Stochastic resonance in an over-damped linear oscillator*

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(Received 18 December 2013; revised manuscript received 18 February 2014; published online 20 June 2014)

For an over-damped linear system subjected to both parametric excitation of colored noise and external excitation of periodically modulated noise, and in the case that the cross-correlation intensity between noises is a time-periodic function, we study the stochastic resonance (SR) in this paper. Using the Shapiro–Loginov formula, we acquire the exact expressions of the first-order and the second-order moments. By the stochastic averaging method, we obtain the analytical expression of the output signal-to-noise ratio (SNR). Meanwhile, we discuss the evolutions of the SNR with the signal frequency, noise intensity, correlation rate of noise, time period, and modulation frequency. We find a new bona fide SR. The evolution of the SNR with the signal frequency presents periodic oscillation, which is not observed in a conventional linear system. We obtain the conventional SR of the SNR with the noise intensity and the correlation rate of noise. We also obtain the SR in a wide sense, in which the evolution of the SNR with time period modulation frequency presents periodic oscillation. We find that the time-periodic modulation of the cross-correlation intensity between noises diversifies the stochastic resonance phenomena and makes this system possess richer dynamic behaviors.

Keywords: stochastic resonance, signal-to-noise ratio, periodically modulated noise, cross-correlation intensity between noises

PACS: 05.40.–a, 05.40.Ca, 02.50.–r **DOI:** 10.1088/1674-1056/23/8/080503

1. Introduction

In 1981, Benzi *et al.*[1] first introduced the concept of stochastic resonance (SR) to explain the periodic recurrence of ice ages on Earth. Since then, the phenomenon of SR has been studied extensively in theory and experiment,[2–20] and has been found in many fields such as laser systems, biological systems, optical systems, and electronic systems. The bona fide SR, which was introduced by Gammaitoni *et al.*,[21] implies that the signal-to-noise ratio (SNR) of the system response versus the frequency of the periodic signal exhibits a peak. The conventional SR refers to the phenomenon that the SNR is non-monotonic with the changes of characteristic parameters of noise (such as the noise intensity and the correlation rate). The SR in a wide sense was introduced by Gitterman,[2] which means that the non-monotonic behaviors of some functions of the system response (such as the moments, the autocorrelation function, the power spectrum, and the SNR) depend on the changes of characteristic parameters (such as the amplitude and the frequency of the input signal, the noise intensity, and the correlation rate, etc.).

According to the early research, the SR phenomenon is typically found in nonlinear dynamical systems driven by periodic signals and noise.[3–5] Nevertheless, recent research shows that SR can also occur in linear systems subjected to a periodic signal and multiplicative noise or linear systems driven just by multiplicative noises.[6–8] In those studies mentioned above, the periodic signal and noise are usually introduced into the systems in the additive form. However, periodically modulated noise often occurs in some actual physical systems (such as an expander in optics or radio astronomy), so many scholars began to study this kind of linear system driven by periodically modulated noise.[9–12]

In 1991, Fulinski and Telejko[13] pointed out that the quantum noise and pump noise in the laser system are cross-correlative, and found that the cross-correlation between noises can change the dynamical behaviors of the system. Since then, the cross-correlation between noises has attracted great attention, and its influences on the SR in both linear and non-linear systems[8–16] have been studied. However, in most of those studies, what is mainly considered is the case that the cross-correlation intensity is constant; the case where the cross-correlation intensity is changed periodically with time is little reported. Specifically, in some actual physical systems (such as the electronic, laser, and quantum systems with cyclical work status like circulation and pulse), the cross-correlation intensity between system thermal noise, input noise, and output noise is transformed periodically with time. Therefore, it is significant to investigate the SR phenomena in the system where the cross-correlation intensity between noises is a periodic function of time, and this research can meet the realistic and objective physical and technical requirements. Considering the correlation intensity be-

*Project supported by the National Natural Science Foundation of China (Grant No. 11171238), the Young Teacher Fund of Fujian Agriculture and Forestry University, China (Grant No. 2011XJJ23), and the Science and Technology Project of the Education Department of Sichuan Province, China (Grant No. 14ZA0050).

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tween noises, which is modulated by the time period, Chen and Wang,[17] and Chen and Zhang[18] discussed the phenomenon of SR in the single-mode laser system; Tessone and Wio,[19] and Tessone *et al.*[20] investigated the effects on SR in a nonlinear system; and Tian *et al.*[16] first studied the effects on SR in an over-damped linear system.

In the present paper, the over-damped linear system, which is driven by correlated additive and multiplicative noises and a periodically modulated noise, is discussed. By using the Shapiro–Loginov formula[22] and the stochastic averaging method, we obtain the exact expressions of the first-order and second-order moments, and the SNR of the output signal. Furthermore, according to the exact expressions, we investigate the non-monotonic behaviors of the SNR with various system parameters, and find that the bona fide SR, conventional SR, and SR in a wider sense appear in this system.

2. System model

We consider an over-damped linear system driven by correlated additive and multiplicative noises, and a periodically modulated noise, which is described by the stochastic differential equation:

$$\dot{x}(t)+\left[\omega_0^2+\xi(t)\right]x(t)=A\sin(\Omega t)\xi(t)+\eta(t), \tag{1}$$

where ω_0 is the intrinsic frequency of the system, $A\sin(\Omega t)\xi(t)$ is a periodically modulated noise, and A and Ω are the amplitude and frequency of the periodic force, respectively, $\eta(t)$ is the white noise, which formally is the derivative of the Wiener process. The mean and correlation of $\eta(t)$ are given by

$$\langle\eta(t)\rangle=0,\ \ \langle\eta(t)\eta(s)\rangle=2D_\eta\delta(t-s), \tag{2}$$

where D_η is the noise intensity of $\eta(t)$. $\xi(t)$ is the stochastic fluctuation of ω_0, and is modeled as asymmetric dichotomous noise, which takes two values B and $-C$. The transition rate of $\xi(t)$ from B to $-C$ is p, and the inverse transition rate is q. The statistical properties of $\xi(t)$ are

$$\langle\xi(t)\rangle=0,\ \ \langle\xi(t)\xi(s)\rangle=D_\xi\lambda_\xi\exp\left[-\lambda_\xi|t-s|\right], \tag{3}$$

with $\lambda_\xi=p+q$, $D_\xi\lambda_\xi=BC$, and $\Lambda_\xi=B-C$, where D_ξ and λ_ξ are the noise intensity and the correlation rate of $\xi(t)$, respectively, and Λ_ξ denotes the asymmetry of $\xi(t)$. Moreover, $\eta(t)$ and $\xi(t)$ are cross-correlative, and the cross-correlation intensity between them is modulated time-periodically and satisfies

$$\langle\xi(t)\eta(s)\rangle=2k(t)\sqrt{D_\xi D_\eta}\,\delta(t-s), \tag{4}$$

where $k(t)$ is the cross-correlation intensity, and $|k(t)|\leq 1$. Here $k(t)$ is a time-periodic function, that is, $k(t)=\lambda\sin(\Theta t+\phi)$, where λ, Θ, and ϕ are the coefficient, frequency, and initial phase of time-periodic modulation, respectively. Here, let $\phi\in[0,\pi]$, $\lambda\in[-1,1]$.

