A sufficient and necessary criterion for separability of bipartite quantum states*

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A sufficient and necessary criterion for separability of bipartite quantum states is presented. We construct a class of mixed states in $2\otimes k$ quantum systems and in $3\otimes k$ quantum systems, respectively and show that these states are separable if and only if they are positive partial transpositions.

Keywords: positive partial transposition, separability

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1. Introduction

Quantum entanglement is one of the most remarkable features of quantum mechanics[1,2] that has found wide applications in the rapidly expanding field of quantum information processing. A fundamental problem in the theory of quantum entanglement is to check whether a given state is entangled or separable. For the case of pure states, determining an entangled state is very easy, since it is based on properties of the Schmidt decomposition. However, for the case of mixed bipartite states, no single practical procedure guaranteed to detect the entanglement of every entangled state has been found so far. There exists a considerable effect to analyze the separability and quantitative character of quantum entanglement[3–16] In Ref. [17], Peres discovered a very strong necessary condition of separability, namely, that the separable states remain positive if they are subjected to a partial transposition (PPT criterion). This was soon shown by Horodecki *et al.* to be sufficient for $2\otimes 2$ and $2\otimes 3$ bipartite systems.[18] The reduction criterion for separability was proposed in Ref. [19]. This criterion is equivalent to the PPT criterion for $2\otimes n$ composite systems, but it is not sufficient for separability in general. More criteria were presented in Refs. [20] and [21], which are necessary conditions for a state to be separable and complement the PPT criterion in certain aspects. The majorization criterion[22] and generalized realignment criterion[23] are also necessary conditions for separability. Some necessary and sufficient operational criteria for low rank density matrices were provided in Refs. [24]–[26]. In Ref. [27] the authors showed that the PPT criterion is both necessary and sufficient for separability in $3\otimes 3$ quantum systems.

In this paper, we study quantum states in which the PPT criterion is both sufficient and necessary for separability. First, we present a sufficient and necessary condition for separability of quantum states in $2\otimes k$ quantum systems. Then we construct a class of mixed states in $3\otimes k$ quantum systems and show that these states are separable if and only if they are PPT. We generalize the results of $3\otimes 3$ systems in Ref. [27] to the case of $3\otimes k$ systems.

2. Separability in $2\otimes k$ quantum systems

In this section, we consider a class of quantum mixed states in $2\otimes k$ systems given by

$$\rho=\sum_{i=1}^{k}\lambda_i|X_i\rangle\langle X_i|, \tag{1}$$

where $\sum_{i=1}^{k}\lambda_i=1$, $0<\lambda_i<1$, $|X_i\rangle$ are orthonormal vectors, $i=1,2,\ldots,k$,

$$\begin{aligned}|X_1\rangle&=(a_1,0,\ldots,0,0,0,0,\ldots,0,a_1')^{\mathrm{T}},\ \ldots,\\ |X_k\rangle&=(0,0,\ldots,0,a_k,0,0,\ldots,a_k',0)^{\mathrm{T}},\end{aligned} \tag{2}$$

where a_i,a_i', $i=1,2,\ldots,k$, are non-zero complex numbers and T stands for transposition.

A permutation operator acting on the second system is defined as

$$P=\begin{pmatrix}0&0&\ldots&0&1\\1&0&\ldots&0&0\\0&1&\ldots&0&0\\\vdots&\vdots&\ldots&\vdots&\vdots\\0&0&\ldots&1&0\end{pmatrix},$$

then we have

$$\begin{aligned}&(I\otimes P)(a_1,a_2,\ldots,a_{k-1},a_k,a_{k+1},a_{k+2},\ldots,a_{2k-1},a_{2k})^{\mathrm{T}}\\ =\ &(a_k,a_1,\ldots,a_{k-2},a_{k-1},a_{2k},a_{k+1},\ldots,a_{2k-2},a_{2k-1})^{\mathrm{T}},\end{aligned} \tag{3}$$

and $P^k=I$, where I denotes identity matrix. The matrix ρ^{T_B} has the form as follows:

