Double coherence resonance of the FitzHugh–Nagumo neuron driven by harmonic velocity noise*

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(Received 17 November 2013; revised manuscript received 18 February 2014; published online 20 June 2014)

The effect of noise frequency on the FitzHugh–Nagumo neuron is investigated by the use of the harmonic velocity noise, which has a direct frequency parameter and no zero frequency part of the power spectrum. It is shown that the neuron has the resonance characteristic strongly responding to the noise with a certain frequency at fixed power, and there is double coherence resonance related to the frequency and the intensity. If the harmonic velocity noise lacks low frequency ingredients, there is no synchronization between the frequency of the neuron and that of the noise. Thus the low frequency part of the noise plays an important role in creating the synchronization.

Keywords: coherence resonance, FitzHugh–Nagumo neuron, harmonic noise, frequency effect

PACS: 05.40.Ca, 05.45.Xt, 87.16.dj **DOI:** 10.1088/1674-1056/23/8/080504

1. Introduction

Stochastic resonance (SR) reveals a positive effect of a noise in an excitable system, i.e., a system will be quite regular at an appropriate (moderate) amount of noise.[1] Since SR was put forward by Benzi *et al.*,[2] many scientists from the circles of biology,[3] physics,[4] and chemistry have paid considerable attention to this counter-intuitive phenomenon. Especially in a neuronal system, it has extensively been reported that the noise can enhance the ability for the neuron to receive and transmit signals.[5–8] Coherence resonance (CR), i.e., the autonomous stochastic resonance, is the SR in the absence of an external periodic signal.[9–11] Since the periodic signal is not common in a real neuronal system, the CR is considered to be more natural and universal.[12,13]

In most of the cases, intensity is an appropriate factor of noise.[1] However, SR or CR can be induced also by other factors. For example, the correlation of a discrete noise can induce CR in the FitzHugh–Nagumo (FHN) neuron,[9] the correlation time of O–U noise can lead to an optical CR,[14] the correlation time of color noise can control internal SR in a chemical model system,[15] and the increase of bounded noise color can enhance the SR in an FHN system.[16] The examples above indicate that the frequency of noise is an influencing factor for SR or CR other than the intensity. It has been proved in experiment that neurons have resonance characteristics, and they respond more strongly to the stimuli with a certain frequency.[17,18] Longtin also pointed out that the frequency sensitivity of the neural response should be taken into account.[19]

Usually, the coordinate variable of a damped oscillator driven by Gaussian white noise is regarded as quasi-monochromatic noise, which is called harmonic noise.[20] Baltanás and Casado studied the bursting behavior of the FHN neuron driven by harmonic noise.[21] The neuron responds strongly to harmonic noise with a certain frequency at fixed power.[22] The harmonic noise has the frequency characteristic, but it lacks the direct frequency parameter.[23] If the time derivative of a harmonic noise is used as the noise source, which is called harmonic velocity noise, a direct frequency parameter exists and its effects can be analyzed. The zero frequency part of such a noise vanishes while the zero frequency part of harmonic noise always exists. In this paper, the CR caused by the harmonic velocity noise in an FHN neuron is investigated.

2. Harmonic velocity noise

The noise oscillator has the following form,

$$\begin{aligned} \dot{x}(t) &= h(t), \\ \dot{h}(t) &= -\Gamma h(t) - \Omega^2 x(t) + \xi(t), \end{aligned} \tag{1}$$

where Ω and Γ are respectively the frequency and the damping parameters. $\xi(t)$ is a Gaussian white noise with $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(s) \rangle = 2\alpha\delta(t-s)$, and α is a parameter related to the intensity of the white noise. If the coordinate variable $x(t)$ of Eq. (1) acts as a noise source, it is the harmonic noise.[20] If the velocity variable $h(t)$ is taken as a noise source, its steady correlation function based on Eq. (1) is,

$$\langle h(t)h(s) \rangle = \frac{\alpha}{\mu_1^2 - \mu_2^2} \left[-\mu_1 \mathrm{e}^{\mu_1 |t-s|} + \mu_2 \mathrm{e}^{\mu_2 |t-s|} \right], \tag{2}$$

where μ_1 and μ_2 are two roots of the characteristic equation $\mu^2 + \Gamma\mu + \Omega^2 = 0$ and we have used the initial distribution $\langle x^2(0) \rangle = \alpha/\Gamma\Omega^2$, $\langle h^2(0) \rangle = \alpha/\Gamma$, and $\langle x(0)h(0) \rangle = 0$.

