New operator-ordering identities and associative integration formulas of two-variable Hermite polynomials for constructing non-Gaussian states*

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For directly normalizing the photon non-Gaussian states (e.g., photon added and subtracted squeezed states), we use the method of integration within an ordered product (IWOP) of operators to derive some new bosonic operator-ordering identities. We also derive some new integration transformation formulas about one- and two-variable Hermite polynomials in complex function space. These operator identities and associative integration formulas provide much convenience for constructing non-Gaussian states in quantum engineering.

Keywords: IWOP method, squeezed states, Hermite polynomials

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1. Introduction

In recent years some theoretical and experimental efforts have been made towards engineering and controlling highly nonclassical, non-Gaussian squeezed states of optical fields.[1–6] Performing photon subtraction or addition on a given Gaussian state, especially on a coherent state and squeezed state, is a possible approach to generating non-Gaussian states. For instance, the n-photon added squeezed state $|\psi_1\rangle$ is expressed as[7]

$$\begin{aligned}|\psi_1\rangle &= C_1|\lambda\rangle_n,\\ |\lambda\rangle_n &= a^{\dagger n}S(\lambda)|0\rangle\\ &= \operatorname{sech}^{1/2}\lambda a^{\dagger n}\exp\left(\frac{1}{2}a^{\dagger 2}\tanh\lambda\right)|0\rangle,\end{aligned}\tag{1}$$

where $S(\lambda)=\exp\left[(\lambda/2)\left(a^{\dagger 2}-a^2\right)\right]$ is the single-mode squeezing operator, C_1 is the normalization constant, $a^\dagger$ is the photon creation operator and a is the annihilation operator, obeying $\left[a,a^\dagger\right]=1$, and $|0\rangle$ is the vacuum. The n-photon subtracted squeezed state is expressed as[8]

$$\begin{aligned}|\psi_2\rangle &= C_2a^nS(\lambda)|0\rangle\\ &= C_2\operatorname{sech}\lambda a^n\exp\left(\frac{1}{2}a^{\dagger 2}\tanh\lambda\right)|0\rangle,\end{aligned}\tag{2}$$

where C_2 is the normalization constant. Generally speaking, for constructing optical non-Gaussian states we need two new operator identities for respectively converting $\mathrm{e}^{ga^2}a^ma^{\dagger n}\mathrm{e}^{fa^{\dagger 2}}$ and $\mathrm{e}^{ga^2}a^{\dagger n}a^m\mathrm{e}^{fa^{\dagger 2}}$ into their normal ordering form. It is commonly acknowledged that this is a tough task. Luckily, Fan *et al.* invented the method of integration within an ordered product (IWOP) of operators,[9,10] which enlightened us to derive new bosonic operator ordering identities for normalizing non-Gaussian states. Correspondingly, we shall deduce some new associative integration transformation formulas about one- and two-variable Hermite polynomials in complex function space. In Sections 2 and 3 we shall use the coherent state representation $|z\rangle$, defined as

$$|z\rangle=\mathrm{e}^{-|z|^2/2+za^\dagger}|0\rangle,\tag{3}$$

and the IWOP method to achieve our goal. For instance, since the vacuum projector normal ordering is $|0\rangle\langle 0|=:\mathrm{e}^{-a^\dagger a}:$, where : : denotes the normal ordering, the overcompleteness relation of $|z\rangle$ can be recast into normally ordered Gaussian form by virtue of the IWOP method[11,12]

$$\begin{aligned}&\int\frac{\mathrm{d}^2z}{\pi}|z\rangle\langle z|=\int\frac{\mathrm{d}^2z}{\pi}:\mathrm{e}^{-(z^*-a^\dagger)(z-a)}:=1,\\ &\mathrm{d}^2z\equiv\mathrm{d}x\mathrm{d}y,\quad z=x+\mathrm{i}y,\end{aligned}\tag{4}$$

and density operator $\mathfrak{P}$-representation can be put into

$$\rho=\int\frac{\mathrm{d}^2z}{\pi}\mathfrak{P}(z)|z\rangle\langle z|=\int\frac{\mathrm{d}^2z}{\pi}\mathfrak{P}(z):\mathrm{e}^{-(z^*-a^\dagger)(z-a)}:,\tag{5}$$

Once $\mathfrak{P}(z)$ of ρ is known, we can derive ρ normal ordering form by performing the integration within normal ordering in Eq. (5).