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$$\rho^{T_B}=\begin{pmatrix}
\lambda_1 a_1\overline{a_1} & 0 & \dots & 0 & 0 & 0 & \lambda_2 a_2\overline{a'_2} & \dots & 0 & 0\\
0 & \lambda_2 a_2\overline{a_2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\
0 & 0 & \dots & \lambda_{k-1}a_{k-1}\overline{a_{k-1}} & 0 & 0 & 0 & \dots & 0 & \lambda_k a_k\overline{a'_k}\\
0 & 0 & \dots & 0 & \lambda_k a_k\overline{a_k} & \lambda_1 a_1\overline{a'_1} & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & \lambda_1 a'_1\overline{a_1} & \lambda_2 a'_2\overline{a'_2} & 0 & \dots & 0 & 0\\
\lambda_2 a'_2\overline{a_2} & 0 & \dots & 0 & 0 & 0 & \lambda_3 a'_3\overline{a'_3} & \dots & 0 & 0\\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \lambda_k a'_k\overline{a'_k} & 0\\
0 & 0 & \dots & \lambda_k a'_k\overline{a_k} & 0 & 0 & 0 & \dots & 0 & \lambda_1 a'_1\overline{a'_1}
\end{pmatrix},$$

where $\overline{a_i}$ and $\overline{a'_i}$ denote the conjugate of a_i and a'_i.

The non-negativity of ρ^{T_B}, i.e., $\rho^{T_B}\geq 0$, is equivalent to the non-negativity of the following 2×2 matrices:

$$A_i=\begin{pmatrix}\lambda_i a_i\overline{a_i} & \lambda_{i+1}a_{i+1}\overline{a'_{i+1}}\\ \lambda_{i+1}a'_{i+1}\overline{a_{i+1}} & \lambda_{i+2}a'_{i+2}\overline{a'_{i+2}}\end{pmatrix},\quad i=1,2,\dots,k, \tag{4}$$

where the indices are $k+1=1$ and $k+2=2$. The non-negativity of A_i is equivalent to the following inequalities

$$\lambda_i\lambda_{i+2}|a_i|^2|a'_{i+2}|^2\geq\lambda_{i+1}^2|a_{i+1}|^2|a'_{i+1}|^2,\quad i=1,2,\dots,k. \tag{5}$$

We can show that the inequalities (5) are equalities, and the proof is shown in Appendix A. Therefore, these equalities can be written as

$$\sqrt{\lambda_i\lambda_{i+2}}a_i a'_{i+2}=\lambda_{i+1}a_{i+1}a'_{i+1}\mathrm{e}^{\mathrm{i}\theta_{i+1}},\quad i=1,2,\dots,k, \tag{6}$$

where all $\theta_i\in[-\pi,\pi]$. Next we show that ρ has a pure state decomposition $\rho=\sum_{l=1}^k|\psi_l\rangle\langle\psi_l|$ such that all states $|\psi_l\rangle$, $l=1,2,\dots,k$, are separable. $|\psi_l\rangle$ can be generally expressed as $|\psi_l\rangle=\sum_{m=1}^k u_{ml}|Y_m\rangle=\sum_{i=1}^2\sum_{j=1}^k a_{ij}^l|ij\rangle$, where u_{ml} are the entries of a $k\times k$ unitary matrix U and $|Y_m\rangle=\sqrt{\lambda_m}|X_m\rangle$. Let $B_l=(a_{ij}^l)_{2\times k}$, and suppose the matrix U has the following form $U=(u_{il})_{k\times k}=(c_{il}\mathrm{e}^{\mathrm{i}\theta_i})_{k\times k}$. According to the unitary condition of U, we have

$$\sum_{l=1}^k c_{ml}c_{nl}=\delta_{mn}. \tag{7}$$

Then B_l, $l=1,2,\dots,k$, has the following form

$$B_l=\begin{pmatrix}\sqrt{\lambda_1}a_1u_{1l} & \sqrt{\lambda_2}a_2u_{2l} & \sqrt{\lambda_3}a_3u_{3l} & \dots & \sqrt{\lambda_k}a_ku_{kl}\\ \sqrt{\lambda_2}a'_2u_{2l} & \sqrt{\lambda_3}a'_3u_{3l} & \sqrt{\lambda_4}a'_4u_{4l} & \dots & \sqrt{\lambda_1}a'_1u_{1l}\end{pmatrix}. \tag{8}$$