2.1. First-order moment of the system stationary state response

Averaging Eq. (1), we obtain

$$\langle\dot{x}(t)\rangle=-\omega_0^2\langle x(t)\rangle-\langle\xi(t)x(t)\rangle. \tag{5}$$

Applying the Shapiro-Loginov formula[22] to $\langle\xi(t)x(t)\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\xi(t)x(t)\rangle=\langle\xi(t)\dot{x}(t)\rangle-\lambda_\xi\langle\xi(t)x(t)\rangle. \tag{6}$$

Multiplying both sides of Eq. (1) by $\xi(t)$ and averaging all the terms, we have

$$\begin{aligned}&\langle\xi(t)\dot{x}(t)\rangle+\omega_0^2\langle\xi(t)x(t)\rangle+\langle\xi^2(t)x(t)\rangle\\&=A\sin(\Omega t)\langle\xi^2(t)\rangle+\langle\xi(t)\eta(t)\rangle.\end{aligned} \tag{7}$$

Here, equation (5) contains the new high order correlation factor $\langle\xi^2(t)x(t)\rangle$, and to reduce the order, we need to use the properties of $\xi(t)$:

$$\begin{aligned}\langle\xi^2(t)x(t)\rangle&=\Lambda_\xi\langle\xi(t)x(t)\rangle+D_\xi\lambda_\xi\langle x(t)\rangle,\\\langle\xi(t)\eta(t)\rangle&=2\lambda\sqrt{D_\xi D_\eta}\sin(\Theta t+\phi).\end{aligned} \tag{8}$$

Inserting Eq. (8) into Eq. (7), we have

$$\begin{aligned}\langle\xi(t)\dot{x}(t)\rangle=&-(\omega_0^2+\Lambda_\xi)\langle\xi(t)x(t)\rangle-D_\xi\lambda_\xi(\langle x(t)\rangle\\&-A\sin(\Omega t))+2\lambda\sqrt{D_\xi D_\eta}\sin(\Theta t+\phi).\end{aligned} \tag{9}$$

Simplifying Eqs. (5), (6), and (9), we obtain the linear differential equations of $\langle x(t)\rangle$ and $\langle\xi(t)x(t)\rangle$ as follows:

$$\begin{cases}\langle\dot{x}(t)\rangle=d_1\langle x(t)\rangle-\langle\xi(t)x(t)\rangle,\\\dfrac{\mathrm{d}}{\mathrm{d}t}\langle\xi(t)x(t)\rangle=-c\langle x(t)\rangle+d_2\langle\xi(t)x(t)\rangle\\\quad+d_3\sin(\Omega t)+d_4\sin(\Theta t+\phi),\end{cases} \tag{10}$$

where $d_1=-\omega_0^2$, $d_2=-(\omega_0^2+\lambda_\xi+\Lambda_\xi)$, $d_3=cA$, $d_4=2\lambda\sqrt{D_\xi D_\eta}$, and $c=D_\xi\lambda_\xi$.

For the sake of the stability of the first-order moment of $x(t)$, we must ensure that the characteristic equation of the corresponding homogeneous equations of the obtained Eq. (10) cannot have positive real roots, that is, the following inequality must hold: $D_\xi\lambda_\xi<\omega_0^2(\omega_0^2+\lambda_\xi+\Lambda_\xi)$. In this paper, we assume that the condition is satisfied. Thus, in the case of the long-time limit $t\to\infty$, the influence of initial conditions will vanish, and the asymptotic expressions of $\langle x(t)\rangle$ and $\langle\xi(t)x(t)\rangle$ are written in the following forms:

$$\begin{aligned}\langle x(t)\rangle_{as}=&\ m_1\sin(\Omega t)+m_2\cos(\Omega t)\\&+m_3\sin(\Theta t)+m_4\cos(\Theta t),\\\langle\xi(t)x(t)\rangle_{as}=&\ m_5\sin(\Omega t)+m_6\cos(\Omega t)\\&+m_7\sin(\Theta t)+m_8\cos(\Theta t),\end{aligned}$$

where

$$m_1 = \frac{d_3(\Omega^2-\beta_2)}{\Omega^4+\beta_1\Omega^2+\beta_2^2},\quad m_2 = -\frac{d_3\Omega(d_1+d_2)}{\Omega^4+\beta_1\Omega^2+\beta_2^2},\quad m_3 = \frac{d_4[\Theta(d_1+d_2)\sin\phi+(\Theta^2-\beta_2)\cos\phi]}{\Theta^4+\beta_1\Theta^2+\beta_2^2},$$
$$m_4 = \frac{d_4[(\Theta^2-\beta_2)\sin\phi-\Theta(d_1+d_2)\cos\phi]}{\Theta^4+\beta_1\Theta^2+\beta_2^2},\quad m_5 = -\frac{d_3(d_2\Omega^2+d_1\beta_2)}{\Omega^4+\beta_1\Omega^2+\beta_2^2},\quad m_6 = \frac{d_3\Omega[\beta_2-\Omega^2-d_1(d_1+d_2)]}{\Omega^4+\beta_1\Omega^2+\beta_2^2},$$
$$m_7 = \frac{d_4\Theta\sin\phi[(\Theta^2-\beta_2)+d_1(d_1+d_2)]-d_4\cos\phi(d_1\beta_2+d_2\Theta^2)}{\Theta^4+\beta_1\Theta^2+\beta_2^2},$$
$$m_8 = -\frac{d_4\sin\phi(d_1\beta_2+d_2\Theta^2)+d_4\Theta\cos\phi[(\Theta^2-\beta_2)+d_1(d_1+d_2)]}{\Theta^4+\beta_1\Theta^2+\beta_2^2},$$
$$\beta_1 = (d_1^2+d_2^2+2c),\quad \beta_2 = (d_1d_2-c).$$

