarf matter in strong magnetic field. In Section 3, the GRE on white dwarfs with strong magnetic field is calculated and discussed. In Section 4, we analyze the cause of the general relativistic effect. In Section 5, the GRE on the maximal mass of the white dwarfs with varying magnetic fields is examined. We conclude with a short summary in Section 6.

2. The modified equation of state of white dwarf matter in strong magnetic field

For the self-completeness of our paper and the convenience of the readers, we first briefly review the modified EOS of white dwarf matter in the strong magnetic field caused by the Landau level, and this section mainly along the line in Refs. [8] and [13]. Consider an electron with mass m_e in a uniform magnetic fields along the z direction. According to relativistic quantum mechanics, the transverse motion is quantized into Landau levels. In this case the energy of a free electron will be modified as[21]

$$E = \left[p_z^2c^2 + m_e^2c^4\left(1 + \nu\frac{2B}{B_c}\right)\right]^{1/2}, \tag{1}$$

where p_z is the momentum along the z direction, B_c is the critical magnetic field with $B_c = m_e^2c^3/\hbar e = 4.414\times10^{13}$ Gs and $\nu = n + 1/2 + \sigma$ with $n = 0, 1, 2\ldots$, $\sigma = \pm1/2$ denoting the Landau level. As a result, the Fermi energy E_F of the electrons for the Landau level ν can be written as follows:

$$E_F^2 = p_F^2(\nu)c^2 + m_e^2c^4(1 + 2\nu B_D), \tag{2}$$

where $B_D = B/B_c$ is the dimensionless magnetic field. Since the square of Fermi momentum $p_F^2(\nu) \geq 0$, we can easily obtain the maximum number of occupied Landau levels from Eq. (2)

$$\nu_m = \frac{\varepsilon_{F_{max}}^2 - 1}{2B_D}, \tag{3}$$

where $\varepsilon_F = E_F/m_ec^2$ is the dimensionless Fermi energy, and $\varepsilon_{F_{max}} = E_{F_{max}}/m_ec^2$ is the dimensionless maximum Fermi energy of a given system determined by B_D and ν. Since we mainly focus on the superstrong magnetic effects of the white dwarfs, we only consider the one Landau level system (higher Landau level corresponding to weaker magnetic field). According to Refs. [8] and [13] the density ρ and pressure p of the degenerate electron gas read respectively

$$\rho = \frac{\mu_e m_H B_D}{2\pi^2\lambda_e^3}\sqrt{\left(\frac{E_F}{m_ec^2}\right)^2 - 1}, \tag{4}$$

$$p = \frac{B_D m_ec^2}{4\pi^2\lambda_e^3}\left(\frac{E_F\sqrt{E_F^2 - m_e^2c^4}}{(m_ec^2)^2} - \ln\left(\frac{E_F + \sqrt{E_F^2 - m_e^2c^4}}{m_ec^2}\right)\right), \tag{5}$$

where $\lambda_e = \hbar/m_ec$ denotes the Compton wavelength of an electron. Since the authors of Ref. [13] using $\varepsilon_F = 200$, $\nu = 1$, $B_D = 19999.5$ to get the new Chandrasekhar mass limit, for convenience of comparison we also first investigate the GRE of SMWD with the same data settings. Under these conditions, combining Eqs. (4) and (5), the ordinary EOS of the white dwarf will be modified as[13]

$$p = K\rho^2, \tag{6}$$

where

$$K = \frac{m_ec^2\pi^2\lambda_e^3}{(\mu_e m_H)^2 B_D}.$$

Figure 1 plots the EOS of white dwarf matter.

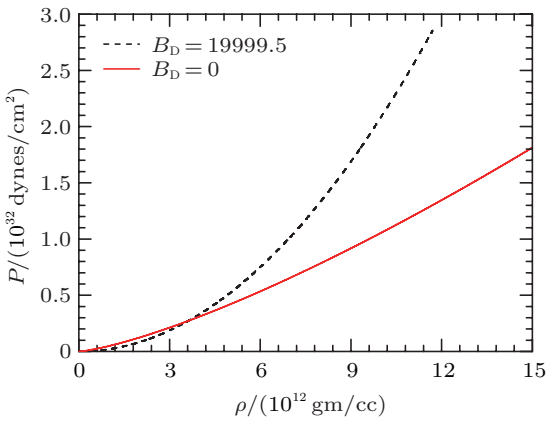


Fig. 1. (color online) The equation of state (EOS) of white dwarf matter, where the dashed line shows the EOS described by Eq. (6) with $\nu = 1$ and $B_D = 19999.5$, while the solid line shows the ordinary EOS ($B_D = 0$).