It is straightforward to verify that the matrix B_l has rank one if

$$c_{1l}c_{3l}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_3)}=c_{2l}^2\mathrm{e}^{\mathrm{i}\theta_2},\ \ c_{1l}c_{4l}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_4)}=c_{2l}c_{3l}. \tag{9}$$

As $0<\mathrm{rank}(B_lB_l^\dagger)\leq\mathrm{rank}(B_l)\mathrm{rank}(B_l^\dagger)$, if the rank of B_l is one, matrix $B_lB_l^\dagger$ has also rank one and $|\psi_l\rangle$ is separable, then ρ has separable pure state decomposition $\rho=\sum_{l=1}^k|\psi_l\rangle\langle\psi_l|$ and ρ is separable if it is PPT.

If ρ is separable, then it is necessary for PPT, which was shown in Ref. [17]; therefore, we have the following theorem.

Theorem 1 If the state ρ satisfies $c_{1l}c_{3l}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_3)}=c_{2l}^2\mathrm{e}^{\mathrm{i}\theta_2}, c_{1l}c_{4l}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_4)}=c_{2l}c_{3l}$, $\theta_i\in[-\pi,\pi]$, then it is separable if and only if it is PPT.

3. Separability in $3\otimes k$ quantum systems

In this section, we consider a class of quantum mixed states in $3\otimes k$ systems given by

$$\rho=\sum_{i=1}^k\lambda_i|X_i\rangle\langle X_i|, \tag{10}$$

where $\sum_{i=1}^k\lambda_i=1$, $0<\lambda_i<1$, $|X_i\rangle$ are orthonormal vectors, $i=1,2,\dots,k$.

$$\begin{aligned}
|X_1\rangle&=(a_1,0,\dots,0,0,0,0,\dots,a'_1,0,0,0,\dots,0,a''_1)^{\mathrm{T}},\\
|X_2\rangle&=(0,a_2,\dots,0,0,0,0,\dots,0,a'_2,a''_2,0,\dots,0,0)^{\mathrm{T}},\\
&\dots\\
|X_{k-1}\rangle&=(0,0,\dots,a_{k-1},0,0,\dots,a'_{k-1},0,0,0,0,\dots,a''_{k-1},0,0)^{\mathrm{T}},\\
|X_k\rangle&=(0,0,\dots,0,a_k,0,\dots,a'_k,00,0,0,\dots,a''_k,0)^{\mathrm{T}},
\end{aligned} \tag{11}$$

where a_i, a'_i, a''_i, $i=1,2,\dots,k$ are non-zero complex numbers, T stands for transposition. We define a permutation operator P acting on the second system

$$\begin{aligned}
&(I\otimes P)(a_1,a_2,\dots,a_{k-1},a_k,a_{k+1},a_{k+2},\dots,\\
&a_{2k-1},a_{2k},a_{2k+1},a_{2k+2},\dots,a_{3k-1},a_{3k})^{\mathrm{T}}\\
=\ &(a_k,a_1,\dots,a_{k-2},a_{k-1},a_{2k},a_{k+1},\dots,\\
&a_{2k-2},a_{2k-1},a_{3k},a_{2k+1},\dots,a_{3k-2},a_{3k-1})^{\mathrm{T}},
\end{aligned} \tag{12}$$

then we have $P^k=I$.

The non-negativity of ρ^{T_B} is equivalent to the non-negativity of the following 3×3 matrices

$$A_i=\begin{pmatrix}\lambda_i a_i\overline{a_i} & \lambda_{i+2}a_{i+2}\overline{a'_{i+2}} & \lambda_{i+1}a_{i+1}\overline{a''_{i+1}}\\ \lambda_{i+2}a'_{i+2}\overline{a_{i+2}} & \lambda_{i+4}a'_{i+4}\overline{a'_{i+4}} & \lambda_{i+3}a'_{i+3}\overline{a''_{i+3}}\\ \lambda_{i+1}a''_{i+1}\overline{a_{i+1}} & \lambda_{i+3}a''_{i+3}\overline{a'_{i+3}} & \lambda_{i+2}a''_{i+2}\overline{a''_{i+2}}\end{pmatrix},$$
$$i=1,2,\dots,k. \tag{13}$$

