Control of fractional chaotic and hyperchaotic systems based on a fractional order controller*

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We present a new fractional-order controller based on the Lyapunov stability theory and propose a control method which can control fractional chaotic and hyperchaotic systems whether systems are commensurate or incommensurate. The proposed control method is universal, simple, and theoretically rigorous. Numerical simulations are given for several fractional chaotic and hyperchaotic systems to verify the effectiveness and the universality of the proposed control method.

Keywords: fractional-order chaotic system, chaos control, fractional-order controller, hyperchaos

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1. Introduction

As a generalization of integer-order dynamic systems, fractional dynamic systems provide better mathematical models for some actual physical and engineering systems.[1,2] The fractional nonlinear dynamic systems have many dynamic behaviors which are similar to the integer-order systems, such as chaos, bifurcation, and attractor.[3–6] Chaotic systems are wildly studied due to their potential applications in physics, biology, information, chemistry, and other fields.[7–9] Meanwhile, hyperchaotic systems are also considered with rapidly increasing interest. A hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent, which implies that its dynamics are expended in several different directions simultaneously. So it has more complex dynamical behaviors than the chaotic system. Typical examples are Chen's four-dimensional hyperchaotic system,[10] the hyperchaotic Lorenz system,[11,12] and the hyperchaotic Rössler system.[13,14]

As we know, chaos control is not only very important but also very difficult in both chaotic systems and hyperchaotic systems. For chaotic systems, there are only a few control methods, such as the linear feedback method,[5] the backstepping method,[15] the sliding mode method,[16] and the Lyapunov equation-based method.[17] Like the chaotic system, there are a few control methods of hyperchaotic systems such as the feedback controller,[18,19] nonlinear controller,[20] and switching control.[21] However, most of the above control methods of systems (whether a chaotic or a hyperchaotic system) concentrate on certain fractional systems. There are few general methods that can be applied to control arbitrary fractional chaotic or hyperchaotic systems. So a general control method of fractional systems is important for both control theory and application of the fractional chaotic system.

On the other hand, the fractional-order control, as a generalization of the traditional integer-order control, is becoming a matter of concern because of its flexibility and integrity.[2,22–24] The TID controller,[22] the $PI^{\lambda}D^{\mu}$ controller,[23] and the CRONE controller[24] are among the well-known fractional-order controllers. In some of those papers, it is verified that the fractional-order controllers are easier to implement and cost less compared with traditional controllers.

However, to the best of our knowledge, little attention has been paid to studying the problem of controlling fractional chaotic systems with the use of fractional-order controllers. Therefore, in this paper, we present a new fractional-order controller based on the Lyapunov stability theory. It can control arbitrary fractional chaotic and hyperchaotic systems to any one of their stable equilibrium points whether systems are commensurate or incommensurate. The proposed controller has the following advantages. 1) It is general and can be applied to almost all chaotic and hyperchaotic systems. 2) It can stabilize any equilibrium point with little control cost and no affect on the parameters of the original systems. 3) It could be easily realized experimentally, and is more suitable for engineering applications. Numerical simulation results of the control of the fractional unified system (incommensurate), the fractional Liu system (commensurate), the hyperchaotic Chen system (commensurate), and the hyperchaotic Lorenz system (incommensurate) demonstrate the effectiveness and the uni-

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versality of the proposed control method.

2. Preliminaries and definitions

2.1. Fractional derivative and numerical solution of differential equation

The fractional calculus plays an important role in modern science. In this paper, we mainly use the Caputo fractional operators.[1,2] The Caputo definition of the fractional derivative, which is called a smooth fractional derivative, is described as

$$ {}_0\mathrm{D}_t^q = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t \frac{f^{(m)}(t)}{(t-\tau)^{q+1-m}}\,\mathrm{d}\tau, & m-1<q<m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & q=m, \end{cases} \tag{1} $$

where m is the first integer that is not less than q, and Γ is the Gamma function,

$$ \Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t. $$

When numerically solving the fractional differential equations, we also adopt the improved version of the Adams–Bashforth–Moulton algorithm[2,25] which is proposed based on the predictor–correctors scheme. To explain the method, we consider the following differential equation:

$$ \begin{aligned} &\mathrm{D}_t^q y(t) = r(t, y(t)), \quad 0 \le t \le T, \\ &y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, m-1. \end{aligned} \tag{2} $$

The differential equation (2) is equivalent to the Volterra integral equation in the following form

$$ y(t) = \sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s, y(s))\,\mathrm{d}s. $$

Now, set $h = T/N$, $t_n = nh$ $(n = 0, 1, \ldots, N)$. The integral equation can be discretized as follows:

$$ \begin{aligned} y_h(t_{n+1}) = &\sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{h^q}{\Gamma(q+2)} r(t_{n+1}, y_h^p(t_{n+1})) \\ &+ \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} r(t_j, y_h(t_j)), \end{aligned} $$

where

$$ y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} r(t_j, y_h(t_j)), $$

and

$$ a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} & \\ \quad -2(n-j+1)^{q+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} $$

$$ b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q), 0 \le j \le n. $$

The error of this approximation is described as follows:

$$ \max_{j=0,1,\cdots,N} |y(t_j) - y_h(t_j)| = O(h^p), $$

where $p = \min(2, 1+q)$.

2.2. Some properties of the fractional derivative

In this paper, we consider the fractional order $0 < q < 1$. There are some general properties of the fractional-order derivative, which are described as follows:[1,2]

Property 1 Caputo fractional derivative is a linear operation, i.e.,

$$ \mathrm{D}_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda \mathrm{D}_t^\alpha f(t) + \mu \mathrm{D}_t^\alpha g(t), \tag{3} $$

where λ and μ are real constants.

Property 2 The Caputo fractional derivative satisfies the additive index law (semigroup property), i.e.,

$$ \mathrm{D}_t^\alpha \mathrm{D}_t^\beta f(t) = \mathrm{D}_t^\beta \mathrm{D}_t^\alpha f(t) = \mathrm{D}_t^{\alpha+\beta} f(t), \tag{4} $$

holds under some reasonable constraints on the function $f(t)$.

Property 3 For the fractional-order nonlinear system $\mathrm{D}_t^\alpha \boldsymbol{x}(t) = \boldsymbol{F}(\boldsymbol{x}(t))$, $\boldsymbol{F}(\boldsymbol{x}(t))$ satisfies the Lipschiz condition with respect to $\boldsymbol{x}$, i.e.,

$$ ||\boldsymbol{F}(\boldsymbol{x}_1(t)) - \boldsymbol{F}(\boldsymbol{x}_2(t))|| \le l||\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)||, \tag{5} $$

where $||\cdot||$ is 1-norm and l is a positive constant; especially if $\boldsymbol{F}(\boldsymbol{x}) = 0$ at $\boldsymbol{x} = 0$. We have

$$ ||\boldsymbol{F}(\boldsymbol{x}(t))|| \le l||\boldsymbol{x}(t)||. \tag{6} $$

In the following, we consider the stability of the fractional-order nonlinear system, which is given as

$$ \mathrm{D}_t^{\boldsymbol{\alpha}} \boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{c}, \tag{7} $$

where $\boldsymbol{F} = (f_1, f_2, \ldots, f_n)^{\mathrm{T}}$ is a vector function and f_i $(i = 1, 2, \ldots, n)$ are continuous differential nonlinear functions; $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$ $(\alpha_i \in (0, 1], i = 1, 2, \ldots, n)$ is the fractional order of derivative; $\boldsymbol{x} = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is the system state variable; $\boldsymbol{x}(0) = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$ is the initial value; and D_t^α denotes the Caputo fractional-order derivative operator. The equilibrium points of this system are calculated by solving $\boldsymbol{F}(\boldsymbol{x}) = 0$. Then we have the following conclusions for the stability of these equilibrium points.