2. Concise operator identity for normally ordering $\mathrm{e}^{ga}a^ma^{\dagger n}\mathrm{e}^{fa^\dagger}$

In this section we shall derive a concise operator identity for normally ordering $\mathrm{e}^{ga}a^ma^{\dagger n}\mathrm{e}^{fa^\dagger}$. From the integration

$$\int\frac{\mathrm{d}^2z}{\pi}\frac{z^mz^{*n}}{\sqrt{n!m!}}\mathrm{e}^{-|z|^2}=\delta_{n,m},\tag{6}$$

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it follows that z^m is the orthonormal basis of a function space, then by considering the independent displacement $(z-\eta/\zeta)\to z$, $(z^*-\xi/\zeta)\to z^*$

$$\begin{aligned} I_1 &\equiv \int \frac{\mathrm{d}^2 z}{\pi} z^m z^{*n} \exp\left(-\zeta|z|^2+\xi z+\eta z^*\right) \\ &= \int \frac{\mathrm{d}^2 z}{\pi} z^m z^{*n} \exp\{-\zeta[(z-\eta/\zeta)(z^*-\xi/\zeta)]+\xi\eta/\zeta\} \\ &= \mathrm{e}^{\xi\eta/\zeta}\int \frac{\mathrm{d}^2 z}{\pi}(z+\eta/\zeta)^m (z^*+\xi/\zeta)^n \mathrm{e}^{-\zeta|z|^2}, \end{aligned} \tag{7}$$

and making the dilation $z\to z'/\sqrt{\zeta}$, $z^*\to z'^*/\sqrt{\zeta}$ in Eq. (7) we obtain

$$\begin{aligned} I_1 &= \mathrm{e}^{\xi\eta/\zeta}\int \frac{\mathrm{d}^2 z'}{\pi\zeta}\left(\frac{z'}{\sqrt{\zeta}}+\eta/\zeta\right)^m \left(\frac{z'^*}{\sqrt{\zeta}}+\xi/\zeta\right)^n \mathrm{e}^{-|z'|^2} \\ &= \mathrm{e}^{\xi\eta/\zeta}\frac{1}{\sqrt{\zeta}^{m+n+2}}\int \frac{\mathrm{d}^2 z}{\pi}\left(z+\eta/\sqrt{\zeta}\right)^m \left(z^*+\xi/\sqrt{\zeta}\right)^n \\ &\quad\times \mathrm{e}^{-|z|^2} \\ &= \frac{1}{\sqrt{\zeta}^{m+n+2}}\mathrm{e}^{\xi\eta/\zeta}\sum_k\sum_l \binom{m}{l}\binom{n}{k}\left(\eta/\sqrt{\zeta}\right)^{m-k} \\ &\quad\times \left(\xi/\sqrt{\zeta}\right)^{n-l}\int \frac{\mathrm{d}^2 z}{\pi} z^{*k} z^l \mathrm{e}^{-|z|^2} \\ &= \mathrm{e}^{\xi\eta/\zeta}\frac{1}{\sqrt{\zeta}^{m+n+2}}\sum_l \binom{m}{l}\binom{n}{l}\left(\eta/\sqrt{\zeta}\right)^{m-l} \\ &\quad\times \left(\xi/\sqrt{\zeta}\right)^{n-l} l! \\ &= (-\mathrm{i})^{m+n}\zeta^{-(n+m)/2-1}\mathrm{e}^{\xi\eta/\zeta}\mathrm{H}_{m,n}\left(\frac{\mathrm{i}\eta}{\sqrt{\zeta}},\frac{\mathrm{i}\xi}{\sqrt{\zeta}}\right), \end{aligned} \tag{8}$$

where in the last step we have introduced the definition of the two-variable Hermite polynomials[13,14]

$$\mathrm{H}_{m,n}(x,y) = \sum_{l=0}^{\min[m,n]} \frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}x^{m-l}y^{n-l}. \tag{9}$$

Equation (8) is a new integration transformation formula about the two-variable Hermite polynomials. According to Eqs. (8) and (4), we can directly use the IWOP method to obtain the new operator identity as follows:

$$\begin{aligned} &\mathrm{e}^{ga}a^m a^{\dagger n}\mathrm{e}^{fa^\dagger} \\ &= \mathrm{e}^{ga}a^m\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z|a^{\dagger n}\mathrm{e}^{fa^\dagger} \\ &= \int \frac{\mathrm{d}^2 z}{\pi}{:}\ \mathrm{e}^{-(z^*-a^\dagger)(z-a)+gz+fz^*}{:}\ z^m z^{*n} \\ &= (-\mathrm{i})^{m+n}{:}\ \mathrm{e}^{fg+ga+fa^\dagger}\mathrm{H}_{m,n}\left[\mathrm{i}(a+f),\mathrm{i}\left(a^\dagger+g\right)\right]{:} \\ &= (-\mathrm{i})^{m+n}{:}\ \mathrm{e}^{fg+ga+fa^\dagger}\mathrm{H}_{n,m}\left[\mathrm{i}\left(a^\dagger+g\right),\mathrm{i}(a+f)\right]{:}. \end{aligned} \tag{10}$$