Where the indices are $k+1=1$, $k+2=2$, and $k+3=3$. The non-negativity of A_i is equivalent to the following inequalities

$$\lambda_i\lambda_{i+4}|a_i|^2|a'_{i+4}|^2 \geq \lambda_{i+2}^2|a_{i+2}|^2|a'_{i+2}|^2, \tag{14}$$

$$\lambda_i\lambda_{i+2}|a_i|^2|a''_{i+2}|^2 \geq \lambda_{i+1}^2|a_{i+1}|^2|a''_{i+1}|^2, \tag{15}$$

$$\lambda_{i+2}\lambda_{i+4}|a'_{i+4}|^2|a''_{i+2}|^2 \geq \lambda_{i+3}^2|a'_{i+3}|^2|a''_{i+3}|^2, \tag{16}$$

$$\begin{aligned}&\lambda_i\lambda_{i+2}\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2\\&+2\lambda_{i+1}\lambda_{i+2}\lambda_{i+3}Re(a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}})\\&-\lambda_{i+2}^3|a_{i+2}|^2|a'_{i+2}|^2|a''_{i+2}|^2-\lambda_i\lambda_{i+3}^2|a_i|^2|a'_{i+3}|^2|a''_{i+3}|^2\\&-\lambda_{i+1}^2\lambda_{i+4}|a_{i+1}|^2|a''_{i+1}|^2|a'_{i+4}|^2\geq 0.\end{aligned} \tag{17}$$

In a similar way, as shown in Section 2, inequalities (14)–(16) can be written as

$$\begin{aligned}\sqrt{\lambda_i\lambda_{i+4}}a_ia'_{i+4}&=\lambda_{i+2}a_{i+2}a'_{i+2}\,\mathrm{e}^{\mathrm{i}\theta'_{i+2}},\\\sqrt{\lambda_i\lambda_{i+2}}a_ia''_{i+2}&=\lambda_{i+1}a_{i+1}a''_{i+1}\,\mathrm{e}^{\mathrm{i}\theta''_{i+1}},\\\sqrt{\lambda_{i+2}\lambda_{i+4}}a'_{i+4}a''_{i+2}&=\lambda_{i+3}a'_{i+3}a''_{i+3}\,\mathrm{e}^{\mathrm{i}\theta'''_{i+3}},\end{aligned} \tag{18}$$

where all $\theta_i\in[-\pi,\pi], i=1,2,\ldots,k$. In fact, $\theta''_{i+2}=\theta'''_{i+2}$, the proof is given in Appendix B. Let θ_{i+2} denote θ''_{i+2} and θ'''_{i+2},

$$\theta''_{i+2}=\theta'''_{i+2}=\theta_{i+2},\ \ i=1,2,\ldots,k. \tag{19}$$

Next we show that ρ has a pure state decomposition $\rho=\sum_{l=1}^k|\psi_l\rangle\langle\psi_l|$ such that all states $|\psi_l\rangle$, $l=1,2,\ldots,k$, are separable. $|\psi_l\rangle$ can be generally expressed as $|\psi_l\rangle=\sum_{m=1}^k u_{ml}|Y_m\rangle=\sum_{i=1}^3\sum_{j=1}^k a_{ij}^l|ij\rangle$, where u_{ml} are the entries of a $k\times k$ unitary matrix U and $|Y_m\rangle=\sqrt{\lambda_m}|X_m\rangle$. Let $B_l=(a_{ij}^l)_{3\times k}$, and suppose the matrix U has the following form $U=(u_{il})_{k\times k}=(c_{il}\,\mathrm{e}^{\mathrm{i}\theta_i})_{k\times k}$. According to the unitary condition of U, we have