2.2. Second-order moment of the system stationary state response

Using a similar method to calculate the stationary state second-order moment of the system response $x(t)$, we obtain the linear differential equations of $\langle x^2(t)\rangle$ and $\langle\xi(t)x^2(t)\rangle$:

$$\begin{cases} \dfrac{\mathrm{d}\langle x^2(t)\rangle}{\mathrm{d}t} \\ = 2d_1\langle x^2(t)\rangle - 2\langle\xi(t)x^2(t)\rangle \\ \quad + 2A\sin(\Omega t)\langle\xi(t)x(t)\rangle + 2\langle\eta(t)x(t)\rangle, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\xi(t)x^2(t)\rangle \\ = -2c\langle x^2(t)\rangle + 2d_5\langle\xi(t)x^2(t)\rangle + 2d_3\langle x(t)\rangle \\ \quad + 2d_6\sin(\Omega t)\langle\xi(t)x(t)\rangle + 2\langle\xi(t)x(t)\eta(t)\rangle, \end{cases} \tag{11}$$

where $d_5 = -(\omega_0^2+\lambda_\xi/2+\Lambda_\xi)$, $d_6 = A\Lambda_\xi$.

In this paper, the following inequality must hold for the sake of the stability of the second-order moment of the system response: $D_\xi\lambda_\xi < \omega_0^2(\omega_0^2+\lambda_\xi/2+\Lambda_\xi)$.

Owing to the fact that equation (11) contains the new coupling terms $\langle\eta(t)x(t)\rangle$ and $\langle\xi(t)x(t)\eta(t)\rangle$, we need to do decoupling. Fulinski regarded the Gaussian white noise as the limit state of symmetric dichotomous noise,[23] and his interpretation of the Gaussian white noise is consistent with that of Stratonovich.[24] Thus, in this paper, we assume that the Gaussian white noise $\eta(t)$ is the limit form of symmetric dichotomous noise $\xi(t)$, whose correlation rate is satisfied when $\lambda_\xi \to \infty$.[16] Consequently, we obtain the linear differential equations of $\langle\eta(t)x(t)\rangle$ and $\langle\xi(t)x(t)\eta(t)\rangle$:

$$\begin{cases} \dfrac{\mathrm{d}\langle\eta(t)x(t)\rangle}{\mathrm{d}t} \\ = -(\omega_0^2+\lambda_\eta)\langle\eta(t)x(t)\rangle - \langle\xi(t)x(t)\eta(t)\rangle \\ \quad + 2A\lambda\sqrt{D_\xi D_\eta}\sin(\Omega t)\sin(\Theta t+\phi) + 2D_\eta, \\ \dfrac{\mathrm{d}\langle\xi(t)x(t)\eta(t)\rangle}{\mathrm{d}t} \\ = -(\omega_0^2+\lambda_\xi+\lambda_\eta)\langle\xi(t)x(t)\eta(t)\rangle - c\langle\eta(t)x(t)\rangle. \end{cases} \tag{12}$$

When $t\to\infty$ and $\lambda_\eta\to\infty$, by solving Eq. (12), we have

$$\langle\eta(t)x(t)\rangle = 0,\quad \xi(t)x(t)\eta(t)\rangle = 0. \tag{13}$$

In the case of the long-time limit $t\to\infty$, substituting Eq. (13) into Eq. (11), we obtain the asymptotic expression of $\langle x^2(t)\rangle$ as follows:

$$\begin{aligned} &\langle x^2(t)\rangle \\ &= m_9\sin(\Omega t)+m_{10}\cos(\Omega t)+m_{11}\sin(\Theta t) \\ &\quad + m_{12}\cos(\Theta t)+m_{13}\sin(2\Omega t)+m_{14}\cos(2\Omega t) \\ &\quad + m_{15}\sin(\Omega+\Theta)t+m_{16}\cos(\Omega+\Theta)t \\ &\quad + m_{17}\sin(\Omega-\Theta)t+m_{18}\cos(\Omega-\Theta)t+m_{19}, \end{aligned} \tag{14}$$

where

$$m_9 = \frac{4d_3[m_1(\Omega^2-\beta_4)+2m_2\Omega(d_1+d_5)]}{\Omega^4+\beta_3\Omega^2+\beta_4^2},\quad m_{10} = \frac{4d_3[-2m_1\Omega(d_1+d_5)+m_2(\Omega^2-\beta_4)]}{\Omega^4+\beta_3\Omega^2+\beta_4^2},$$
$$m_{11} = \frac{4d_3[m_3(\Theta^2-\beta_4)+2m_4\Theta(d_1+d_5)]}{\Theta^4+\beta_3\Theta^2+\beta_4^2},\quad m_{12} = \frac{4d_3[-2m_3\Theta(d_1+d_5)+m_4(\Theta^2-\beta_4)]}{\Theta^4+\beta_3\Theta^2+\beta_4^2},$$
$$m_{13} = -\frac{m_5\Omega\left[d_6(d_1+d_5)+A(\Omega^2+d_5^2+c)\right]+m_6\left[d_6(\beta_2-\Omega^2)+A(d_1\Omega^2+d_5\beta_2)\right]}{2(\Omega^4+\beta_1\Omega^2+\beta_2^2)},$$
$$m_{14} = \frac{m_5\left[d_6(\beta_2-\Omega^2)+A(d_1\Omega^2+d_5\beta_2)\right]-m_6\Omega\left[d_6(d_1+d_5)+A(\Omega^2+d_5^2+c)\right]}{2(\Omega^4+\beta_1\Omega^2+\beta_2^2)},$$
$$m_{15} = -\frac{m_7(\Omega+\Theta)\left\{4d_6(d_1+d_5)+A\left[(\Omega+\Theta)^2+4(d_5^2+c)\right]\right\}+2m_8\left\{d_6\left[\beta_4-(\Omega+\Theta)^2\right]+A\left[d_1(\Omega+\Theta)^2+d_5\beta_4\right]\right\}}{(\Omega+\Theta)^4+\beta_3(\Omega+\Theta)^2+\beta_4^2},$$
$$m_{16} = \frac{2m_7\left\{d_6\left[\beta_4-(\Omega+\Theta)^2\right]+A\left[d_1(\Omega+\Theta)^2+d_5\beta_4\right]\right\}-m_8(\Omega+\Theta)\left[4d_6(d_1+d_5)+A(\Omega+\Theta)^2+4A(d_5^2+c)\right]}{(\Omega+\Theta)^4+\beta_3(\Omega+\Theta)^2+\beta_4^2},$$