The above velocity noise is called harmonic velocity noise (HVN). The power spectrum of HVN is the Fourier

*Project supported by the National Natural Science Foundation of China (Grant No. 11175128).

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transformation of Eq. (2), and expressed as

$$S(\omega)=\int_{-\infty}^{\infty}\langle h(t)h(0)\rangle\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t=\frac{2\alpha\omega^2}{\Gamma^2\omega^2+(\omega^2-\Omega^2)^2}. \quad (3)$$

The power of HVN can be deduced by the residue theorem as

$$W=\int_0^{\infty}S(\omega)\,\mathrm{d}\omega=\frac{\alpha\pi}{\Gamma}. \quad (4)$$

Here, $\alpha=D\Gamma$, and D is the white noise intensity. Such a choice can ensure that the power of HVN relates only to D, and Ω or Γonly affects the form of the power spectrum. In Ref. [21], Ω is fixed, and $\alpha=D\Gamma$ is chosen for the same reason.

The power spectrum contains all the information of a noise. Figure 1 indicates that Ω controls the central frequency of HVN and Γ controls the frequency bandwidth. HVN is a band-passing noise with the frequency peak at $\omega=\Omega$. It also indicates that there is $S(\omega)=0$ at $\omega=0$, namely, the zero frequency part of HVN vanishes, which is absolutely different from the harmonic noise, whose zero frequency part is always nonzero.[23] A comparison of HVN with the harmonic noise at the same power, reveals the more obvious difference between them for the larger Γ and the smaller Ω (Fig. 2). These two noises can induce absolutely disparate results over a long time in a free diffuse system.[23]

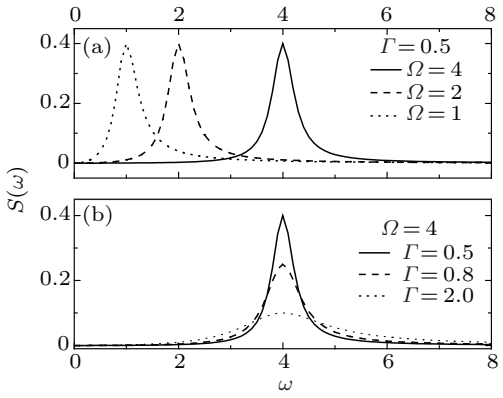


Fig. 1. Power spectra of HVN with different values of Ω (a) and Γ (b), and $D=0.1$.

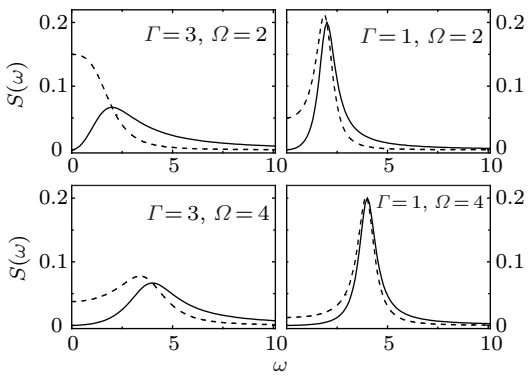


Fig. 2. Comparisons between HVN (solid line) harmonic noise (dotted line) at different values of Ω and Γ for $D=0.1$.

3. Neuron model

The dimensionless stochastic FHN model[24] is given by

$$\varepsilon\frac{\mathrm{d}v}{\mathrm{d}t}=f(v)-u+h(t), \quad (5a)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=v-u-a_0, \quad (5b)$$

where $v(t)$ and $u(t)$ are respectively the voltage and the recovery variable. a_0 and ε are constants, with ε being the time scale that is much smaller than one, implying that $v(t)$ is the fast and $u(t)$ is the slow variable. The nonlinear function is $f(v)=v(v-0.5)(1-v)$, and $h(t)$ is an HVN in Eq. (1).