Theorem 1[2] When we consider the commensurate fractional-order system, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the equilibrium points $\boldsymbol{x}^*$ of system (7) are globally asymptotically stable if all the eigenvalues λ_i $(i = 1, 2, \ldots, n)$ of the Jacobian matrix $\boldsymbol{J} = \partial \boldsymbol{F}/\partial \boldsymbol{x}$, evaluated at $\boldsymbol{x}^*$, satisfy the condition

$$ |\arg(\lambda_i)| > \alpha\pi/2, \quad i = 1, 2, \ldots, n. \tag{8} $$

Theorem 2[2] When we consider the incommensurate fractional-order system, i.e., $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_n$. Suppose that M is the LCM of the denominators u_i's of α_i's, where $\alpha_i = v_i/u_i, v_i, u_i \in Z^+$ for $i = 1,2,\ldots,n$. The system (7) is asymptotically stable if

$$|\arg(\lambda)| > \pi/(2M), \tag{9}$$

for all roots λ of the following equation

$$\det(\mathrm{diag}(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \ldots, \lambda^{M\alpha_n}) - \boldsymbol{J}) = 0. \tag{10}$$

For the arbitrary dimensional fractional-order nonlinear system (7), we consider this form in the following:

$$\begin{aligned} \mathrm{D}_t^{\alpha}\boldsymbol{x} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}), \\ \mathrm{D}_t^{\beta}\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}), \end{aligned} \tag{11}$$

where $\boldsymbol{x} \in R^n$ and $\boldsymbol{y} \in R^m$ are the state vectors; $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)^{\mathrm{T}}$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m)^{\mathrm{T}}$ are the fractional order of derivatives; $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)^{\mathrm{T}}$ and $\boldsymbol{g} = (g_1, g_2, \ldots, g_m)^{\mathrm{T}}$ are continuous differential nonlinear vector functions. To describe the new design and analysis, we give the following assumption.

Assumption 1 Function $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ is smooth in a neighborhood of $\boldsymbol{y} = \boldsymbol{y}^*$ and the subsystem $\mathrm{D}_t^{\alpha}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}^*)$ is asymptotically stable about the equilibrium $\boldsymbol{x} = \boldsymbol{x}^*$ for all $\boldsymbol{x}$.

Remark 1 For Assumption 1, system (11) is very general, which contains almost all fractional-order chaotic systems. According to this assumption, the subsystem $\mathrm{D}_t^{\alpha}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}^*)$ is asymptotically stable about the equilibrium $\boldsymbol{x} = \boldsymbol{x}^*$ for all $\boldsymbol{x}$, which implies that there is a Lyapunov function $V_0(\boldsymbol{x}-\boldsymbol{x}^*)$ and a positive numbers λ_1, such that $\dot{V}_0(\boldsymbol{x}-\boldsymbol{x}^*) = (\partial V_0/\partial(\boldsymbol{x}-\boldsymbol{x}^*))\dot{\boldsymbol{x}} \leq -\lambda_1||\boldsymbol{x}-\boldsymbol{x}^*||^2$. Since every $g_i(\boldsymbol{x},\boldsymbol{y})$ $(i = 1,2,\ldots,m)$ is smooth function and system (11) is chaotic (or hyperchaotic), there are two positive numbers λ_{i1} and λ_{i2}, such that $||g_i(\boldsymbol{x},\boldsymbol{y})|| \leq \lambda_{i1}||\boldsymbol{x}|| + \lambda_{i2}||\boldsymbol{y}||$, $i = 1,\ldots,m$. Here $||\cdot||$ is also 1-norm.

3. Chaos control based on a fractional-order controller

In this section, we propose a new fractional-order controller to achieve chaos control of the fractional-order chaotic and hyperchaotic system. For the Assumption 1, the system (11) is composed of two subsystem: one is an asymptotically stable system and the other is an augment system. We add a fractional-order controller in the augment system. Then the controller can control the augment system response to the asymptotically stable equilibrium point, and at the same time, the asymptotically stable system response to the asymptotically stable equilibrium point too. Hence, in order to control system (11) to its equilibrium point $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, we add the following fractional-order controller $\boldsymbol{\varphi}(\boldsymbol{u})$ to system (11), then the controlled system would be given as

$$\begin{aligned} \mathrm{D}_t^{\alpha}\boldsymbol{x} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}), \\ \mathrm{D}_t^{\beta}\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{\varphi}(\boldsymbol{u}). \end{aligned} \tag{12}$$

The control function $\boldsymbol{\varphi}(\boldsymbol{u})$ is given as

$$\boldsymbol{\varphi}(\boldsymbol{u}) = -\boldsymbol{k}\boldsymbol{u}, \tag{13}$$

where $\boldsymbol{k}$ is a control parameter matrix, which is always diagonal and $\boldsymbol{u}$ is defined as

$$\mathrm{D}_t^{1-\beta}\boldsymbol{u} = \boldsymbol{\omega}(\boldsymbol{y}-\boldsymbol{y}^*), \tag{14}$$

with $\boldsymbol{\omega} = \mathrm{diag}(\omega_1,\ldots,\omega_m)$ being a positive diagonal matrix.

According to the Properties 1 and 2 of the Caputo fractional derivative operator, we can get $\dot{\boldsymbol{u}} = \mathrm{D}_t^1\boldsymbol{u} = \mathrm{D}_t^{\beta}\mathrm{D}_t^{1-\beta}\boldsymbol{u} = \boldsymbol{\omega}\mathrm{D}_t^{\beta}\boldsymbol{y}$. In the same way, according to Eq. (14) and Property 3 of the Caputo fractional derivative, there is a positive number λ_2, such that $||\boldsymbol{\omega}(\boldsymbol{y}-\boldsymbol{y}^*)|| \leq \lambda_2||\boldsymbol{u}||$. Let $\omega^* = \min\{|\omega_i|, i = 1,\ldots,m\}$, then $||\boldsymbol{y}-\boldsymbol{y}^*|| \leq \lambda_2/\omega^*||\boldsymbol{u}||$ and if $\boldsymbol{u} = 0$, then $\boldsymbol{y} = \boldsymbol{y}^*$.

Theorem 3 The fractional-order controller can control the system (11) response to the asymptotically stable equilibrium points $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, if the control parameter matrix $\boldsymbol{k} = \mathrm{diag}(k_1,\ldots,k_m)$ $(k_i > 0, i = 1,\ldots,m)$ and $\boldsymbol{\omega} = \mathrm{diag}(\omega_1,\ldots,\omega_m)$ $(\omega_i > 0, i = 1,\ldots,m)$ satisfy the following conditions:

$$\text{i)} \quad k_i \geq \lambda_{i2}\lambda_2/\omega^* + m^2\omega_i\lambda_{i1}^2/(4\lambda_1), \tag{15}$$

$$\text{ii)} \quad (k_i - \lambda_{i2}\lambda_2/\omega^*)(k_j - \lambda_{j2}\lambda_2/\omega^*) \geq m^2\lambda_2^2(\omega_i\lambda_{i2} + \omega_j\lambda_{j2})^2/(4\omega_i\omega_j\omega^{*2}), \tag{16}$$

where $i = 1,\ldots,m$ and $1 \leq i < j \leq m$.

Proof Without loss of generality, let $(\boldsymbol{x}^*,\boldsymbol{y}^*) = 0$ be an equilibrium point of the fractional chaotic(or hyperchaotic) system (11). If $(\boldsymbol{x}^*,\boldsymbol{y}^*) \neq 0$ is the equilibrium point of the system, the method of proof can be easily used by carrying out an additional coordinate transformation $\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^*$ and $\bar{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{y}^*$.