$$m_{17}=\frac{m_7(\Omega-\Theta)\left\{4d_6(d_1+d_5)+A\left[(\Omega-\Theta)^2+4(d_5^2+c)\right]\right\}-2m_8\left\{d_6\left[\beta_4-(\Omega-\Theta)^2\right]+A\left[d_1(\Omega-\Theta)^2+d_5\beta_4\right]\right\}}{(\Omega-\Theta)^4+\beta_3(\Omega-\Theta)^2+\beta_4^2},$$

$$m_{18}=-\frac{2m_7\left\{d_6\left[\beta_4-(\Omega-\Theta)^2\right]+A\left[d_1(\Omega-\Theta)^2+d_5\beta_4\right]\right\}+m_8(\Omega-\Theta)\left\{4d_6(d_1+d_5)+A\left[(\Omega-\Theta)^2+4(d_5^2+c)\right]\right\}}{(\Omega-\Theta)^4+\beta_3(\Omega-\Theta)^2+\beta_4^2},$$

$$m_{19}=-\frac{2m_5(Ad_5+d_6)}{\beta_4},\quad \beta_3=4(d_1^2+d_5^2+2c),\quad \beta_4=4(d_1d_5-c).$$

Because the second-order moment $\langle x^2(t)\rangle$ depends on time t, we take the average over time t within one period and obtain the following results:

(i) When $\Theta=\Omega$, we have

$$\langle x^2(t)\rangle_{\rm st}=m_{18}+m_{19}; \tag{15}$$

(ii) When $\Theta=N\Omega$, $N\in$ N and $N\neq 1$, we have

$$\langle x^2(t)\rangle_{\rm st}=m_{19}; \tag{16}$$

(iii) When $\Theta\neq N\Omega$ and $N\in$ N, we have

$$\begin{aligned}\langle x^2(t)\rangle_{\rm st}=&\frac{m_{11}\Omega}{2\pi\Theta}\left[1-\cos\left(\frac{2\pi\Theta}{\Omega}\right)\right]+\frac{m_{12}\Omega}{2\pi\Theta}\sin\left(\frac{2\pi\Theta}{\Omega}\right)\\&+\frac{m_{15}\Omega}{2\pi(\Omega+\Theta)}\left[1-\cos\left(\frac{2\pi\Theta}{\Omega}\right)\right]\\&+\frac{m_{16}\Omega}{2\pi(\Omega+\Theta)}\sin\left(\frac{2\pi\Theta}{\Omega}\right)\\&+\frac{m_{17}\Omega}{2\pi(\Omega-\Theta)}\left[1-\cos\left(\frac{2\pi\Theta}{\Omega}\right)\right]\\&-\frac{m_{18}\Omega}{2\pi(\Omega-\Theta)}\sin\left(\frac{2\pi\Theta}{\Omega}\right)+m_{19}.\end{aligned} \tag{17}$$

2.3. Correlation function of the system stationary state response

Solving Eq. (1), we obtain the universal expression of the solution as follows:

$$\begin{aligned}x(t+\tau)=&\,x(t)g(\tau)\exp(-\omega_0^2\tau)\\&+A\int_0^\tau\exp(-\omega_0^2v)h(v)\sin[\Omega(t+\tau-v)]\,{\rm d}v\\&+\int_0^\tau\exp(-\omega_0^2v)f(v)\,{\rm d}v,\end{aligned} \tag{18}$$

where

$$\begin{aligned}g(v)&=\left\langle\exp\left[-\int_0^v\xi(u)du\right]\right\rangle,\\h(t-v)&=\left\langle\xi(v)\exp\left[-\int_v^t\xi(u)du\right]\right\rangle,\\f(t-v)&=\left\langle\eta(v)\exp\left[-\int_v^t\xi(u)du\right]\right\rangle.\end{aligned}$$

Because equation (18) contains an exponential integral of multiplicative and additive noises, we transform $g(v)$, $h(v)$, and $f(v)$ into the forms of series, and obtain the asymptotic expressions as follows:[16]

$$g(v)=\frac{N_1}{N_1-N_2}\exp[-N_2v]-\frac{N_2}{N_1-N_2}\exp[-N_1v],$$

$$\begin{aligned}h(t-v)&=D_\xi\exp[-\lambda_\xi(t-v)-1]g(t-v),\\f(t-v)&=0,\end{aligned} \tag{19}$$

where

$$N_{1,2}=\frac{d_1-d_2}{2}\pm\sqrt{\left(\frac{d_1-d_2}{2}\right)^2+c}.$$

By inserting Eq. (19) into Eq. (18), we can have a new expression of the solution. Multiplying both sides of the new expression by $x(t)$ and averaging all the terms, we obtain the asymptotic expression of the correlation function as

$$\begin{aligned}&\langle x(t+\tau)x(t)\rangle\\=&\,\langle x^2(t)\rangle g(\tau)\exp[-\omega_0^2\tau]\\&+\frac{\langle x(t)\rangle AD_\xi}{e(N_1-N_2)}[f_1\sin(\Omega t)+f_2\cos(\Omega t)],\end{aligned} \tag{20}$$

where

$$\begin{aligned}f_1=&\,a_1\sin(\Omega\tau)+a_2\cos(\Omega\tau)\\&+a_3e^{-(b_1+\lambda_\xi)\tau}+a_4e^{-(b_2+\lambda_\xi)\tau},\\f_2=&\,c_1\sin(\Omega\tau)+c_2\cos(\Omega\tau)\\&+c_3e^{-(b_1+\lambda_\xi)\tau}+c_4e^{-(b_2+\lambda_\xi)\tau},\\a_1=&\frac{N_1\Omega}{(b_2+\lambda_\xi)^2+\Omega^2}-\frac{N_2\Omega}{(b_1+\lambda_\xi)^2+\Omega^2},\\a_2=&\frac{N_1(b_2+\lambda_\xi)}{(b_2+\lambda_\xi)^2+\Omega^2}-\frac{N_2(b_1+\lambda_\xi)}{(b_1+\lambda_\xi)^2+\Omega^2},\\a_3=&\frac{N_2(b_1+\lambda_\xi)}{(b_1+\lambda_\xi)^2+\Omega^2},\quad a_4=\frac{-N_1(b_2+\lambda_\xi)}{(b_2+\lambda_\xi)^2+\Omega^2},\\c_1=&\,a_2,\quad c_2=-a_1,\quad c_3=\frac{-N_2\Omega}{(b_1+\lambda_\xi)^2+\Omega^2},\\c_4=&\frac{N_1\Omega}{(b_2+\lambda_\xi)^2+\Omega^2},\quad b_{1,2}=\omega_0^2+N_{1,2}.\end{aligned}$$