3. General relativistic effects on white dwarfs with strong magnetic field

Based on the modified EOS obtained in the last section, we can calculate the mass–radius relation of the SMWD. First we review the results under the Newtonian framework,[8,13] then we take into account the GRE, and make a comparison between these two formalisms. In the framework of Newtonian gravity theory, the structure of the white dwarf is determined by the following hydrostatic equilibrium equation:

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}, \tag{7}$$

$$\frac{dm}{dr} = 4\pi\rho r^2. \tag{8}$$

Combining Eqs. (7) and (8) yields

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2\,\mathrm{d}p}{\rho\,\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{9}$$

If the EOS takes the form of $p = K\rho^{1+1/n}$, introducing $\rho = \rho_c\theta^n$ with ρ_c being the central density of the white dwarf and $r = a\xi$ with

$$a = \sqrt{\frac{(n+1)K\rho^{1-n/n}}{4\pi G}},$$

then the above equation reduces to the famous Lane–Emden equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}p}{\mathrm{d}\xi}\right) = -\theta^n\xi^2. \tag{10}$$

Note that, when the magnetic field is strong enough, the EOS of Eq. (6) implies $n = 1$. Thus by solving the $n = 1$ Lane–Emden equation analytically,[8,13] the mass and radius of the SMWD read respectively

$$M = \frac{1}{(\mu_e m_H)^2}\left(\frac{hc}{2G}\right)^{3/2} \approx 2.58M_\odot, \tag{11}$$

and

$$R = \sqrt{\frac{K\pi}{2G}} \approx 0.07\times 10^8\ \mathrm{cm}, \tag{12}$$

while from Eqs. (3) and (4) we can get the maximum central density

$$\rho_{c\max} = \frac{\mu_e m_H B_D}{2\pi^2\lambda_e^3}\sqrt{2B_D}. \tag{13}$$

It is worth noting that if the EOS does not take the form of a polytope, then equations (7) and (8) can only be solved by a numerical method with the help of the boundary conditions: $m = 0$ at $r = 0$ and $p = 0$ at $r = R$. As a convenient comparison, we plot the numerical result of the Newtonian formalism in Fig. 2 with the dot–dashed line. Now we turn to the GRE of the SMWD. Since we only consider the uniform and static magnetic field, in order to incorporate the GRE, the Newtonian hydrostatic equation can be replaced by the Tolman–Oppenheimer–Volkoff (TOV) equation[16,17]

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -G\frac{(p+\rho c^2)(m+4\pi r^3 p/c^2)}{r(rc^2-2Gm)}, \tag{14}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho. \tag{15}$$

With the help of the boundary conditions, $m = 0$ at $r = 0$ and $p = 0$ at $r = R$, we can solve the above equations numerically to obtain the mass–radius relation of the SMWD. The numerical results are also plotted in Fig. 2 with the solid line; it is shown that the maximum mass now reduced to $2.16M_\odot$ due to the GRE.

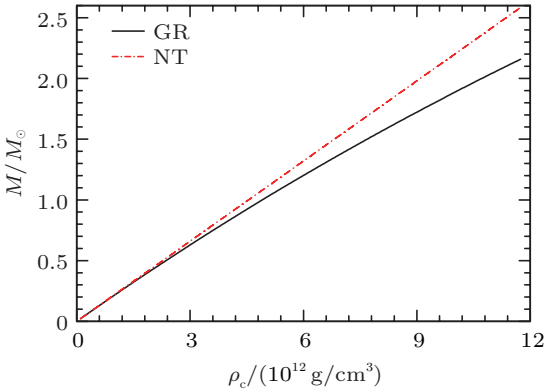


Fig. 2. (color online) Comparison of the relations of stellar mass versus central density in the two gravity theories. The dot–dashed line represents the relation under NT, while the solid line represents the relation under the GR. Both of the lines use the same conditions: $\nu = 1$ and $B_D = 19999.5$.