We introduce a Lyapunov function

$$V = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} + V_0(\boldsymbol{x}), \tag{17}$$

according to Remark 1 and Property 2, then

$$\begin{aligned} \dot{V} &= \frac{1}{2}\dot{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\dot{\boldsymbol{u}} + \dot{V}_0(\boldsymbol{x}) \\ &= \boldsymbol{u}^{\mathrm{T}}\dot{\boldsymbol{u}} + \dot{V}_0(\boldsymbol{x}) \\ &= (u_1,\ldots,u_m)\mathrm{diag}(\omega_1,\ldots,\omega_m)\mathrm{D}_t^{\beta}\boldsymbol{y} + \dot{V}_0(\boldsymbol{x}) \\ &= (\omega_1 u_1,\ldots,\omega_m u_m)(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{k}\boldsymbol{u}) + \dot{V}_0(\boldsymbol{x}) \\ &= \sum_{i=1}^{m}\omega_i u_i(g_i(\boldsymbol{x},\boldsymbol{y}) - k_i u_i) + \dot{V}_0(\boldsymbol{x}) \\ &= \sum_{i=1}^{m}\omega_i u_i g_i(\boldsymbol{x},\boldsymbol{y}) - \sum_{i=1}^{m}\omega_i k_i u_i^2 + \dot{V}_0(\boldsymbol{x}) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m}\omega_i|u_i|||g_i(\boldsymbol{x},\boldsymbol{y})|| - \sum_{i=1}^{m}\omega_i k_i u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&\le \sum_{i=1}^{m}\omega_i|u_i|(\lambda_{i1}||\boldsymbol{x}|| + \lambda_{i2}||\boldsymbol{y})||) - \sum_{i=1}^{m}\omega_i k_i u_i^2\\
&\quad -\lambda_1||\boldsymbol{x}||^2\\
&\le \sum_{i=1}^{m}\omega_i|u_i|(\lambda_{i1}||\boldsymbol{x}|| + \lambda_{i2}\lambda_2/\omega^*||\boldsymbol{u}||)\\
&\quad -\sum_{i=1}^{m}\omega_i k_i u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&= \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i=1}^{m}\omega_i\lambda_{i2}\lambda_2/\omega^*|u_i|||\boldsymbol{u}||\\
&\quad -\sum_{i=1}^{m}\omega_i k_i u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&= \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i=1}^{m}\sum_{j=1}^{m}\omega_i\lambda_{i2}\lambda_2/\omega^*|u_i||u_j|\\
&\quad -\sum_{i=1}^{m}\omega_i k_i u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&= \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i\ne j}\omega_i\lambda_{i2}\lambda_2/\omega^*|u_i||u_j|\\
&\quad +\sum_{i=1}^{m}\omega_i\lambda_{i2}\lambda_2/\omega^* u_i^2 - \sum_{i=1}^{m}\omega_i k_i u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&= \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i\ne j}\omega_i\lambda_{i2}\lambda_2/\omega^*|u_i||u_j|\\
&\quad -\sum_{i=1}^{m}(\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*)u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&= \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i<j}(\omega_i\lambda_{i2} + \omega_j\lambda_{j2})\lambda_2/\omega^*\\
&\quad \cdot|u_i||u_j| - \sum_{i=1}^{m}(\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*)u_i^2 - \lambda_1||\boldsymbol{x}||^2\\
&\le \sum_{i=1}^{m}\omega_i\lambda_{i1}|u_i|||\boldsymbol{x}|| + \sum_{i<j}(\omega_i\lambda_{i2} + \omega_j\lambda_{j2})\lambda_2/\omega^*\\
&\quad \cdot|u_i||u_j| - \sum_{i<j}2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\\
&\quad \cdot\sqrt{\omega_j k_j - \omega_j\lambda_{j2}\lambda_2/\omega^*}|u_i||u_j|\\
&\quad -\sum_{i=1}^{m}2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\sqrt{\lambda_1}|u_i|||\boldsymbol{x}||\\
&= -\sum_{i<j}[2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\\
&\quad \cdot\sqrt{\omega_j k_j - \omega_j\lambda_{j2}\lambda_2/\omega^*} - (\omega_i\lambda_{i2} + \omega_j\lambda_{j2})\lambda_2/\omega^*]\\
&\quad \cdot|u_i||u_j| - \sum_{i=1}^{m}(2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\sqrt{\lambda_1}\\
&\quad -\omega_i\lambda_{i1})|u_i|||\boldsymbol{x}||.
\end{aligned}
$$

The conclusion $\dot{V}\le 0$ would be obtained, if the control parameters $\boldsymbol{k}$ and $\boldsymbol{\omega}$ satisfy the following conditions:

(i) $\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^* \ge 0$,

(ii) $2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\sqrt{\omega_j k_j - \omega_j\lambda_{j2}\lambda_2/\omega^*}$
$-(\omega_i\lambda_{i2} + \omega_j\lambda_{j2})\lambda_2/\omega^* \ge 0$,

(iii) $2/m\sqrt{\omega_i k_i - \omega_i\lambda_{i2}\lambda_2/\omega^*}\sqrt{\lambda_1} - \omega_i\lambda_{i1} \ge 0$,

for all $1\le i\le m$ and $1\le i<j\le m$. After calculation, the above conditions are equivalent to

$$\text{a)}\quad k_i \ge \lambda_{i2}\lambda_2/\omega^* + m^2\omega_i\lambda_{i1}^2/(4\lambda), \tag{18}$$

$$\text{b)}\quad (k_i - \lambda_{i2}\lambda_2/\omega^*)(k_j - \lambda_{j2}\lambda_2/\omega^*) \ge m^2\lambda_2^2(\omega_i\lambda_{i2} + \omega_j\lambda_{j2})^2/(4\omega_i\omega_j\omega^{*2}), \tag{19}$$

for all $1\le i\le m$ and $1\le i<j\le m$. Hence we have $\dot{V}\le 0$. Obviously, $\dot{V}=0$ if and only if $\boldsymbol{u}=0$ (or $\boldsymbol{y}=0$) and $\boldsymbol{x}=0$. Then set $E=\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})|\dot{V}=0\}=\{0\}$ is the largest invariant set for the augment system. According to the LaSalle invariance principle, Theorem 3 is proved.

The fractional system is general in Theorem 3. If the fractional-order system is a three-dimensional (3D) chaotic system or a four-dimensional (4D) hyperchaotic system, the same conclusion would be obtained. We consider a 3D chaotic system, which usually has this form

$$\begin{aligned}
&\mathrm{D}_t^{\alpha}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},y),\\
&\mathrm{D}_t^{\alpha}y = g(\boldsymbol{x},y),
\end{aligned}\tag{20}$$

where $\boldsymbol{x}\in R^2$ and $y\in R$ are the state vectors, and $\boldsymbol{f}=(f_1,f_2)^{\mathrm{T}}$ and g are continuous differential nonlinear functions. In the same way, we add the following fractional-order controller $\varphi(u)$ in system (22), then the controlled system would be obtained

$$\begin{aligned}
&\mathrm{D}_t^{\alpha}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},y),\\
&\mathrm{D}_t^{\beta}y = g(\boldsymbol{x},y) + \varphi(u).
\end{aligned}\tag{21}$$

The control function $\varphi(u)$ is given as $\varphi(u)=-ku$, where k is a control parameter and u is defined as $\mathrm{D}_t^{1-\beta}u=\omega(y-y^*)$, with ω being a positive real constant. Then the following corollary would be obtained.

Corollary 1 The fractional-order controller, which only has a single variable, can control the system (20) response to the asymptotically stable equilibrium point $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, if the control parameters k ($k>0$) and ω ($\omega>0$) satisfy the following condition:

$$k \ge \lambda_{12}\lambda_2/\omega + \omega\lambda_{11}^2/(4\lambda_1). \tag{22}$$

If the system is a 4D hyperchaotic system, which has this form:

$$\begin{aligned}
&\mathrm{D}_t^{\alpha}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}),\\
&\mathrm{D}_t^{\alpha}\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}),
\end{aligned}\tag{23}$$

where $\boldsymbol{x}\in R^2$ and $\boldsymbol{y}\in R^2$ are the state vectors, and $\boldsymbol{f}=(f_1,f_2)^{\mathrm{T}}$ and $\boldsymbol{g}=(g_1,g_2)^{\mathrm{T}}$ are continuous differential nonlinear vector functions. In the same way, we add the following

fractional-order controller $\boldsymbol{\varphi}(\boldsymbol{u})$ in system (23), then the controlled system would be obtained

$$\begin{aligned} \mathrm{D}_t^{\alpha}\boldsymbol{x} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}),\\ \mathrm{D}_t^{\beta}\boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})+\boldsymbol{\varphi}(\boldsymbol{u}). \end{aligned} \tag{24}$$

The control function $\boldsymbol{\varphi}(\boldsymbol{u})$ is given as $\boldsymbol{\varphi}(\boldsymbol{u}) = -\boldsymbol{k}\boldsymbol{u}$ where $\boldsymbol{k} = \mathrm{diag}(k_1, k_2)$ is a control parameter matrix and $\boldsymbol{u}$ is defined as $\mathrm{D}_t^{1-\beta}\boldsymbol{u} = \boldsymbol{\omega}(\boldsymbol{y}-\boldsymbol{y}^*)$, with $\boldsymbol{\omega} = \mathrm{diag}(\omega_1, \omega_2)$ being a positive diagonal matrix. Then the following corollary would be obtained.