From Eq. (20) it follows that the correlation function $\langle x(t+\tau)x(t)\rangle$ depends on both t and τ. We take the average over t within one period and obtain the following results:

i) When $\Theta=\Omega$, we have

$$\begin{aligned}&\langle x(t+\tau)x(t)\rangle_{\rm st}\\=&\,(m_{18}+m_{19})g(\tau)\exp[-\omega_0^2\tau]\\&+\frac{AD_\xi}{2e(N_1-N_2)}[(m_1+m_3)f_1+(m_2+m_4)f_2];\end{aligned} \tag{21}$$

ii) When $\Theta=N\Omega$, $N\in$ N, and $N\neq 1$, we have

$$\langle x(t+\tau)x(t)\rangle_{\rm st}=m_{19}g(\tau)\exp[-\omega_0^2\tau]$$

$$+\frac{AD_\xi}{2e(N_1-N_2)}(m_1f_1+m_2f_2); \quad (22)$$

iii) When $\Theta = N\Omega$ and $N \in N$, we have

$$\begin{aligned}&\langle x(t+\tau)x(t)\rangle_{\rm st}\\ &= \langle x^2(t)\rangle_{\rm st} g(\tau)\exp(-\omega_0^2\tau)\\ &\quad + \frac{AD_\xi[(m_1+l_1+l_2)f_1+(m_2+l_3+l_4)f_2]}{2e(N_1-N_2)}, \quad (23)\end{aligned}$$

where $\langle x^2(t)\rangle_{\rm st}$ is given by Eq. (17),

$$\begin{aligned}l_1 &= -\frac{m_3\Omega^2}{\pi(\Omega^2-\Theta^2)}\sin\left(\frac{2\pi\Theta}{\Omega}\right),\\ l_2 &= -\frac{m_4\Omega^2}{\pi(\Omega^2-\Theta^2)}\left[\cos\left(\frac{2\pi\Theta}{\Omega}\right)-1\right],\\ l_3 &= \frac{m_3\Omega\Theta}{\pi(\Omega^2-\Theta^2)}\left[\cos\left(\frac{2\pi\Theta}{\Omega}\right)-1\right],\\ l_4 &= -\frac{m_4\Omega\Theta}{\pi(\Omega^2-\Theta^2)}\sin\left(\frac{2\pi\Theta}{\Omega}\right).\end{aligned}$$

2.4. Signal-to-noise ratio (SNR) of the output signal

The power spectrum $S(\omega)$ is defined as the Fourier transform of the correlation function $\langle x(t+\tau)x(t)\rangle_{\rm st}$.

(I) When $\Theta = \Omega$, calculating the Fourier transform of Eq. (21), we have

$$\begin{aligned}S(\omega) &= \int_{-\infty}^{+\infty}\langle x(t+\tau)x(t)\rangle_{\rm st}\exp(-{\rm i}\omega\tau)\,{\rm d}\tau\\ &= S_1(\omega)+S_2(\omega), \quad (24)\end{aligned}$$

where

$$\begin{aligned}S_1(\omega) &= \frac{\pi AD_\xi}{2e(N_1-N_2)}[(m_1+m_3)a_2+(m_2+m_4)c_2]\delta(\omega-\Omega),\\ S_2(\omega) &= \frac{(m_{18}+m_{19})}{N_1-N_2}\left(\frac{N_1b_2}{b_2^2+\omega^2}-\frac{N_2b_1}{b_1^2+\omega^2}\right)\\ &\quad + \frac{AD_\xi[(m_1+m_3)a_3+(m_2+m_4)c_3]}{2e(N_1-N_2)}\\ &\quad \times\left[\frac{b_1+\lambda_\xi}{(b_1+\lambda_\xi)^2+\omega^2}\right]\\ &\quad + \frac{AD_\xi[(m_1+m_3)a_4+(m_2+m_4)c_4]}{2e(N_1-N_2)}\\ &\quad \times\left[\frac{b_2+\lambda_\xi}{(b_2+\lambda_\xi)^2+\omega^2}\right].\end{aligned}$$

(II) When $\Theta = N\Omega$, $N \in N$, and $N \neq 1$, calculating the Fourier transform of Eq. (22), we have

$$\begin{aligned}S(\omega) &= \int_{-\infty}^{+\infty}\langle x(t+\tau)x(t)\rangle_{\rm st}\exp(-{\rm i}\omega\tau)\,{\rm d}\tau\\ &= S_1(\omega)+S_2(\omega), \quad (25)\end{aligned}$$

where

$$\begin{aligned}S_1(\omega) &= \frac{\pi AD_\xi}{2e(N_1-N_2)}[m_1a_2+m_2c_2]\delta(\omega-\Omega),\\ S_2(\omega) &= \frac{m_{19}}{N_1-N_2}\left(\frac{N_1b_2}{b_2^2+\omega^2}-\frac{N_2b_1}{b_1^2+\omega^2}\right)\\ &\quad + \frac{AD_\xi[m_1a_3+m_2c_3]}{2e(N_1-N_2)}\left[\frac{b_1+\lambda_\xi}{(b_1+\lambda_\xi)^2+\omega^2}\right]\\ &\quad + \frac{AD_\xi[m_1a_4+m_2c_4]}{2e(N_1-N_2)}\left[\frac{b_2+\lambda_\xi}{(b_2+\lambda_\xi)^2+\omega^2}\right].\end{aligned}$$

(III) When $\Theta \neq N\Omega$ and $N \in N$, calculating the Fourier transform of Eq. (23), we have

$$\begin{aligned}S(\omega) &= \int_{-\infty}^{+\infty}\langle x(t+\tau)x(t)\rangle_{\rm st}\exp(-{\rm i}\omega\tau)\,{\rm d}\tau\\ &= S_1(\omega)+S_2(\omega), \quad (26)\end{aligned}$$

where

$$\begin{aligned}S_1(\omega) &= \frac{\pi AD_\xi}{2e(N_1-N_2)}[(m_1+l_1+l_2)a_2+(m_2+l_3+l_4)c_2]\\ &\quad \times\delta(\omega-\Omega),\\ S_2(\omega) &= \frac{\langle x^2(t)\rangle_{\rm st}}{N_1-N_2}\left(\frac{N_1b_2}{b_2^2+\omega^2}-\frac{N_2b_1}{b_1^2+\omega^2}\right)\\ &\quad + \frac{AD_\xi[(m_1+l_1+l_2)a_3+(m_2+l_3+l_4)c_3]}{2e(N_1-N_2)}\\ &\quad \times\left[\frac{b_1+\lambda_\xi}{(b_1+\lambda_\xi)^2+\omega^2}\right]\\ &\quad + \frac{AD_\xi[(m_1+l_1+l_2)a_1+(m_2+l_3+l_4)c_4]}{2e(N_1-N_2)}\\ &\quad \times\left[\frac{b_2+\lambda_\xi}{(b_2+\lambda_\xi)^2+\omega^2}\right],\end{aligned}$$

and $\langle x^2(t)\rangle_{\rm st}$ is given by Eq. (17).