According to the nonlinear theory, the unique equilibrium point (v_0,u_0) of Eq. (1) is

$$v_0=0.5+\sqrt[3]{-c_2+\sqrt{c_2^2+c_1^3}}+\sqrt[3]{-c_2-\sqrt{c_2^2+c_1^3}}, \quad (6a)$$

$$u_0=0.5-a_0+\sqrt[3]{-c_2+\sqrt{c_2^2+c_1^3}}+\sqrt[3]{-c_2-\sqrt{c_2^2+c_1^3}}, \quad (6b)$$

where

$$c_1=\frac{1}{4},\quad c_2=\frac{0.5-a_0-h(t)}{2}.$$

Let

$$p=-\left[\frac{\partial F}{\partial v}+\frac{\partial G}{\partial u}\right]_{v_0,u_0}=1+\frac{1}{\varepsilon}\left(3v_0^2-3v_0+0.5\right),$$

$$q=\left[\frac{\partial F}{\partial v}\frac{\partial G}{\partial u}-\frac{\partial G}{\partial v}\frac{\partial F}{\partial u}\right]_{v_0,u_0}=\frac{1}{\varepsilon}\left(3v_0^2-3v_0+1.5\right),$$

and $\Delta=p^2-4q$ is a function of $a_0,\varepsilon,h(t)$.

The excitability of the FHN system is associated with a supercritical Hopf-bifurcation.[1] We select $a_0=0.2466$, and $\varepsilon=0.01$. There are $p>0$ and $\Delta=p^2-4q<0$ without $h(t)$, thus the equilibrium point is a stable focus associated with a stable limit circle, and the corresponding intrinsic circular frequency is $|\sqrt{p^2-4q}|/2=9.98$. There exists $p\approx 0$ at $h(t)=0.0186$ where the supercritical Hopf-bifurcation will appear. When $h(t)>0.0186$, the limit cycle becomes unstable and the spikes come into being. Briefly, the occurrence of the spike is controlled by $h(t)$ which is described by the three parameters, i.e., D, Ω, and Γ.

Equations (1) and (2) can be simulated numerically by using a fourth-order stochastic Runge–Kutta algorithm.[17] In the calculations, we use the time step $\Delta t=0.003$ and $a_0=0.2466$, $\varepsilon=0.01$.

4. Numerical results and analyses

4.1. Spike train with different values of Ω

The HVNs and the spike trains with different values of Ω are shown in Fig. 3. It can be seen that there exists a certain Ω of the noise at which the spike train is most regular. It is shown that the neuron responds more strongly to noise with a certain frequency (i.e. the frequency CR exists).