Corollary 2 The fractional-order controller can control the system (23) response to the asymptotically stable equilibrium point $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, if the control parameters matrix $\boldsymbol{k} = \mathrm{diag}(k_1, k_2)$ $(k_i > 0, i = 1, 2)$ and $\boldsymbol{\omega} = \mathrm{diag}(\omega_1, \omega_2)$ $(\omega_i > 0, i = 1, 2)$ satisfy the following conditions:

$$\text{(I)} \quad k_i \geq \lambda_{i2}\lambda_2/\omega^* + \omega_i\lambda_{i1}^2/\lambda_1, i = 1, 2, \tag{25}$$

$$\text{(II)} \quad (k_1 - \lambda_{12}\lambda_2/\omega^*)(k_2 - \lambda_{22}\lambda_2/\omega^*) \geq \lambda_2^2(\omega_1\lambda_{12} + \omega_2\lambda_{22})^2/(\omega_1\omega_2\omega^{*2}). \tag{26}$$

4. Simulation and analysis

In this section, two 3D fractional-order chaotic systems and two 4D fractional-order hyperchaotic systems (two are commensurate and the others are incommensurate) are used as examples to show the effectiveness of the fractional-order controller. Firstly, we consider the nonlinear system as form (11), and we get the subsystem, which is asymptotically stable about the equilibrium point, by Theorems 1 and 2. Then, by adding a fractional controller in the system and letting the control parameters satisfy the conditions of Theorem 3, this system can be stabilized to the equilibrium point by a fractional-order controller.

4.1. Stabilization of the fractional-order unified system

The unified system,[26] which is introduced by Lü *et al.* in 2002, can unify the Lorenz, the Chen, and the Lü systems. The expression for the fractional unified system is given by[27]

$$\begin{aligned} \mathrm{D}_t^{\alpha_1}x_1 &= (25a+10)(x_2 - x_1),\\ \mathrm{D}_t^{\alpha_2}x_2 &= (28-35a)x_1 + (29a-1)x_2 - x_1x_3,\\ \mathrm{D}_t^{\alpha_3}x_3 &= x_1x_2 - (a+8)/3x_3, \end{aligned} \tag{27}$$

where $a \in R$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ is the fractional order of derivatives. When $a \in [0, 0.8)$, system (27) is the fractional generalized Lorenz system; when $a = 0.8$, it becomes the fractional Lü system; when $a \in (0.8, 1]$, it is the fractional generalized Chen system. If selecting $a = 1$, there are three equilibrium points $(0, 0, 0)$, $(3\sqrt{7}, 3\sqrt{7}, 21)$, and $(-3\sqrt{7}, -3\sqrt{7}, 21)$ in the system. With the fractional order $(\alpha_1, \alpha_2, \alpha_3) = (0.86, 0.9, 0.94)$, the incommensurate unified system can generate chaos which is shown in Fig. 1.

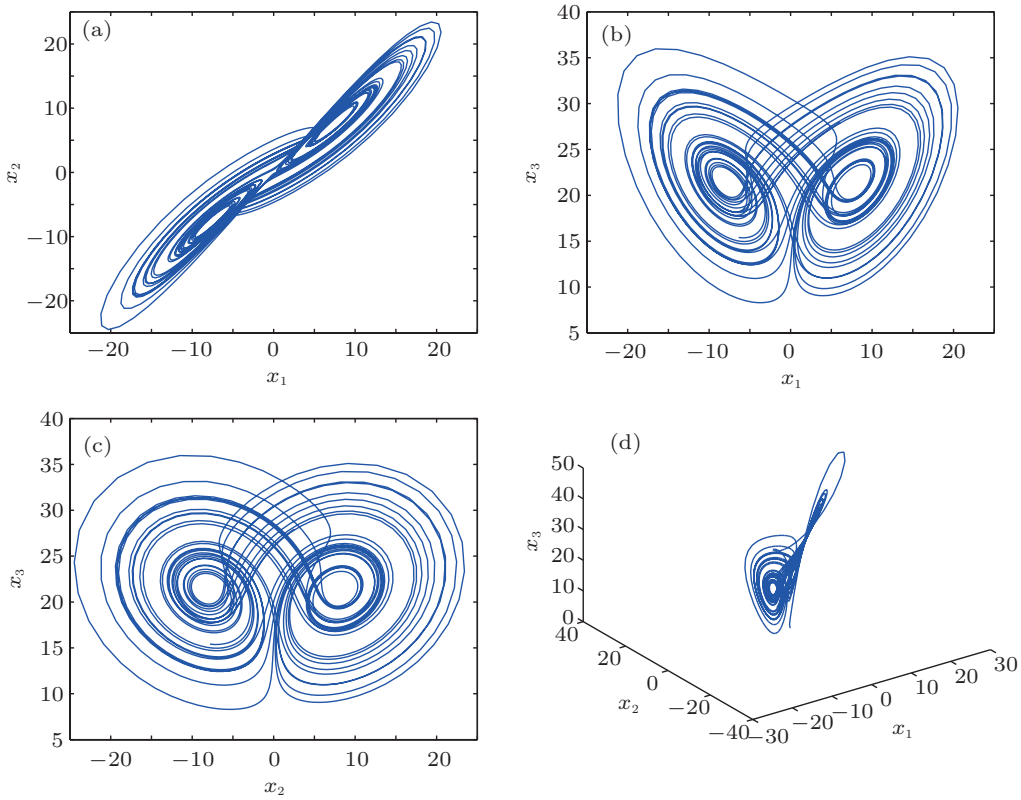


Fig. 1. (color online) Chaotic attractors in the fractional unified system with $(\alpha_1, \alpha_2, \alpha_3) = (0.86, 0.9, 0.94)$. The panels (a), (b), (c), and (d) show the x_2–x_1, x_3–x_1, x_3–x_2, and 3D views, respectively.

We can easily get the two-dimensional subsystem of system (27) if $x_2 = x_2^*$,

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= (25a+10)(x_2^* - x_1), \\ \mathrm{D}_t^{\alpha_3} x_3 &= x_1 x_2^* - (a+8)/3 x_3. \end{aligned} \tag{28}$$

According to the Theorem 2, we can solve all roots of Eq. (10), and the root $\lambda_0 = 1.0214 + 0.0684\sqrt{-1}$, which has the smallest compound angle, satisfies the condition of Theorem 2, i.e, $|\arg\lambda_0| = 0.0668 > \pi/(2M)$, where $M = 50$. Then the subsystem is asymptotically stable about the equilibrium point $x_1 = x_1^*$ and $x_3 = x_3^*$. Therefore, let $\boldsymbol{x} = [x_1, x_3]^{\mathrm{T}}$ and $y = x_2$, system (27) satisfies Assumption 1. Thus, we add a fractional controller in system (27) as follows:

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= (25a+10)(x_2 - x_1), \\ \mathrm{D}_t^{\alpha_2} x_2 &= (28-35a)x_1 + (29a-1)x_2 - x_1 x_3 - \varphi(u), \\ \mathrm{D}_t^{\alpha_3} x_3 &= x_1 x_2 - (a+8)/3 x_3, \end{aligned} \tag{29}$$

where control function $\varphi(u) = -ku$ and $D_t^{1-\alpha_2} u = \omega(x_2 - x_2^*)$. When the control parameters k and ω satisfy the condition of Corollary 1, system (29) can be stabilized to the equilibrium point $\boldsymbol{x}^*$ by a single state fractional-order controller.

When selecting $x_2^* = -3\sqrt{7}$ and the parameters $k = 300$, $\omega = 3$, and $u(0) = 0$, the numerical results, illustrated in Fig. 2, show that the fractional-order unified system is driven to its equilibrium point $(-3\sqrt{7}, -3\sqrt{7}, 21)$ asymptotically as $t \to \infty$. When x_2^* changes, the controller can drive the system to other equilibrium points.