The signal-to-noise ratio (SNR) is defined as

$${\rm SNR} = \frac{\int_0^\infty S_1(\omega)\,{\rm d}\omega}{S_2(\omega=\Omega)}. \quad (27)$$

We emphasize that the above results are correct only under the assumption of the stability of solutions, that is, the two above-mentioned inequalities hold.

3. Numerical discussion

Now we do the numerical work on the above analytical expressions in Eqs. (24)–(27), which show the behaviors of SNR for any combination of the parameters Ω, D_ξ, λ_ξ, Λ_ξ, D_η, λ, Θ, ϕ.

3.1. Bona fide stochastic resonance

In Fig. 1, we plot the curves of the SNR each as a function of the frequency of periodic signal Ω with different values of parameters (including λ_ξ, Λ_ξ, and Θ).

As shown in Fig. 1, all the curves show that the evolution of the SNR with Ω presents periodic oscillation. Namely, one-main-peak SR phenomenon takes place, indicating that the bona fide stochastic resonance appears. In Fig. 1(a), with the increase of λ_ξ, the maximum of SNR decreases and the

position of the main peak remains unchanged. There exists a critical frequency Ω_c. When $\Omega < \Omega_c$, the SNR gradually lowers with the increase of λ_ξ; when $\Omega > \Omega_c$, the SNR gradually increases with the increase of λ_ξ. In Fig. 1(b), the shape of each SNR curve does not change with Λ_ξ. But with the increase of Λ_ξ, the whole curve moves upwards. In Fig. 1(c), with the increase of Θ, the value of each $\mathrm{SNR}(\Omega)_{\max}$ remains unchanged. However, the main peak shifts toward the right, and appears in $\Omega = \Theta$.

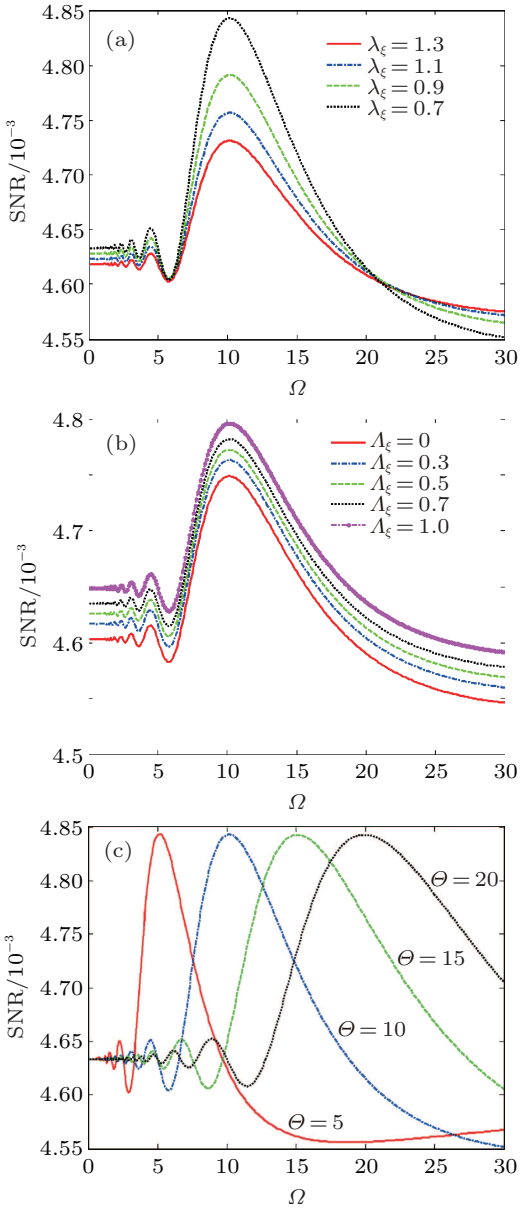


Fig. 1. (color online) Curves of SNR versus Ω for various values of (a) λ_ξ, (b) Λ_ξ, and (c) Θ, with other parameter values being (a) $\omega = 10$, $A = 10$, $D_\xi = 4$, $D_\eta = 0.1$, $\lambda = 1$, $\phi = 0$, $\Lambda_\xi = 0.5$, $\Theta = 10$; (b) $\omega = 10$, $A = 10$, $D_\xi = 4$, $D_\eta = 0.1$, $\lambda = 1$, $\phi = 0$, $\lambda_\xi = 1$, $\Theta = 10$; (c) $\omega = 10$, $A = 10$, $D_\xi = 4$, $D_\eta = 0.1$, $\lambda = 1$, $\phi = 0$, $\lambda_\xi = 0.7$, $\Lambda_\xi = 0.5$.

Specially, as shown in Fig. 1, there is more than one peak in each curve of the SNR versus Ω, i.e., stochastic multi-resonance (SMR) phenomenon occurs, which is not observed in a conventional linear system.

3.2. Conventional stochastic resonance

In Fig. 2, we present the curves of the SNR versus noise intensity D_ξ with different values of parameters (including λ_ξ, and Λ_ξ).