This result is rather interesting. It is shown that the GRE becomes notable as the mass of the SMWD increases, which is contrast with our usual expectation that the GRE can be neglected in the calculation of the Chandrasekhar limit. To get a deeper understanding of this result, we let

$$\Delta_m\% = \frac{M - M_{GR}}{M}\times 100\%$$

represent the percentage of relative mass difference at a fixed central density, where R and M are the radius and mass of the white dwarf under the Newtonian hydrostatic equation, and M_{GR} is the mass of the white dwarf under general relativity. Our calculation shows that the maximum percentage of relative mass difference can reach $\Delta_m\% \approx 16.5\%$. It is worth noting that in Ref. [22] the authors already realized that the GRE would become obvious when $R/R_g \le 10$. One can see clearly in Fig. 3 that $R/R_g \approx 9$ when the mass of the SMWD approaches the new Chandrasekhar limit.

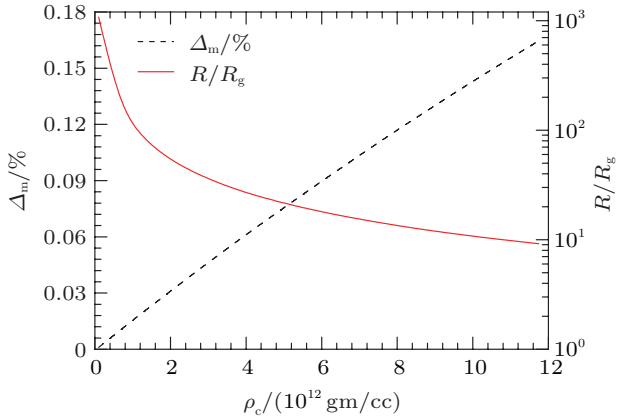


Fig. 3. (color online) The dashed line represents the percentage of relative mass difference at the same central density (labeled by the left y axis). The solid line represents the ratio of R/R_g (labeled by the right y axis). Both the lines have the same horizontal coordinate as the central density with the condition that $\nu = 1$ and $B_D = 19999.5$.

4. Analyze the cause of the general relativistic effect

In order to know the origin of the GRE, we have to analyze the two different equations (Newtonian hydrostatic equilibrium equation versus TOV equation). Here we mainly focus on the maximum central density ρ_c, which makes the largest mass difference between the two gravity theories. Comparing Eq. (7) of NT with Eq. (14) in GR, one can easily see that the gradient of p becomes smaller when one takes GRE into account. Hence under the same EOS, the radius of the white dwarf calculated with the TOV equation is smaller than that calculated with the Newtonian hydrostatic equilibrium equation. Similarly, the mass and density become smaller when one takes GRE into account. Figure 4 shows the pressure and mass varying with respect to the radius at the maximum central density. One can see clearly that the GRE will make pressure decrease a little faster from the center to the surface.

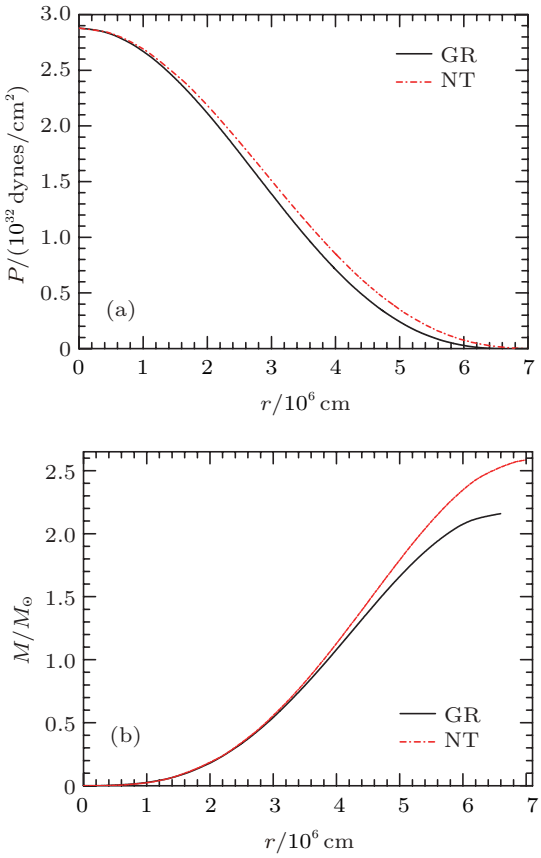


Fig. 4. (color online) (a), (b) The pressure and the mass distributions as functions of r within a white dwarf with $\rho_c = 1.1742 \times 10^{13}$ g cm^{-3}, $\nu = 1$, and $B_D = 19999.5$, respectively.