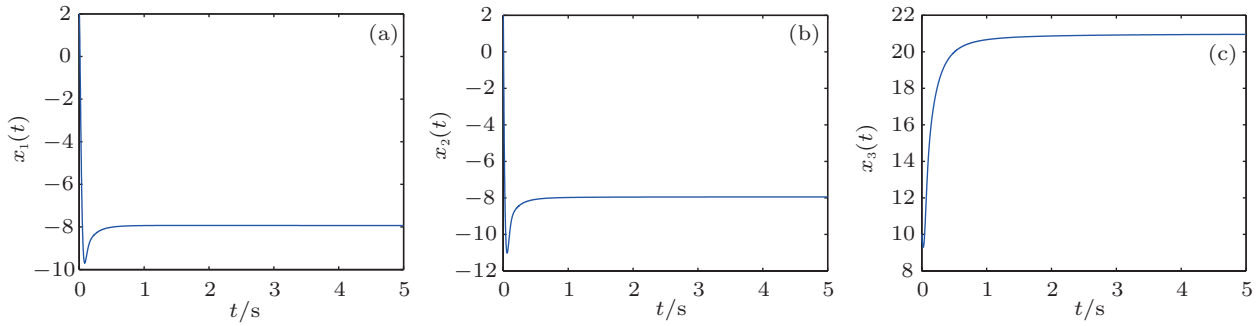


Fig. 2. (color online) Time waveforms of state variables x_1 (a), x_2 (b), and x_3 (c) of the controlled fractional unified system.

4.2. Stabilization of the fractional-order Liu's system

In 2009, Liu *et al.* introduced a 3D chaotic dynamical system, which is called Liu's system.[28] Its fractional-order version was described by Gejji and Bhalekar,[29] and it has the form:

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1(t) &= -a x_1 - e x_2^2, \\ \mathrm{D}_t^{\alpha_2} x_2(t) &= b x_2 - k x_1 x_3, \\ \mathrm{D}_t^{\alpha_3} x_3(t) &= -c x_3 + m x_1 x_2, \end{aligned} \tag{30}$$

where $a, b, c, e, k, m \in R$. When $a = e = 1$, $b = 2.5$, $k = m = 4$, and $c = 5$, the system has five equilibrium points. Two of them are complex and three are real equilibriums $(0,0,0)$, $(-0.883883, 0.940151, -0.664787)$, and $(-0.883883, -0.940151, -0.664787)$. If we consider the commensurate system, the smallest order of this system to generate chaos is 0.916.[2] With the order $\alpha_1 = \alpha_2 = \alpha_3 = 0.94$, the system (30) can generate chaos, which is shown in Fig. 3.

We can easily get the two-dimensional subsystem of system (30) if $x_2 = x_2^*$,

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1(t) &= -a x_1 - e x_2^{*2}, \\ \mathrm{D}_t^{\alpha_3} x_3(t) &= -c x_3 + m x_1 x_2^*. \end{aligned} \tag{31}$$

It is asymptotically stable about the equilibrium point $x_1 = x_1^*$ and $x_3 = x_3^*$, because the eigenvalues of the Jacobian matrix of system (31), evaluated at (x_1^*, x_3^*) are $\lambda_1 = -a < 0$ and $\lambda_2 = -c < 0$, which satisfy Eq. (8), i.e., $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$. Therefore, let $\boldsymbol{x} = [x_1, x_3]^{\mathrm{T}}$ and $y = x_2$, system (30) satisfies Assumption 1. Thus, we add a controller in system (30) as follows:

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1(t) &= -a x_1 - e x_2^2, \\ \mathrm{D}_t^{\alpha_2} x_2(t) &= b x_2 - k x_1 x_3 - \varphi(u), \\ \mathrm{D}_t^{\alpha_3} x_3(t) &= -c x_3 + m x_1 x_2, \end{aligned} \tag{32}$$

where control function $\varphi(u) = -ku$ and $D_t^{1-\alpha_2} u = \omega(x_2 - x_2^*)$. When the control parameters k and ω satisfy the condition of Corollary 1, system (32) can be stabilized to the equilibrium point $\boldsymbol{x}^*$ by a single state fractional-order controller.

When selecting $x_2^* = 0.940151$ and the parameters $k = 50$, $\omega = 1$, and $u(0) = 0$, the numerical results, illustrated in Fig. 4, show that the fractional-order Liu system is driven to its equilibrium point $(-0.883883, 0.940151, -0.664787)$ asymptotically as $t \to \infty$. When x_2^* changes, the controller can drive the system to other equilibrium points.

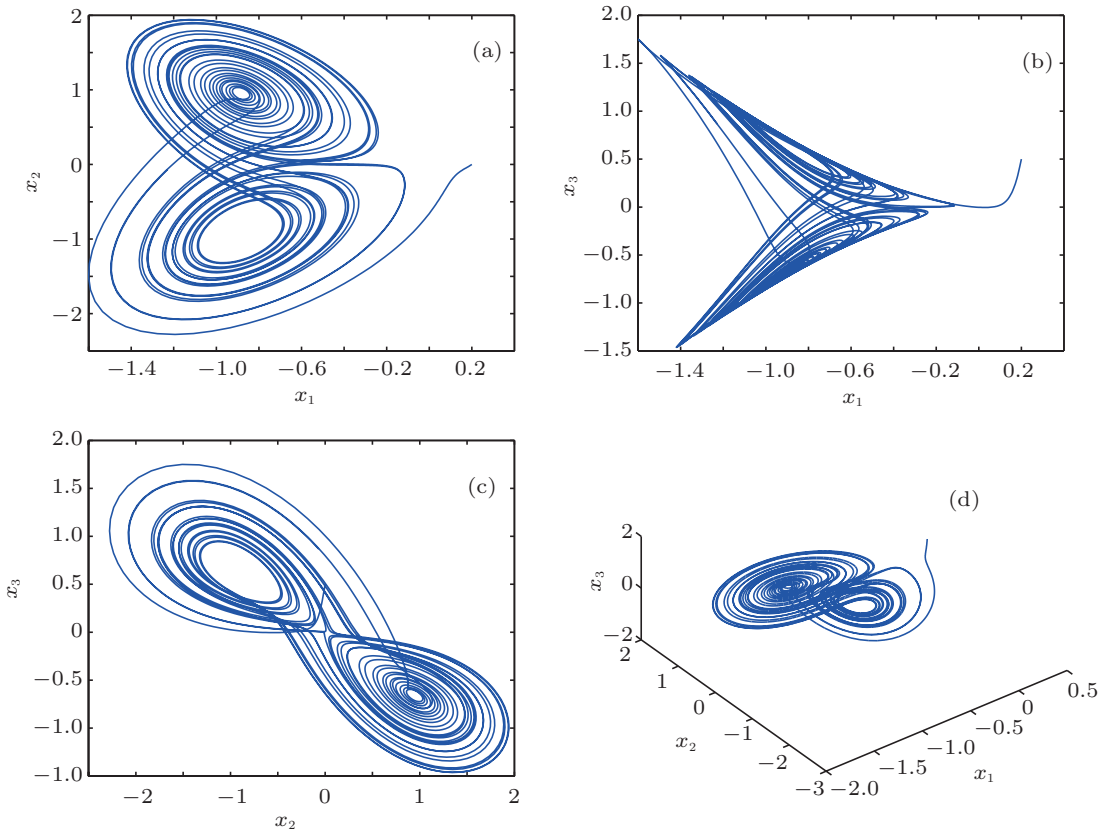


Fig. 3. (color online) Chaotic attractors in the fractional Liu's system with $(\alpha_1, \alpha_2, \alpha_3) = (0.94, 0.94, 0.94)$. The panels (a), (b), (c), and (d) show the x_2–x_1, x_3–x_1, x_3–x_2, and 3D views, respectively.