This operator identity is useful for normalizing the photon-added coherent states $C_0 a^{\dagger n}|z\rangle$, and here we assume

$$\begin{aligned} |C_0|^2\langle z|a^n a^{\dagger n}|z\rangle &= \mathrm{e}^{-|z|^2}|C_0|^2\langle 0|\mathrm{e}^{z^*a}a^n a^{\dagger n}\mathrm{e}^{za^\dagger}|0\rangle \\ &= 1, \end{aligned} \tag{11}$$

where according to Eq. (10)

$$\begin{aligned} &\mathrm{e}^{z^*a}a^n a^{\dagger n}\mathrm{e}^{za^\dagger} \\ &= (-1)^n{:}\ \mathrm{e}^{|z|^2+z^*a+za^\dagger}\mathrm{H}_{n,n}\left[\mathrm{i}\left(a^\dagger+z^*\right),\mathrm{i}(a+z)\right]{:}. \end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (11) yields

$$|C_0|^2\langle z|a^n a^{\dagger n}|z\rangle = |C_0|^2(-1)^n\mathrm{H}_{n,n}(\mathrm{i}z^*,\mathrm{i}z) = 1. \tag{13}$$

By comparing the definition of Laguerre polynomials

$$\mathrm{L}_n(x) = \sum_{l=0}\binom{n}{l}\frac{(-1)^l}{l!}x^l$$

with Eq. (9) we see

$$\mathrm{L}_n(xy) = \frac{(-1)^n}{n!}\mathrm{H}_{n,n}(x,y), \tag{14}$$

so $C_0 = 1/\sqrt{n!\mathrm{L}_n(-|z|^2)}$.

In Eq. (8), when $\zeta=1$ and $\xi=\eta^*$, we have the following integration transformation for converting $z^m z^{*n}$ into $\mathrm{H}_{m,n}(\mathrm{i}\eta,\mathrm{i}\eta^*)$:

$$\begin{aligned} &\int \frac{\mathrm{d}^2 z}{\pi} z^m z^{*n}\exp\left(-|z|^2+\eta^* z+\eta z^*\right) \\ &= (-\mathrm{i})^{m+n}\mathrm{e}^{|\eta|^2}\mathrm{H}_{m,n}(\mathrm{i}\eta,\mathrm{i}\eta^*). \end{aligned} \tag{15}$$

In Eq. (10), when $f=0$ and $g=0$, we see the concise operator indentity as follows:

$$a^m a^{\dagger n} = (-\mathrm{i})^{m+n}{:}\ \mathrm{H}_{n,m}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right){:}, \tag{16}$$

it follows

$$\begin{aligned} &\sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}a^n a^{\dagger m} \\ &= \sum_{n,m=0}^{\infty}\frac{(-\mathrm{i}\tau)^n(-\mathrm{i}t)^m}{n!m!}{:}\ \mathrm{H}_{n,m}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right){:}. \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} &\sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}a^n a^{\dagger m} \\ &= \mathrm{e}^{\tau a}\mathrm{e}^{ta^\dagger} =: \exp\left(\tau a+ta^\dagger+\tau t\right): \\ &= :\exp\left[-(-\mathrm{i}\tau)(-\mathrm{i}t)+(-\mathrm{i}t)\left(\mathrm{i}a^\dagger\right)+(-\mathrm{i}\tau)(\mathrm{i}a)\right]:, \end{aligned} \tag{18}$$

comparing Eq. (17) with Eq. (18), we see that the generating function of two-variable Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ should be

$$\sum_{n,m=0}^{\infty}\frac{t^m\tau^n}{m!n!}\mathrm{H}_{m,n}(x,y) = \exp(-t\tau+tx+\tau y), \tag{19}$$

which is the generating function of $\mathrm{H}_{m,n}(x,y)$, or

$$\mathrm{H}_{m,n}(x,y) = \frac{\partial^{m+n}}{\partial t^m\partial\tau^n}\exp(-t\tau+tx+\tau y)|_{t,\tau=0}. \tag{20}$$