5. General relativistic effects on the maximal mass of the white dwarfs with varying magnetic fields

In order to highlight the GRE appearing in the SMWD, we use the same method as mentioned above to calculate the maximal mass of the white dwarf with varying magnetic fields. We let the magnetic field strength B_D vary in the range 4×10^4 Gs–2×10^4 Gs. For a given B_D, we can calculate the EOS from Eqs. (4) and (5).

To obtain similar results for comparison, we again calculate the maximal mass of the white dwarf with two different structure equations (Newtonian hydrostatic equilibrium equation and TOV equation). By a series of detailed numerical calculations, we plot the maximal mass of the white dwarf with varying magnetic fields in Fig. 5. To make a clearer comparison, we also plot the percentage of relative mass difference $\Delta_m\%$ in the same figure. For example, when $B_D = 199.5$, the maximum masses are $2.52M_\odot$ and $2.47M_\odot$ corresponding to the Newtonian hydrostatic equation and TOV equation, respectively. The mass difference $\Delta_m\% \approx 1.92\%$, which is much smaller than the percentage of relative mass difference (16.5%) under the condition $B_D = 19999.5$. Obviously, at a lower magnetic field the difference of the stellar masses in two gravity theories is very weak. That is, a manifest GRE does not exist for the white dwarfs with a weaker magnetic field.

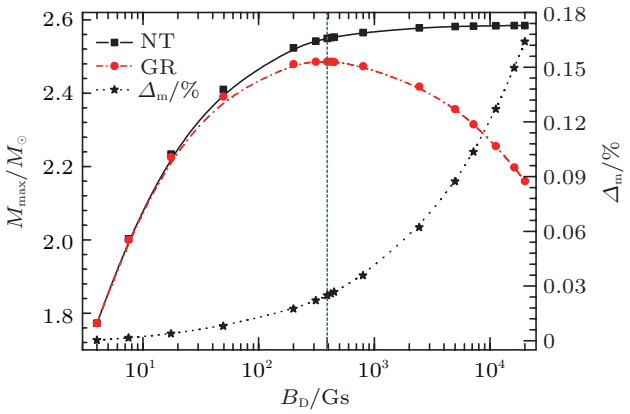


Fig. 5. (color online) The solid line represents M_{max} under the Newtonian hydrostatic equilibrium equation, the dot–dashed line represents M_{max} under the TOV equation, and the dotted line is the percentage of the relative mass difference. Where the left y axis denotes the maximum mass of the white dwarf in the condition of various B_D, the right y axis represents the mass difference between two gravity theories under the same B_D, and the vertical line represents the magnetic field corresponding to the maximum M_{max} under the TOV equation.

More interestingly, this figure shows that, when one takes the GRE into account, the maximal mass will be first increased when the magnetic field strength keeps on increasing and reaches the maximal value $M = 2.48M_\odot$ with $B_D = 391.5$. Then if we further increase the magnetic fields, surprisingly, the maximal mass of the white dwarf will decrease when one takes the GRE into account. One can also read from Fig. 5 that the GRE will become more significant when the magnetic field B_D increases. In other words, the GRE not only modifies the maximal mass of the SMWD, but also sets a mass limit for the SMWD at $B_D = 391.5$, which corresponds to the magnetic field $B = 1.73 \times 10^{16}$ Gs. In order to give the readers

a more intuitive picture, we list some representative points of our figure in Table 1.

Table 1. The values of $B_{\rm D}$, $M_{\rm max}$, $M_{\rm TOVmax}$, and $\Delta_m/\%$.