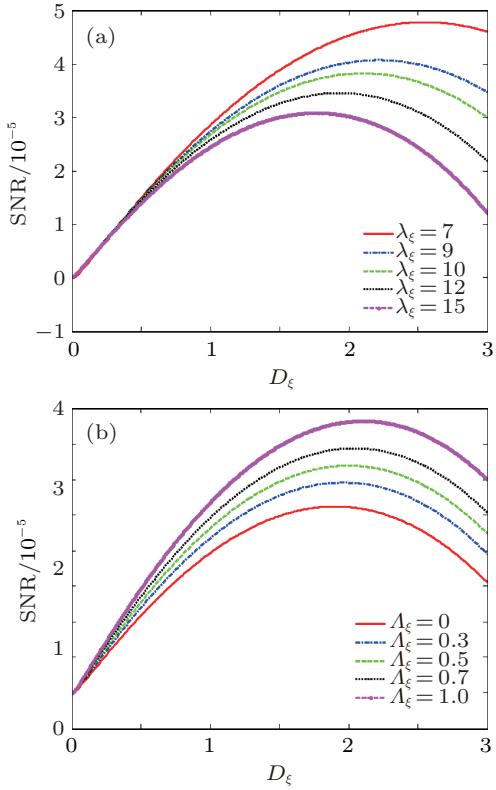


Fig. 2. (color online) Curves of SNR versus D_ξ for various values of (a) λ_ξ and (b) Λ_ξ, with other parameter values being (a) $\omega = 2$, $A = 10$, $\Omega = 50$, $D_\eta = 0.1$, $\lambda = 1$, $\Theta = 10$, $\phi = 0$, $\Lambda_\xi = 1$; (b) $\omega = 2$, $A = 10$, $\Omega = 50$, $D_\eta = 0.1$, $\lambda = 1$, $\Theta = 10$, $\phi = 0$, $\lambda_\xi = 10$.

As shown in Fig. 2, each curve shows that the SNR attains a maximum value with increasing D_ξ, indicating that the conventional SR phenomenon takes place. In Fig. 2(a), with the increase of λ_ξ, the maximum of SNR decreases and the position of the peak shifts toward the left. However, in Fig. 2(b), with the increase of Λ_ξ, the maximum of SNR increases and the position of the peak shifts toward the right. That is, we can increase the SNR of output signal by reducing λ_ξ or increasing Λ_ξ.

Figure 3 shows the curves of the SNR versus correlation rate λ_ξ for different values of parameters (including D_ξ and D_η).

As shown in Fig. 3, each curve shows that the SNR attains a maximum value by increasing λ_ξ, indicating that the conventional SR occurs. In Fig. 3(a), with the increase of D_ξ, the maximum of SNR increases and the position of the peak stays unchanged. However, as shown in Fig. 3(b), with the increase of D_η, the maximum of SNR decreases and the position of the peak shifts toward the right. Figure 3 shows that when the other parameters are fixed, D_ξ and D_η exert different influences on the SNR differently. That is, with the increase of

the SNR of the output signal increases, but with the increase of D_η the SNR of the output signal decreases.

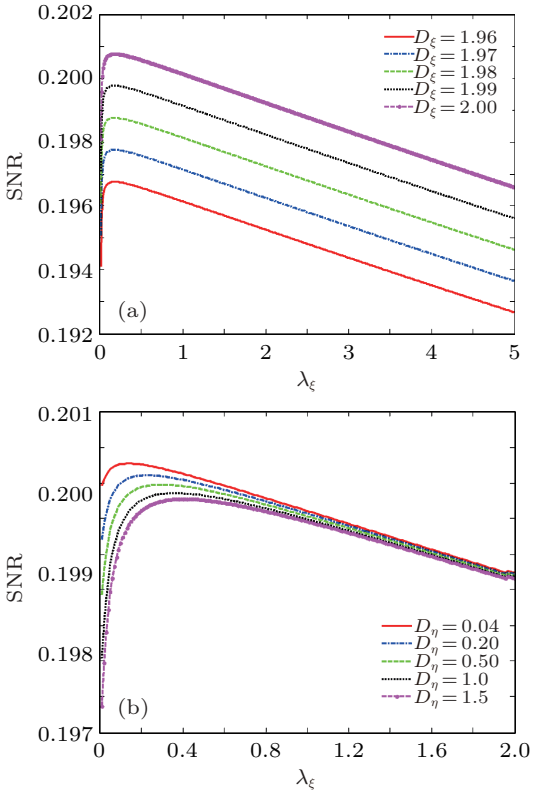


Fig. 3. (color online) Curves of SNR versus λ_ξ for various values of (a) D_ξ and (b) D_η, with other parameter values being (a) $\omega = 10$, $A = 800$, $\Omega = 30$, $\Lambda_\xi = 0$, $\lambda = 1$, $\Theta = 10$, $\phi = 0$, $D_\eta = 0.04$; (b) $\omega = 10$, $A = 800$, $\Omega = 30$, $\Lambda_\xi = 0$, $\lambda = 1$, $\Theta = 10$, $\phi = 0$, $D_\xi = 2$.

3.3. Stochastic resonance in the wide sense

In this part, we aim to examine the dependence of the SNR of the output signal on cross-correlation modulation frequency Θ.

Figure 4 shows the curves of the SNR versus Θ for different values of λ and ϕ. From Fig. 4, we see that the evolution of the output SNR with Θ presents periodic oscillation, where the resonance and the inhibition both exist at the same time. Moreover, when ϕ and λ are given different values, there are three types of periodical-oscillatory SR phenomena: one-main-peak SR, one-main-valley SR, one-main-peak, and one-main-valley SR. When $\phi = 0$ and $\lambda = 1$, the curve of the SNR versus Θ manifests as the one-main-peak SR; when $\phi = 0$ and $\lambda = -1$, the one-main-valley SR phenomenon takes place; in addition, when $\phi = \pi/2$ and $\lambda = 1(-1)$, then each curve of SNR–Θ exhibits the one-main-peak and one-main-valley SR. Finally, when $\lambda = 0$, the curve of the SNR versus Θ is a horizontal line. The additive white noise and the multiplicative color noise are uncorrelated, so the curve of the SNR takes this horizontal line as a symmetric axis when $\lambda > 0$ and $\lambda < 0$.

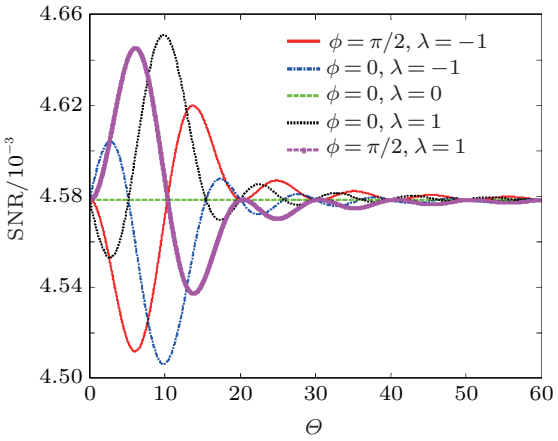


Fig. 4. (color online) Curves of SNR versus Θ for various values of ϕ and λ, with other parameter values being $\omega = 10$, $A = 10$, $\Omega = 10$, $\Lambda_\xi = 0$, $\lambda_\xi = 2$, $D_\xi = 4$, $D_\eta = 0.1$.