It then follows that

$$\begin{aligned}\sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}a^{\dagger n}a^m &= \mathrm{e}^{\tau a^\dagger}\mathrm{e}^{ta}\\ &= \mathrm{e}^{-t\tau}\mathrm{e}^{ta}\mathrm{e}^{\tau a^\dagger}\\ &= {:}\exp\left(-t\tau+ta+\tau a^\dagger\right){:}\\ &= \sum_{n,m=0}^{\infty}\frac{\tau^n t^m}{n!m!}{:}\mathrm{H}_{m,n}\left(a,a^\dagger\right){:}, \end{aligned}\tag{21}$$

where $:\ :$ denotes antinormal ordering, so the concise antinormal ordering form of $a^{\dagger n}a^m$ is

$$a^{\dagger n}a^m = {:}\mathrm{H}_{m,n}\left(a,a^\dagger\right){:} = {:}\mathrm{H}_{n,m}\left(a^\dagger,a\right){:}. \tag{22}$$

3. Concise operator identity for normally ordering $\mathrm{e}^{fa^2}a^m a^{\dagger n}\mathrm{e}^{ga^{\dagger 2}}$

For constructing non-Gaussian states (say, photon-added squeezed states) with the aid of the IWOP method we will need to derive an associative integration formula

$$\begin{aligned}I_2 &\equiv \int\frac{\mathrm{d}^2z}{\pi}z^n z^{*m}\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)\\ &= A^{-1/2}\left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^n\left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m\sum_{k=0}^{\min[n,m]}\left(\frac{\zeta}{\sqrt{fg}}\right)^k\\ &\quad\times\frac{n!m!}{k!(n-k)!(m-k)!}\\ &\quad\times\mathrm{H}_{n-k}\left(\frac{2g\xi-\zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)\mathrm{H}_{m-k}\left(\frac{2f\eta-\zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)\mathrm{e}^W, \end{aligned}\tag{23}$$

where

$$A=\zeta^2-4fg,\ \ W=\frac{-\zeta\xi\eta+g\xi^2+f\eta^2}{A}, \tag{24}$$

and H_{n-k} is the usual single-variable Hermite polynomial.

Proof By introducing auxiliary parameters t and τ as the following expression

$$\begin{aligned}I_2 &= \frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\int\frac{\mathrm{d}^2z}{\pi}\\ &\quad\times\exp\left[\zeta|z|^2+(\xi+t)z+(\eta+\tau)z^*+fz^2+gz^{*2}\right]|_{t,\tau=0}\end{aligned}\tag{25}$$

and using the known integration formula[15]

$$\begin{aligned}&\int\frac{\mathrm{d}^2z}{\pi}\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)\\ &=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\left[\frac{-\zeta\xi\eta+f\eta^2+g\xi^2}{\zeta^2-4fg}\right],\end{aligned}\tag{26}$$

where ζ, f, and g are assumed to be the values such that the above integration is convergent whose convergent condition is $\mathrm{Re}(\zeta\pm f\pm g)<0$ and

$$\mathrm{Re}\left(\frac{\zeta^2-4fg}{\zeta\pm f\pm g}\right)<0,$$

we perform the integration in Eq. (25)

$$I_2=\frac{1}{\sqrt{A}}\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\exp\left[\frac{-\zeta t\tau+(2f\eta-\zeta\xi)\tau+(2g\xi-\zeta\eta)t+f\tau^2+gt^2}{A}\right]\Bigg|_{t,\tau=0}\mathrm{e}^W. \tag{27}$$

Expanding the term $\exp[-(\zeta/A)t\tau]$ into a power series we have

$$\begin{aligned}I_2 &= \frac{1}{\sqrt{A}}\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-\zeta t\tau}{A}\right)^k\exp\left[\frac{(2f\eta-\zeta\xi)\tau+(2g\xi-\zeta\eta)t+f\tau^2+gt^2}{A}\right]\Bigg|_{t,\tau=0}\mathrm{e}^W\\ &= \frac{1}{\sqrt{A}}\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-\zeta}{A}\right)^k\frac{\partial^{2k}}{\partial\left(\frac{2g\xi-\zeta\eta}{A}\right)^k\partial\left(\frac{2f\eta-\zeta\xi}{A}\right)^k}\exp\left[\frac{(2f\eta-\zeta\xi)\tau+(2g\xi-\zeta\eta)t+f\tau^2+gt^2}{A}\right]\Bigg|_{t,\tau=0}\mathrm{e}^W. \end{aligned}\tag{28}$$