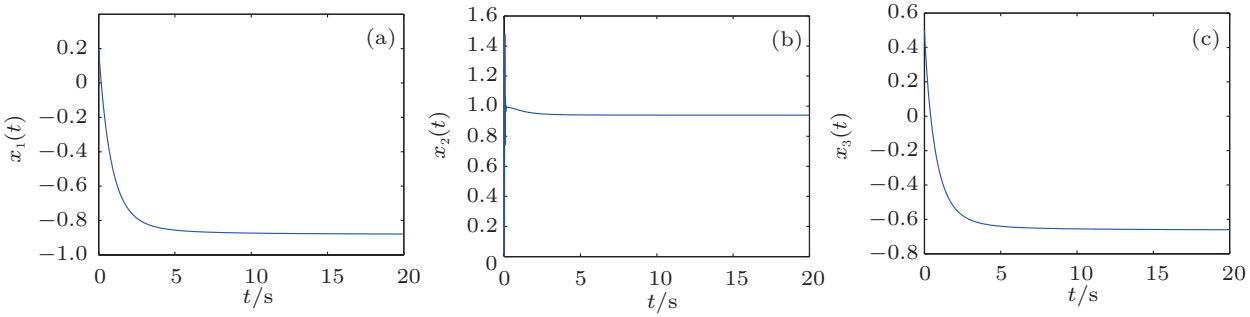


Fig. 4. (color online) Time waveforms of state variables x_1 (a), x_2 (b), and x_3 (c) of the controlled fractional Liu's system.

4.3. Stabilization of the fractional-order hyperchaotic Chen system

In 2005, Li *et al.*[10] proposed a new hyperchaotic Chen system from the Chen system, which is described by

$$\begin{aligned} \dot{x_1} &= a(x_2 - x_1) + x_4, \\ \dot{x_2} &= dx_1 - x_1x_3 + cx_2, \\ \dot{x_3} &= x_1x_2 - bx_3, \\ \dot{x_4} &= x_2x_3 + rx_4, \end{aligned} \tag{33}$$

where $a, b, c, d, k \in R$. When $a = 35$, $b = 3$, $c = 12$, $d = 7$, and $0.085 < r \leq 0.798$, system (33) is hyperchaotic.[30] Then we consider the fractional version of this system, which is given by

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= a(x_2 - x_1) + x_4, \\ \mathrm{D}_t^{\alpha_2} x_2 &= dx_1 - x_1x_3 + cx_2, \\ \mathrm{D}_t^{\alpha_3} x_3 &= x_1x_2 - bx_3, \\ \mathrm{D}_t^{\alpha_4} x_4 &= x_2x_3 + rx_4, \end{aligned} \tag{34}$$

where α_i $(0 < \alpha_i \leq 1, i = 1, 2, 3, 4)$ is the fractional order. When $a = 35$, $b = 3$, $c = 12$, $d = 7$, and $k = 0.5$, this system has only one equilibrium point at $(0, 0, 0, 0)$. If we consider the commensurate system, according to the numerical simulations, the lowest order of this system to generate hyperchaos is 0.946. If selecting the order $\alpha = 0.96$ and initial value $(2, 2, 1, -1)$, the system exhibits hyperchaotic behavior,

as shown in Fig. 5.

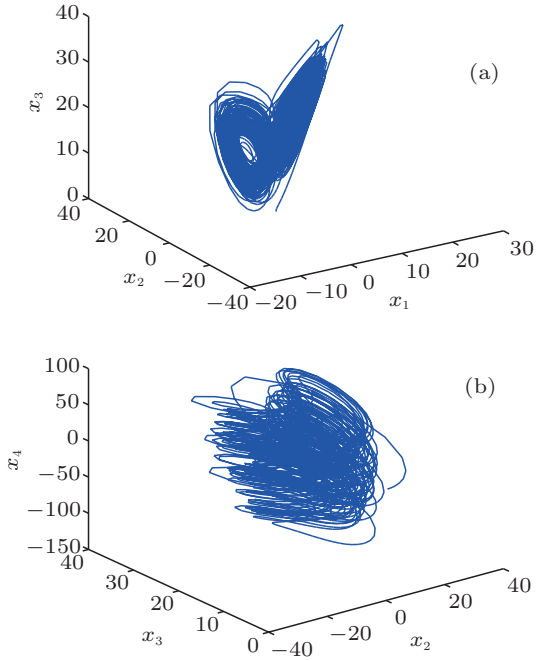


Fig. 5. (color online) Chaotic attractors in the fractional-order hyperchaotic Chen system with $\alpha = 0.96$. The panels (a) and (b) show the 3D views of x_3–x_2–x_1 and x_4–x_3–x_2, respectively.

We can easily get the two-dimensional subsystem of system (34) if $x_2 = x_4 = 0$

$$\begin{aligned} \mathrm{D}_t^{\alpha} x_1 &= -a x_1, \\ \mathrm{D}_t^{\alpha} x_3 &= -b x_3. \end{aligned} \tag{35}$$

It is asymptotically stable about the equilibrium point $x_1 = x_3 = 0$, because the eigenvalues of the Jacobian matrix of system (35), evaluated at $(0,0)$, are $\lambda_1 = -a < 0$ and $\lambda_2 = -b < 0$, which satisfy Eq. (8), i.e., $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$. Therefore, let $\boldsymbol{x} = [x_1, x_3]^{\mathrm{T}}$ and $\boldsymbol{y} = [x_2, x_4]^{\mathrm{T}}$, system (34) satisfies Assumption 1. Thus, we add controllers in system (34) as follows:

$$\begin{aligned} \mathrm{D}_t^{\alpha} x_1 &= a(x_2 - x_1) + x_4, \\ \mathrm{D}_t^{\alpha} x_2 &= d x_1 - x_1 x_3 + c x_2 + \varphi_1(u_1), \\ \mathrm{D}_t^{\alpha} x_3 &= x_1 x_2 - b x_3, \\ \mathrm{D}_t^{\alpha} x_4 &= x_2 x_3 + r x_4 + \varphi_2(u_2), \end{aligned} \tag{36}$$

where $\varphi_i(u_i) = -k_i u_i$ $(i = 1, 2)$, $D_t^{1-\alpha} u_1 = \omega_1 x_2$, $D_t^{1-\alpha} u_2 = \omega_2 x_4$. When the control parameters k_1, k_2 and ω_1, ω_2 satisfy the conditions of Corollary 2, system (36) can be stabilized by a fractional-order controller.

When selecting parameters $k_1 = k_2 = 100$, $\omega_1 = \omega_2 = 1$, $\boldsymbol{u}(0) = 0$, the numerical results, illustrated in Fig. 6, show that the fractional-order system is driven to its equilibrium point $(0,0,0,0)$ asymptotically as $t \to \infty$.

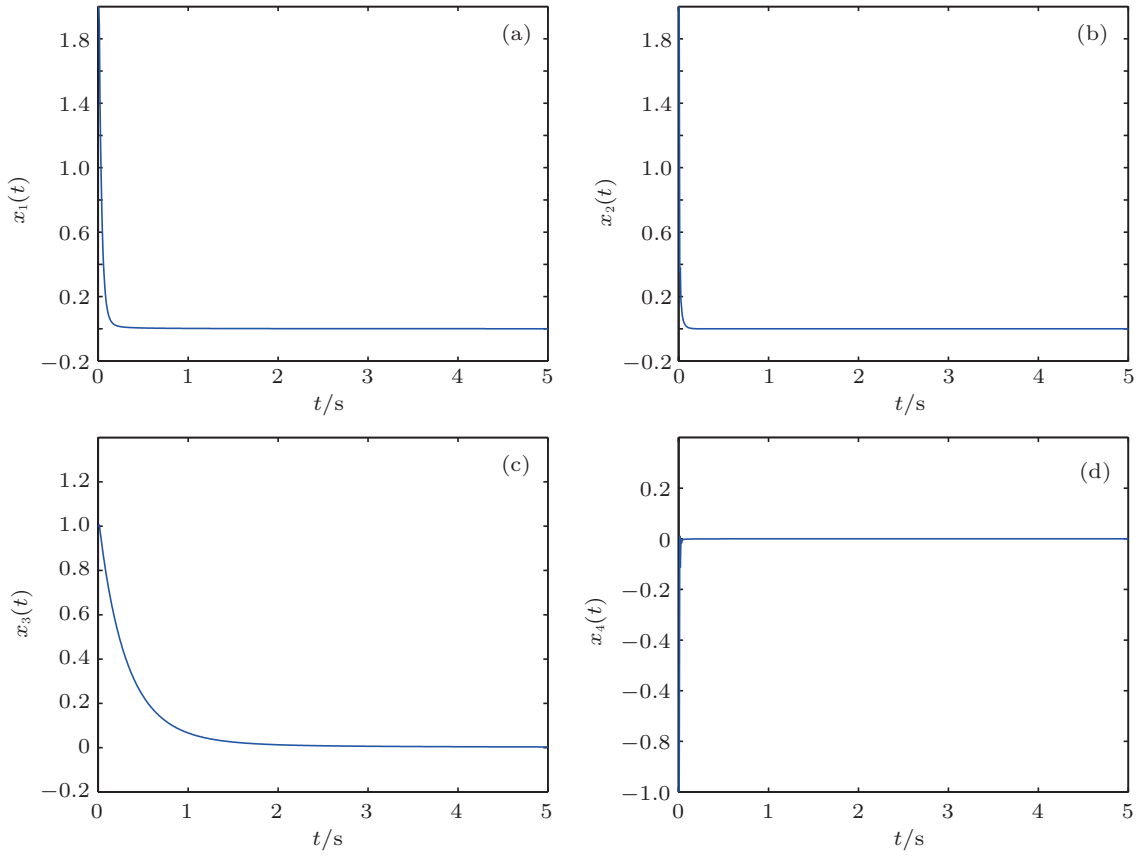


Fig. 6. (color online) Time waveforms of state variables x_1 (a), x_2 (b), x_3 (c), and x_4 (d) of the controlled fractional hyperchaotic Chen system.