Figure 5 shows the curves of the SNR versus Θ for different values of λ. From Fig. 5(a), we see that when $\phi = 0$, if $\lambda > 0$, there is one-main-peak SR; if $\lambda < 0$, there is one-main-valley SR. In addition, the positions of the main peaks and the main valleys are the same, which are at $\Theta = \Omega$. As shown in Fig. 5(b), all the curves show that when $\phi = \pi/2$, each curve presents one-main-peak and one-main valley SR. It is worth emphasizing that the value of λ does not change the type of the periodical-oscillatory SR nor the positions of the main peak and main valley. However, with the increase

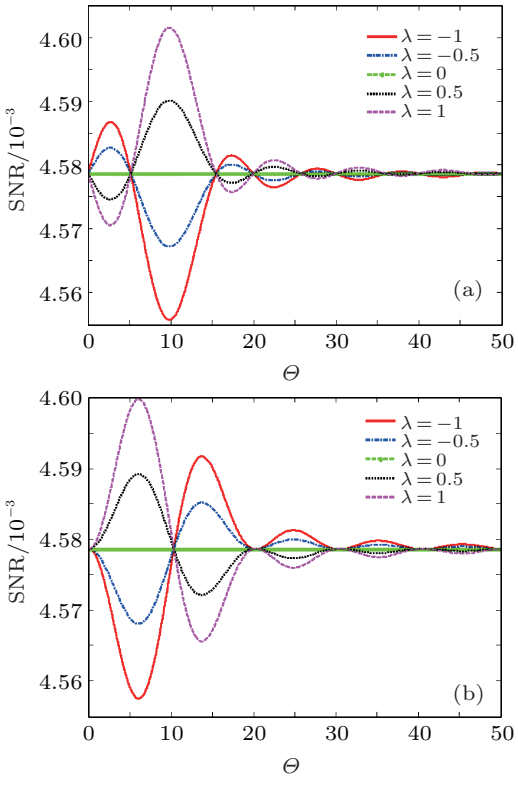


Fig. 5. (color online) Curves of SNR versus Θ for various values λ, with other parameter values being $\omega = 10$, $A = 10$, $\Omega = 10$, $\Lambda_\xi = 0$, $\lambda_\xi = 2$, $D_\xi = 4$, $D_\eta = 0.01$, (a) $\phi = 0$, (b) $\phi = \pi/2$.

of $|\lambda|$, the value of the main peak increases and the value of the main valley decreases. That is, properly adjusting λ can strengthen the effect of enhancement or inhibition on the SNR at the positions of the main peak and main valley. Finally, with the increase of Θ, all the curves of the SNR will reach a steady value, which is the value of the SNR when $\lambda = 0$.

Figure 6 shows the curves of the SNR versus Θ for different values of ϕ and $\lambda = 1$. From Figs. 6(a) and 6(b), we find that as ϕ changes, the types of the SR transfer among the three types of periodical-oscillatory SRs mentioned above. Because of the symmetric situations between $\lambda < 0$ and $\lambda > 0$, we can obtain the corresponding changing result of the SNR when $\lambda = -1$.

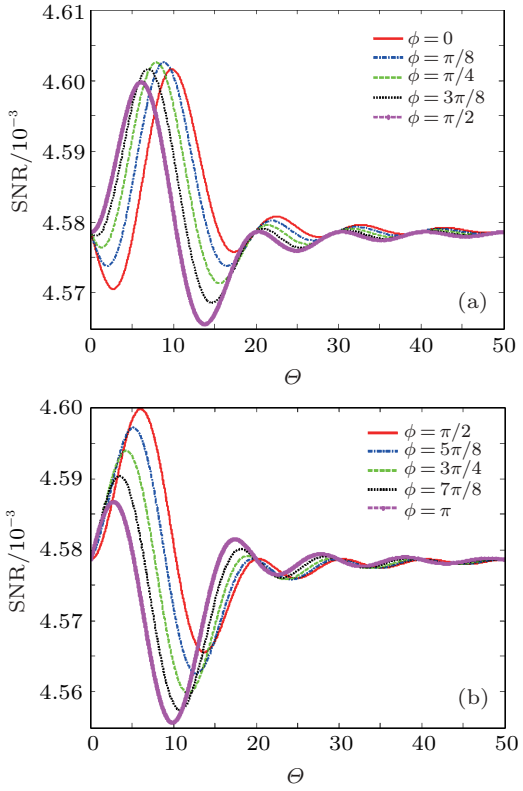


Fig. 6. (color online) Curves of SNR versus Θ for various values of ϕ, with other parameter values being $\omega = 10$, $A = 10$, $\Omega = 10$, $\Lambda_\xi = 0$, $\lambda_\xi = 2$, $D_\xi = 4$, $D_\eta = 0.01$, $\lambda = 1$.

As shown in Figs. 4–6, each curve of SNR versus Θ shows that the output SNR fluctuates periodically with the increase of Θ. Namely, a periodical-oscillatory SR phenomenon takes place. This interesting SR phenomenon is a kind of SR in the wider sense. Compared with the case that noises are uncorrelated or that the cross-correlation intensity is constant, the time-periodic modulation of cross-correlation intensity between noises leads to rich dynamics properties. In addition, the time-periodic modulation directly affects the periodic property of noise, and the coupling effect between noises is affected directly by their phase difference, frequency difference, and amplitude difference. Therefore, the appearance and properties of the SR phenomenon of the output signal are directly related to the three parameters, i.e., Θ, ϕ, and λ. Moreover, the adjustments of those modulated parameters lead to the diversity of the SR phenomenon. Specifically, ϕ affects the phase diagram of the curve of the SNR–Θ; with the adjustment of λ, the phase diagram of the curve of the SNR–Θ remains unchanged while the values of the main peak and the main valley change.

4. Conclusions

In this paper, we investigated the phenomenon of stochastic resonance in a linear system driven by correlated additive and multiplicative noises and a periodically modulated noise, and detect three different forms of SRs existing in this linear system: bona fide SR, conventional SR, and SR in the wider sense. Specially, the evolution of the output SNR with Ω presents periodic oscillation, which has not been reported in linear systems before. Moreover, we find that with the increases of D_ξ and Λ_ξ, and with the decreases of D_η and λ_ξ, the output SNR will increase.

In conclusion, with the proper adjustments of the parameters mentioned above, we can effectively control the stochastic resonance of this over-damped linear system within a certain range, and further broaden the application scope of the SR phenomenon in physics and engineering.

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