$\varepsilon_{\rm F}$	$B_{\rm D}$	$M/M_\odot$	$M_{\rm TOV}/M_\odot$	$\Delta_m/\%$
3	4	1.7729	1.7677	0.293
4	7.5	2.0028	1.9949	0.394
6	17.5	2.2341	2.2208	0.595
10	49.5	2.4104	2.3864	0.996
20	199.5	2.5232	2.4747	1.922
25	312	2.5417	2.4811	2.384
28	391.5	2.5491	2.4813	2.659
30	449.5	2.5530	2.4804	2.844
40	799.5	2.5654	2.4693	3.746
70	2449.5	2.5780	2.4138	6.369
100	4999.5	2.5816	2.3523	8.882
150	11250	2.5838	2.2521	12.838
200	19999.5	2.5844	2.1573	16.526

As to the origination of such a superstrong magnetic field in the white dwarfs, we prefer the explanation of Das and Mukhopadhyay.[13] Both in Newtonian gravity and in general relativity, for an accreting magnetized white dwarf, the increasing total mass strengthens the gravity and hence pulls the star to a smaller size. Due to the conservation of the total magnetic flux in such an accretion process, the magnetic field in the star will increase continuously, until the inner magnetic field exceeds the critical strength $B_{\rm c}$ and the Landau quantization becomes significant. In Newtonian gravity, as shown by the solid line in Fig. 5 (also shown by the line marked with filled circles in Fig. 2 in Ref. [13]), as the magnetic field strength increases in the evolutionary track, the stellar mass will progressively increase to a maximal value around $2.58M_\odot$. In general relativity, the evolutionary track of the white dwarf with increasing the magnetic field will be along the dash–dot line in Fig. 5, where the maximal mass is about $2.48M_\odot$ at $B_{\rm D} = 391.5$. Similar to the Newtonian gravity case discussed by Das and Mukhopadhyay,[13] when the continuous accretion process makes the mass of the white dwarf approach $2.48M_\odot$, a violent thermonuclear reaction will be sparked off and then a type-Ia supernova takes place. Therefore, it is natural to conclude that the mass limit $2.48M_\odot$ is the absolute limit; any further increase of the magnetic field strength (and thus decrease of stellar mass) is impossible. Furthermore, if a white dwarf is born with a lower mass and with a higher magnetic field strength, it will be unstable and explode immediately for any additional mass accretion.

6. Conclusions

In this paper, we investigated the issue of the GRE on SMWDs. The achievement is that, in contrast with usual expectation, we found that when the magnetic field is strong enough the GRE will become significant. In addition, we compared our results for the mass–radius relation with previous results which calculated within the framework of NT. Our calculations indicate that the maximum mass of the SMWD obtained by Das and Mukhopadhyay[13] will reduce by 16.5% if we take the GRE into account. Besides, we also calculated the pressure and mass varying with radius at the maximum central density, and the results show that there is indeed a significant GRE in the SMWD. In order to justify when the GRE becomes notable, we further calculated the maximal mass of the white dwarfs with varying magnetic fields, i.e. we let $B_{\rm D} \in (4, 2\times10^4)$, and find that when one takes GRE into account the maximal mass will be first increased when the magnetic field strength keeps on increasing and reaches the maximal value $M_{\rm max} = 2.48M_\odot$ with $B_{\rm D} = 391.5$. Then if we further increase the magnetic field, surprisingly, the maximal mass of the white dwarf will decrease when one takes the GRE into account. In other words, the GRE not only modifies the maximal mass of the SMWD but also sets a severe constraint on the magnetic field, namely, when the magnetic field exceeds the threshold $B = 1.73\times10^{16}$ Gs, the maximal mass of the SMWD will decrease rather than increase. For a stable white dwarf, when the continuous accretion process makes the stellar mass approach $2.48M_\odot$, a violent thermonuclear reaction will be sparked off and a type-Ia supernova will explode. Therefore, the mass limit $2.48M_\odot$ is the absolute limit; any further increase of the magnetic field strength (and thus decrease of stellar mass) is impossible. In addition, if a white dwarf is born with a lower mass and with a higher magnetic field strength, any additional mass accretion will make the white dwarf explode immediately. This result is also consistent with the conclusion of Refs. [23] and [24], where the authors constrain the magnetic fields from the stability issue.

It is worth noting that there are still many aspects of this theory that deserve further investigation. For instance, apart from a superstrong magnetic field inside the white dwarf, can we find other possible types of mechanism to go beyond the Chandrasekhar limit? Furthermore, in this paper, we only consider a uniform and static magnetic field; it is also interesting to generalize our framework to incorporate varying magnetic field effects. We leave these interesting questions for future study.

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