4.4. Stabilization of the fractional-order hyperchaotic Lorenz system

Wang *et al.*[30] added a nonlinear controller x_4 to the first equation of the Lorenz system,[2] and let $\dot{x}_4 = -x_2x_3 + rx_4$, then a new system is obtained

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) + x_4,\\ \dot{x}_2 &= cx_1 - x_1x_3 - x_2,\\ \dot{x}_3 &= x_1x_2 - bx_3,\\ \dot{x}_4 &= -x_2x_3 + rx_4.\end{aligned} \tag{37}$$

When $a = 10$, $b = 8/3$, $c = 28$, and $r = -1$, the system exhibits a hyperchaotic behavior. Wang *et al.*[31] considered the 4D fractional-order Lorenz system, which can be described as

$$\begin{aligned}\mathrm{D}_t^{\alpha_1}x_1 &= a(x_2 - x_1) + x_4,\\ \mathrm{D}_t^{\alpha_2}x_2 &= cx_1 - x_1x_3 - x_2,\\ \mathrm{D}_t^{\alpha_3}x_3 &= x_1x_2 - bx_3,\\ \mathrm{D}_t^{\alpha_4}x_4 &= x_2x_3 + rx_4.\end{aligned} \tag{38}$$

When selecting the parameters as $a = 10$, $b = 8/3$, $c = 28$, $r = -1$, and $\alpha_i = 0.98$ $(i = 1,2,3,4)$, system (37) is hyperchaotic.[31] Then we consider the fractional version of this system, which is incommensurate. By calculation, this system has five equilibrium points. Two of them are complex and three are real equilibriums $(0,0,0)$, $(-1.166531, -21.626769, 9.460608, 204.602380)$, and $(1.166531, 21.626769, 9.460608, -204.602380)$. If selecting the order $\alpha_1 = 0.94$, $\alpha_2 = 0.96$, $\alpha_3 = 0.97$, $\alpha_4 = 0.99$, and initial value $(2,-2,1,-1)$, the system exhibits hyperchaotic behavior, as shown in Fig. 7.

We can easily get the two-dimensional subsystem of system (39) if $x_2 = x_2^*$ and $x_4 = x_4^*$

$$\begin{aligned}\mathrm{D}_t^{\alpha_1}x_1 &= a(x_2^* - x_1) + x_4^*,\\ \mathrm{D}_t^{\alpha_3}x_3 &= x_1x_2^* - bx_3.\end{aligned} \tag{39}$$

According to Theorem 2, we can solve all roots of Eq. (10), and the root $\lambda_0 = 1.0096 - 0.0327\sqrt{-1}$, which has the smallest compound angle, satisfies the condition of Theorem 2, i.e., $|\arg\lambda_0| = 0.0324 > \pi/(2M)$, where $M = 100$. Then the subsystem is asymptotically stable about the equilibrium point $x_1 = x_1^*$ and $x_3 = x_3^*$. Therefore, let $\boldsymbol{x} = [x_1, x_3]^{\mathrm{T}}$ and $\boldsymbol{y} = [x_2, x_4]^{\mathrm{T}}$, system (39) satisfies Assumption 1. Thus, we add controllers in system (38) as follows:

$$\begin{aligned}\mathrm{D}_t^{\alpha_1}x_1 &= a(x_2 - x_1) + x_4,\\ \mathrm{D}_t^{\alpha_2}x_2 &= dx_1 - x_1x_3 + cx_2 + \varphi_1(u_1),\\ \mathrm{D}_t^{\alpha_3}x_3 &= x_1x_2 - bx_3,\\ \mathrm{D}_t^{\alpha_4}x_4 &= x_2x_3 + rx_4 + \varphi_2(u_2),\end{aligned} \tag{40}$$

where $\varphi_i(u_i) = -k_iu_i$ $(i = 1,2)$, $D_t^{1-\alpha_2}u_1 = \omega_1x_2$, $D_t^{1-\alpha_4}u_2 = \omega_2x_4$. When the control parameters k_1, k_2 and ω_1, ω_2 satisfy the conditions of Corollary 2, the system can be stabilized by the fractional-order controller.

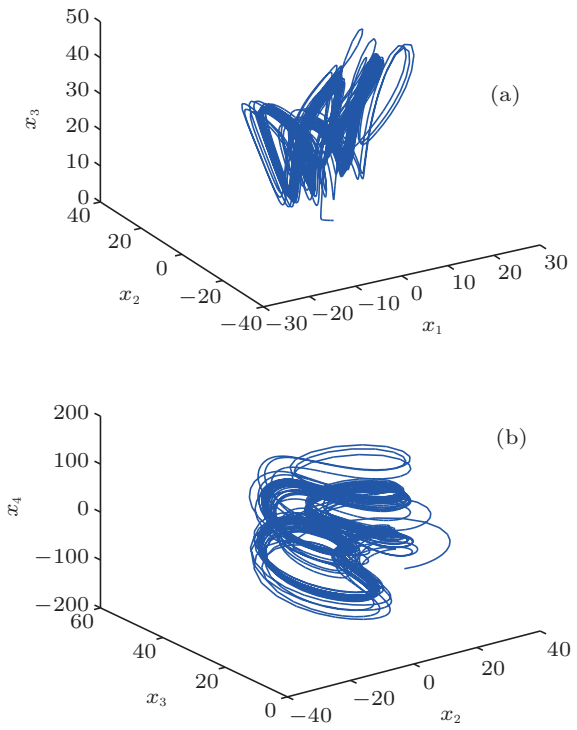


Fig. 7. (color online) Chaotic attractors in the fractional hyperchaotic Lorenz system with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.94, 0.96, 0.97, 0.99)$. Panels (a) and (b) show the 3D views of x_3–x_2–x_1 and x_4–x_3–x_2, respectively.

When selecting the equilibrium point $x_2^* = 21.626769$, $x_4^* = -204.602390$, the parameters $k_1 = k_2 = 100$, $\omega_1 = \omega_2 = 1$, and initial value $\boldsymbol{u}(0) = 0$, the numerical results are illustrated in Fig. 8, which shows that the fractional-order hyperchaotic Lorenz system is driven to its equilibrium point $(-1.166531, 21.626769, 9.460608, -204.602390)$ asymptotically as $t \to \infty$. When x_2^* and x_4^* change, the controller can drive the system to other equilibrium points.

4.5. Comparing fractional-order controller with traditional controllers

Finally, in order to compare the fractional-order controller with traditional controllers, we also apply the linear feedback control method in Ref. [15] to control Chen's system to its equilibrium point $(-3\sqrt{7}, -3\sqrt{7}, 21)$ and apply the nonlinear feedback control in Ref. [31] to control the hyperchaotic Lorenz system to its equilibrium point $(1.167, 21.627, 9.461, -204.602)$. The numerical results, illustrated in Fig. 9 and Fig. 10, show that the fractional-order Chen system and hyperchaotic Lorenz system are driven to their equilibrium points asymptotically as $t \to \infty$, respectively.