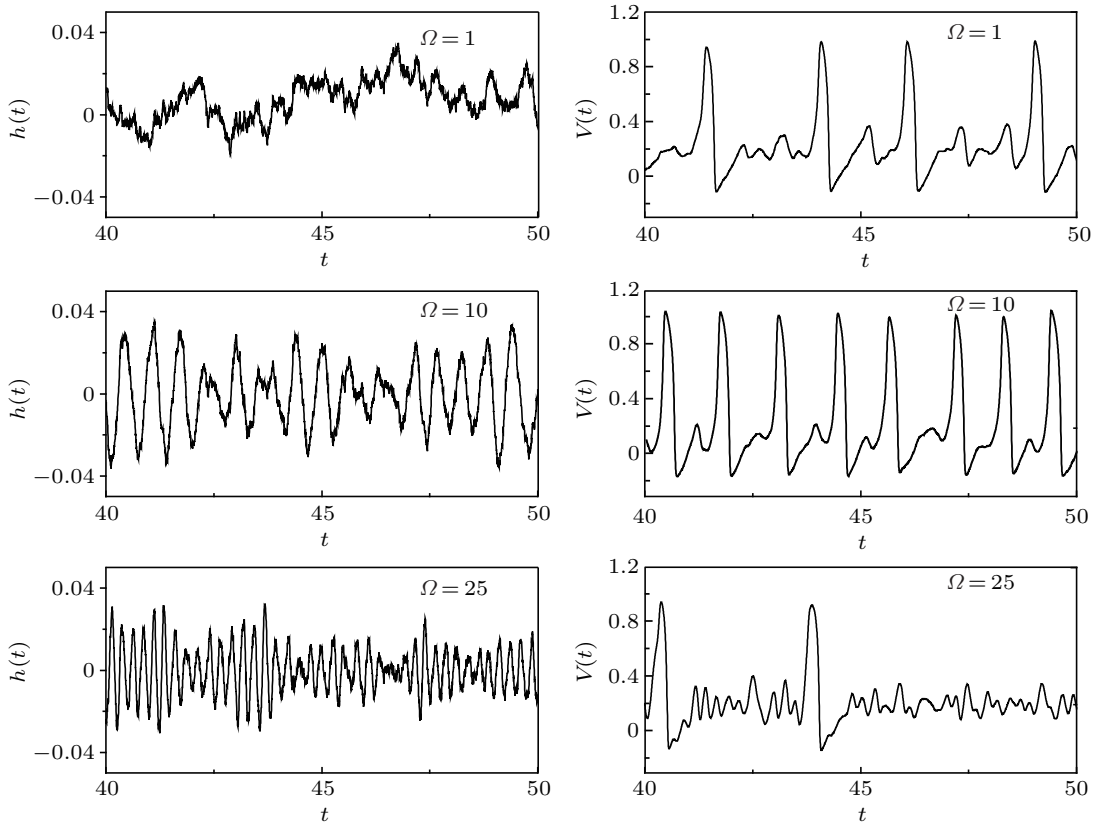


Fig. 3. Time-dependent HVNs (left) and the corresponding spike trains (right) at different values of Ω, $D = 0.0002$, and $\Gamma = 1$.

4.2. Double coherence resonance

The coefficient of the variation, R, is the most common measure of the spike train regularity,[1] which is defined as the ratio of the standard deviation of the interspike intervals (ISIs) to its mean, i.e.,

$$R = \frac{\sqrt{\langle T^2 \rangle - \langle T \rangle^2}}{\langle T \rangle}, \tag{7}$$

where

$$\langle T^2 \rangle = \lim_{N\to\infty} \sum_N (t_{i+1} - t_i)^2 / N, \quad \langle T \rangle = \lim_{N\to\infty} \sum_N (t_{i+1} - t_i)/N.$$

The value of R will be smaller as the train becomes more regular, and a strictly periodic spiking has $R = 0$. The total number of ISIs is fixed at $N = 5 \times 10^4$ in the present case.

The coherence resonance is manifested by the minimum of R, i.e., R has a non-monotonic transition.[13] In Fig. 4, we can see that R has a minimum at about $\Omega = 10$, which is just the value corresponding to the intrinsic frequency 9.98 of the limit circle without $h(t)$. It is also shown that the frequency CR exists. The intrinsic frequency of the dynamical system plays a very important role in governing the driven dynamics.[24] A limit cycle with a time period exists in a dynamical system. Applying an external driving force with the same period, the external energy can be well transferred to the system through the resonance mechanism.

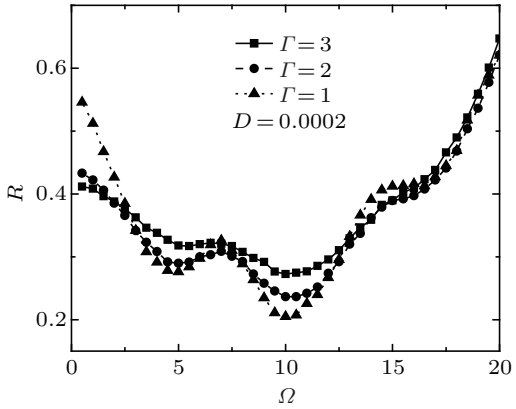


Fig. 4. Curves of the coefficient of variation R versus Ω at different values of Γ and $D = 2 \times 10^{-4}$.