$$\sum_{l=1}^k c_{ml}c_{nl}=\delta_{mn}. \tag{20}$$

Then

$$B_l=\begin{pmatrix}\sqrt{\lambda_1}a_1u_{1l} & \sqrt{\lambda_2}a_2u_{2l} & \sqrt{\lambda_3}a_3u_{3l} & \ldots & \sqrt{\lambda_k}a_ku_{kl}\\ \sqrt{\lambda_3}a'_3u_{3l} & \sqrt{\lambda_4}a'_4u_{4l} & \sqrt{\lambda_5}a'_5u_{5l} & \ldots & \sqrt{\lambda_2}a'_2u_{2l}\\ \sqrt{\lambda_2}a''_2u_{2l} & \sqrt{\lambda_3}a''_3u_{3l} & \sqrt{\lambda_4}a''_4u_{4l} & \ldots & \sqrt{\lambda_1}a''_1u_{1l}\end{pmatrix}. \tag{21}$$

If

$$\begin{aligned}c_{1l}c_{3l}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_3)}&=c_{2l}^2\,\mathrm{e}^{\mathrm{i}\theta_2},\\c_{1l}c_{5l}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_5+\theta'_3)}&=c_{3l}^2\,\mathrm{e}^{\mathrm{i}2\theta_3},\\c_{1l}c_{4l}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_4)}&=c_{2l}c_{3l},\end{aligned} \tag{22}$$

the matrix B_l has rank one. Therefore, ρ has separable pure state decomposition, and ρ is separable if it is PPT.

If ρ is separable then it is necessary for PPT,[17] therefore, we have the following theorem:

Theorem 2 If the state ρ satisfies Eqs. (22), then it is separable if and only if it is PPT.

Remark 1 If $k=3$, the results are the same as those in Ref. [27]. We thus generalized the results of $3\otimes 3$ systems in Ref. [27] to the case of $3\otimes k$ systems.

4. Conclusion and remarks

We have studied the quantum states for which the PPT criterion is both sufficient and necessary for separability. First, we presented a sufficient and necessary condition for separability of quantum states in $2\otimes k$ systems, which is not considered in Ref. [27]. It has been shown that all $2\times k$ NPT (negative under partial transposition) states are distillable.[28] Therefore, if the state ρ does not satisfy $c_{1l}c_{3l}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_3)}=c_{2l}^2\,\mathrm{e}^{\mathrm{i}\theta_2}, c_{1l}c_{4l}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_4)}=c_{2l}c_{3l}$, $\theta_i\in[-\pi,\pi]$, ρ is not only entangled but also distillable.

Then we constructed a class of mixed states in $3\otimes k$ quantum systems and showed that these states are separable if and only if they are PPT. The results of $3\otimes 3$ systems in Ref. [27] are generalized to the case of $3\otimes k$ systems. Separable states contain no entanglement, so it is impossible to convert them into entangled states by LOCC (local operations and classical communication) operations. Therefore, the states we constructed cannot be distilled. Moreover, in a similar way, the sufficient and necessary condition for separability of quantum mixed states in higher dimensions can be obtained. We hope that our results will help further investigations of multipartite quantum systems.

Appendix A

In fact, for the case of $i=1$ suppose that

$$\lambda_2^2|a_2|^2|a'_2|^2<\lambda_1\lambda_3|a_1|^2|a'_3|^2, \tag{A1}$$

by multiplying $\lambda_3|a_3|^2$ on both sides of inequality (A1), and using inequality (5) for $i=2$ we have

$$\lambda_2\lambda_3|a_3|^2|a''_2|^2<\lambda_1\lambda_4|a_1|^2|a''_4|^2, \tag{A2}$$

then by multiplying $\lambda_4|a_4|^2$ on both sides of inequality (A2), and using inequality (5) for $i=3$ we have

$$\lambda_2\lambda_4|a_4|^2|a''_2|^2<\lambda_1\lambda_5|a_1|^2|a''_5|^2. \tag{A3}$$