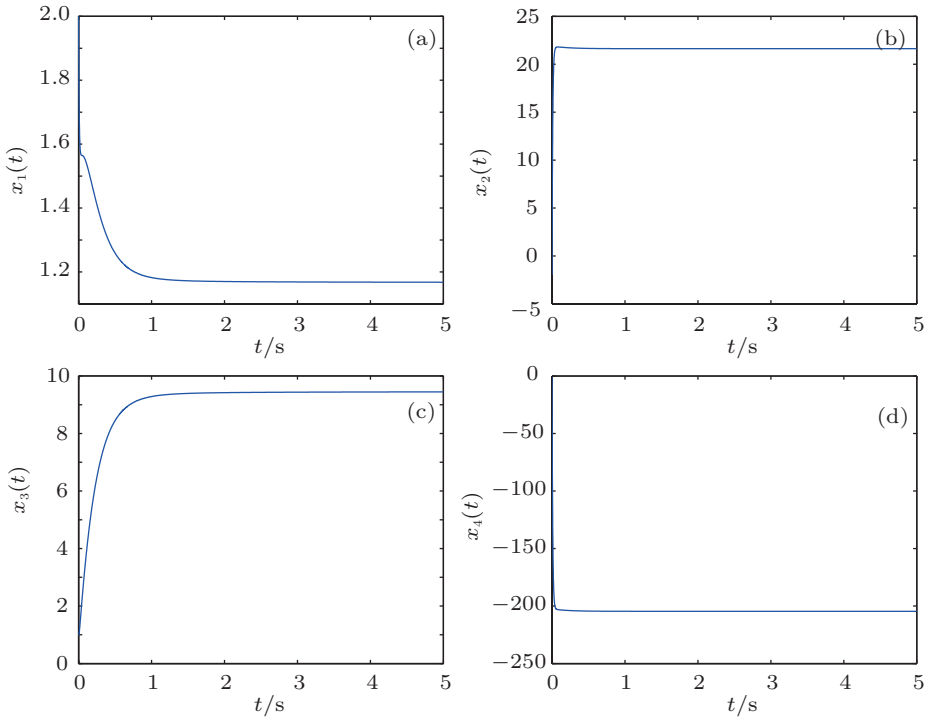


Fig. 8. (color online) Time waveforms of state variables x_1 (a), x_2 (b), x_3 (c), and x_4 (d) of the controlled fractional hyperchaotic Lorenz system.

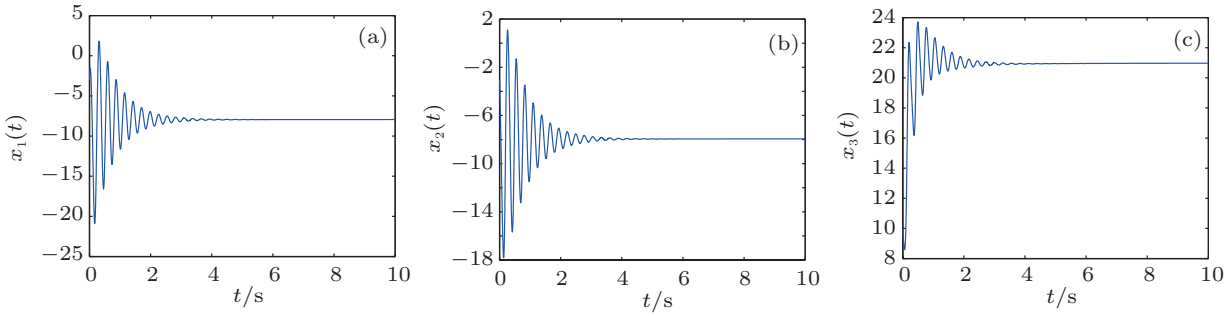


Fig. 9. (color online) Time waveforms of state variables x_1 (a), x_2 (b), and x_3 (c) of the fractional chaotic Chen system controlled by the linear feedback controller.

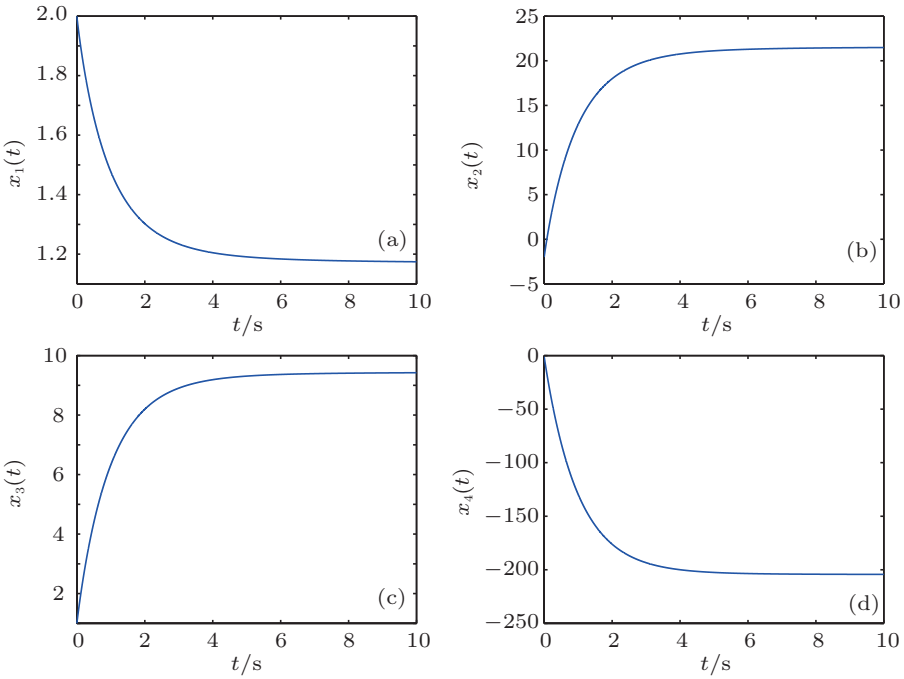


Fig. 10. (color online) Time waveforms of state variables x_1 (a), x_2 (b), x_3 (c), and x_4 (d) of the fractional hyperchaotic Lorenz system controlled by the nonlinear feedback controller.

By comparing Fig. 2 and Fig. 9, we can see that the fractional-order controller controls the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ to the equilibrium point at about 0.5 s, 0.5 s, and 1.2 s respectively, but the linear feedback control is at about 2.5 s in the same precision. By comparing Fig. 8 and Fig. 10, we can see that the fractional-order controller controls the state variables $x_1(t), x_2(t), x_3(t), x_4(t)$ to the equilibrium point at about 1.5 s, 0.2 s, 1.5 s, and 0.2 s respectively, but the nonlinear feedback control is at about 6 s in the same precision. These also show that the fractional-order controller can control the fractional-order chaotic and hyperchaotic systems more effectively and quickly than traditional controllers.

5. Conclusion

In this letter, we proposed a fractional-order controller and presented a control law for the arbitrary dimensional commensurate (or incommensurate) fractional-order systems. According to the Lyapunov stability theorem of the fractional-order systems, the control law achieved the stabilization of the fractional-order chaotic and hyperchaotic systems. The proposed method provides a theoretical basis for the applications of the fractional-order control in fractional dynamic systems. Our proposed controller is very simple, universal and could be easily realized experimentally. Furthermore, this method can control fractional-order chaotic and hyperchaotic systems more effectively and quickly than traditional controllers. We have implemented and verified our method for other fractional-order chaotic and hyperchaotic systems,[2,13,24] namely: the fractional-order Newton–Leipnik's system, Lü's chaotic system, financial system, hyperchaotic Rössler system, etc. The numerical simulation results indicate that the proposed controller can effectively control the fractional-order chaotic and hyperchaotic systems, and the proposed method provides a theoretical basis for applications of the fractional-order control in fractional dynamic systems.

In future works on this topic, we will consider whether the proposed method can be extended to the control of other complex chaotic systems, such as fractional multi-scroll chaotic systems,[32] and networked fractional chaotic systems.[33–36] Multi-scroll chaotic systems have non-smooth functions and complex multi-scroll chaotic attractors, which make systems much more complex than the similar fractional chaotic systems. The networked fractional chaotic systems are composed of many nodes, and each node receives information from its adjacent nodes. Extending the proposed method to these systems is important for both control theory and applications of fractional chaotic systems.

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