The larger the damping parameter Γ, the flatter the curve will be. The reason is that due to the wider frequency bandwidth and the greater amount of different ingredients for larger Γ, the response of neurons to a certain frequency becomes weaker. Furthermore, R has two relative minima respectively at $\Omega = 5$ and $\Omega = 15$. Since the intrinsic frequency of the limit circle is about 10, there are 1:2 and 3:2 frenquency resonances.[24] Integer ratio frequency resonance is usually related to the same ratio phase-locking (i.e. synchronization), which appears in Arnold's tongues. The term 'Arnold's tongue' is coined to describe the regions in the parameter space, where an oscillator is locked onto the phase of an external forcing oscillator. However, we find that the synchroniza-

tion is not always accompanied with the frequency resonance in the later analysis.

At the same time as the frequcency CR, the FHN system still has the CR relating to the noise power (i.e. noise intensity). In Fig. 5, the values of R are all non-monotonic functions of D. Thus, the FHN neuron driven by HVN has a double coherence resonance, as shown in Fig. 6. The lowest point of the double coherence resonance is $R = 0.195$ at $\Omega = 11.5$ and $D = 5 \times 10^{-4}$.

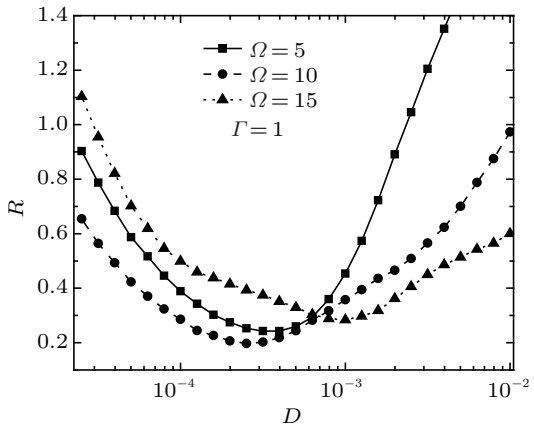


Fig. 5. Curves of the coefficient of variation R versus D at different values of Ω and $\Gamma = 1$.

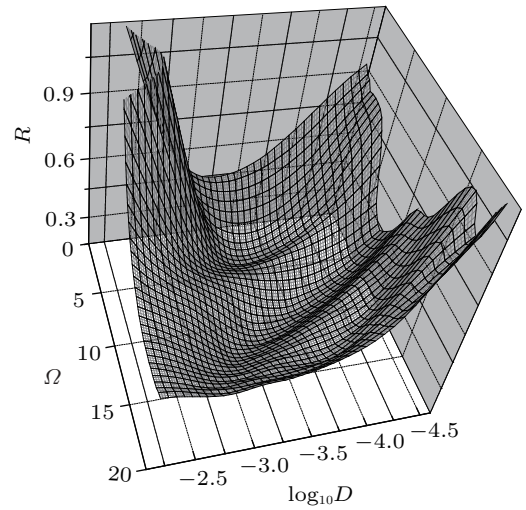


Fig. 6. Double coherence resonance with Ω and D at $\Gamma = 1$.

4.3. Condition of synchronization

When a nonliner system is related to an external frequency, the synchronization is often taken into consideration. CR is usually related to the stochastic phase-locking, i.e. the imperfect synchronization.[19,24] It has been proved that the imperfect synchronization exists in the FHN neuron driven by a harmonic noise.[21,22]

Now, we come to investigate the percentage of ISIs with a certain frequency ω in the spike train:[24]

$$P(\omega) = N(\omega)/N. \tag{8}$$

The total number of ISIs is still fixed at $N = 5 \times 10^4$. $N(\omega)$ represents the number of ISIs that are equal to a certain time scale T, with the corresponding frequency $\omega = 2\pi/T$. For $N(\omega_0)$, only those frequencies that fall into a narrow region around ω_0 are counted, i.e., $|\omega - \omega_0| < \Delta$ and Δ is fixed at $\Delta = 0.1$.