By repeating the above procedure $k-3$ times, at last, we have

$$\lambda_2\lambda_{k-1}|a'_2|^2|a_{k-1}|^2<\lambda_1\lambda_k|a_1|^2|a'_k|^2, \tag{A4}$$

by multiplying $\lambda_k|a_k|^2$ on both sides of inequality (A4), and using inequality (5) for $i=k-1,k$ successively we have

$$\begin{aligned}&\lambda_2\lambda_{k-1}\lambda_k|a'_2|^2|a_{k-1}|^2|a_k|^2<\lambda_1\lambda_k^2|a_1|^2|a'_k|^2|a_k|^2\\&\leq\lambda_2\lambda_{k-1}\lambda_k|a'_2|^2|a_{k-1}|^2|a_k|^2,\end{aligned} \tag{A5}$$

which contradicts itself. So

$$\lambda_2^2|a_2|^2|a'_2|^2=\lambda_1\lambda_3|a_1|^2|a'_3|^2.$$

Appendix B

Using Eqs. (18), inequality (17) becomes

$$\begin{aligned}&\mathrm{Re}(\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}})\\&\geq \lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2.\end{aligned}\tag{B1}$$

According to Eqs. (18), we can also obtain

$$\begin{aligned}&|\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}}|\\&=\sqrt{\lambda_{i+1}\lambda_{i+3}}|a_{i+1}a''_{i+3}|\sqrt{\lambda_{i+1}\lambda_{i+3}}|\overline{a''_{i+1}a'_{i+3}}||a'_{i+2}\overline{a_{i+2}}|\\&=\lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2.\end{aligned}\tag{B2}$$

Therefore,

$$\begin{aligned}&\mathrm{Re}(\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}})\\&=\lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2.\end{aligned}\tag{B3}$$

By using Eq. (B3) we have

$$\begin{aligned}&\frac{\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}}}{\lambda_{i+2}\lambda_{i+4}a_{i+2}\overline{a''_{i+2}}a'_{i+3}\overline{a_{i+3}}a''_{i+4}\overline{a'_{i+4}}}\\&=\frac{\lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2}{\lambda_{i+1}\lambda_{i+5}|a_{i+1}|^2|a''_{i+3}|^2|a'_{i+5}|^2},\end{aligned}$$

then

$$a_{i+1}a'_{i+2}a''_{i+3}a''_{i+4}=a''_{i+1}a''_{i+2}a_{i+3}a'_{i+4},\quad i=1,2,\ldots,k.$$

By using Eqs. (18), we get

$$\begin{aligned}&\frac{\lambda_{i+2}}{\sqrt{\lambda_{i+1}\lambda_{i+3}}}\frac{\lambda_{i+3}}{\sqrt{\lambda_{i+2}\lambda_{i+4}}}a_{i+3}a''_{i+3}a'_{i+2}a''_{i+2}\,\mathrm{e}^{\mathrm{i}(\theta''_{i+2}+\theta''_{i+3})}\\&=\frac{\lambda_{i+3}}{\sqrt{\lambda_{i+2}\lambda_{i+4}}}\frac{\lambda_{i+2}}{\sqrt{\lambda_{i+1}\lambda_{i+3}}}a_{i+3}a''_{i+3}a'_{i+2}a''_{i+2}\,\mathrm{e}^{\mathrm{i}(\theta'''_{i+3}+\theta'''_{i+2})},\end{aligned}\tag{B4}$$

then we have

$$\theta''_{i+2}-\theta'''_{i+2}=\theta''_{i+3}-\theta'''_{i+3},\quad i=1,2,\ldots,k.$$

On the other hand, we have

$$\begin{aligned}&\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}}\\&=\mathrm{Re}(\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}})\\&=\lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2,\end{aligned}\tag{B5}$$

and

$$\begin{aligned}&\lambda_{i+1}\lambda_{i+3}a_{i+1}\overline{a''_{i+1}}a'_{i+2}\overline{a_{i+2}}a''_{i+3}\overline{a'_{i+3}}\\&=\sqrt{\lambda_{i+1}\lambda_{i+3}}a_{i+1}a''_{i+3}\sqrt{\lambda_{i+1}\lambda_{i+3}}\overline{a''_{i+1}a'_{i+3}}a'_{i+2}\overline{a_{i+2}}\\&=\lambda_i\lambda_{i+4}|a_i|^2|a''_{i+2}|^2|a'_{i+4}|^2\,\mathrm{e}^{\mathrm{i}(\theta''_{i+2}-\theta'''_{i+2})}.\end{aligned}\tag{B6}$$

Therefore, $\theta''_{i+2}=\theta'''_{i+2}$.

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