The dependence of $P(\omega)$ on ω for three different values of Ω are shown in Fig. 7. $P(\omega)$ for $\Omega = 10$ has a very sharp and narrow peak, corresponding to the minimum of R in Fig. 4 at $\Omega = 10$. The peak of $\Omega = 10$ also appears at $\omega \approx 5$, which means that the frequency of most spikes is inconsistent with the noise frequency, thus the synchronization is absent. As can be seen also at $\Omega = 10$ in Fig. 3, the period of HVN is almost twice the value of $v(t)$, that is to say, the frequency CR is not always accompanied by the synchronization.

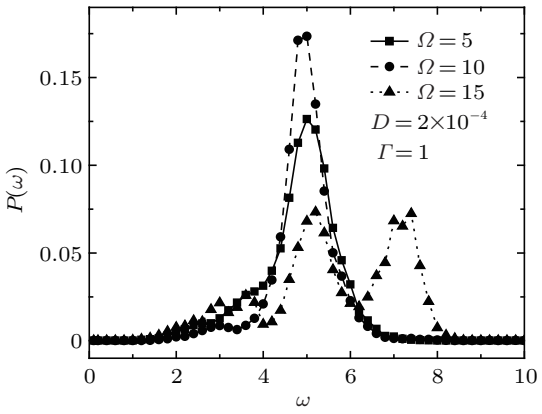


Fig. 7. Curves of $P(\omega)$ versus ω at different values of Ω, $\Gamma = 1$, and $D = 2 \times 10^{-4}$.

An imperfect synchronization exists in the FHN neuron driven by a harmonic noise.[21,22] The main difference between HVN and the harmonic noise is the absence of the low frequency part in HVN. We plot $P(\omega)$ at different smaller values of Ω in Fig. 8 and find the imperfect synchronizations, i.e., the maxima of $P(\omega)$ appear at $\omega = \Omega$. The reason is that the HVN at a smaller Ω still has a certain number of low frequency ingredients so as to maintain the synchronization. The synchronization will disappear if $\Omega \geq 8$. Therefore, the low frequency part of the noise may play an important role in creating the synchronization.

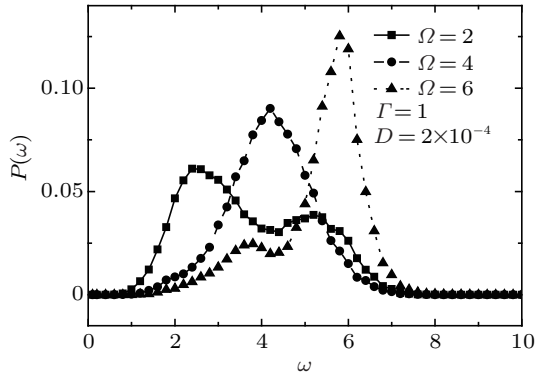


Fig. 8. Curves of $P(\omega)$ versus ω at different smaller values of Ω, $\Gamma = 1$, $D = 2 \times 10^{-4}$.

5. Summary

We investigated the frequency in the HVN, the velocity variable of a noise oscillator, and its effects on the FHN neuron model. It is indicated that the frequency CR exists. Driven by HVN at a fixed power and a certain frequency, the spike train becomes most regular, and the coefficient of the variation R reaches a minimum. At the same time, the intensity CR still coexists with the frequency CR, and then there is a double CR. The double CR provides a frequency method to control the regularity of spikes more than the noise power.

At a smaller Ω, there is a synchronization between the frequency of the neuron and that of the noise, as HVN still has a certain number of low frequency ingredients. The synchronization will disappear when Ω becomes larger. We believe that the low frequency part of a noise is an important condition under which the synchronization is created. For the same reason, the synchronization exists in the FHN neuron driven by the harmonic noise that always has the low frequency part. This may be a contribution to reducing the excessive synchronization in the neural system that is related to certain diseases, such as parkinsonism, epilepsy and so on.

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