Then we use the generating function formula of single-variable Hermite polynomials $\mathrm{H}_n(x)$

$$\mathrm{H}_n(x)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\exp\left(2xt-t^2\right)|_{t=0} \tag{29}$$

to obtain

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\exp\left[\frac{(2g\xi-\zeta\eta)t+gt^2}{A}\right]|_{t=0}\\ &=\left(\frac{\mathrm{i}\sqrt{g}}{\sqrt{A}}\right)^n\frac{\partial^n}{\partial\left(\mathrm{i}\sqrt{g}t/\sqrt{A}\right)^n}\\ &\quad\times\exp\left[2\frac{2g\xi-\zeta\eta}{2\mathrm{i}\sqrt{gA}}\frac{\mathrm{i}\sqrt{g}t}{\sqrt{A}}-\left(\frac{\mathrm{i}\sqrt{g}t}{\sqrt{A}}\right)^2\right]\Bigg|_{t=0}\\ &=\left(\frac{\mathrm{i}\sqrt{g}}{\sqrt{A}}\right)^n\mathrm{H}_n\left(\frac{2g\xi-\zeta\eta}{2i\sqrt{gA}}\right)\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\frac{\partial^m}{\partial\tau^m}\exp\left[\frac{(2f\eta-\zeta\xi)\tau+f\tau^2}{A}\right]\Bigg|_{\tau=0}\\ &=\left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m\mathrm{H}_m\left(\frac{2f\eta-\zeta\xi}{2\mathrm{i}\sqrt{fA}}\right).\end{aligned}\tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (28) we obtain

$$\begin{aligned} I_2 = & \frac{1}{\sqrt{A}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\zeta}{A}\right)^k \\ & \times \frac{\partial^{2k}}{\partial \left(\dfrac{2g\xi - \zeta\eta}{A}\right)^k \partial \left(\dfrac{2f\eta - \zeta\xi}{A}\right)^k} \\ & \times \left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m \left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^n \\ & \times \mathrm{H}_m\left(\frac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right) \mathrm{H}_n\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right). \end{aligned} \tag{32}$$

Further, using the recurrence relation of single-variable Hermite polynomials

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}\mathrm{H}_n(x) = \frac{2^l n!}{(n-l)!}\mathrm{H}_{n-l}(x), \tag{33}$$

we have

$$\begin{aligned} & \frac{\partial^k}{\partial \left(\dfrac{2f\eta - \zeta\xi}{A}\right)^k} \mathrm{H}_m\left(\frac{2f\eta - \zeta\xi}{2i\sqrt{fA}}\right) \\ = & \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{f}}\right)^k \frac{\partial^k}{\partial \left(\dfrac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)^k} \mathrm{H}_m\left(\frac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right) \\ = & \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{f}}\right)^k \frac{2^k m!}{(m-k)!} \mathrm{H}_{m-k}\left(\frac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} & \frac{\partial^k}{\partial \left(\dfrac{2g\xi - \zeta\eta}{A}\right)^k} \mathrm{H}_n\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right) \\ = & \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{g}}\right)^k \frac{\partial^k}{\partial \left(\dfrac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)^k} \mathrm{H}_n\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right) \\ = & \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{g}}\right)^k \frac{2^k n!}{(n-k)!} \mathrm{H}_{n-k}\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right). \end{aligned} \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (32) leads to

$$\begin{aligned} I_2 = & \frac{\left(\mathrm{i}\sqrt{g/A}\right)^n \left(\mathrm{i}\sqrt{f/A}\right)^m}{\sqrt{A}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\zeta}{A}\right)^k \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{f}}\right)^k \\ & \times \left(\frac{\sqrt{A}}{2\mathrm{i}\sqrt{g}}\right)^k \frac{\partial^{2k}}{\partial \left(\dfrac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)^k \partial \left(\dfrac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)^k} \\ & \times \mathrm{H}_n\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right) \mathrm{H}_m\left(\frac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right) \mathrm{e}^W \\ = & A^{-1/2} \left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^n \left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m \sum_{k=0}^{\min[n,m]} \left(\frac{\zeta}{\sqrt{fg}}\right)^k \\ & \times \frac{n!m!}{k!(n-k)!(m-k)!} \mathrm{H}_{n-k}\left(\frac{2g\xi - \zeta\eta}{2\mathrm{i}\sqrt{gA}}\right) \\ & \times \mathrm{H}_{m-k}\left(\frac{2f\eta - \zeta\xi}{2\mathrm{i}\sqrt{fA}}\right) \mathrm{e}^W, \end{aligned} \tag{36}$$

thus equation (23) is proved.

Using Eq. (30) and the IWOP method, we have the new operator identity

$$\begin{aligned} & \mathrm{e}^{fa^2} a^m a^{\dagger n} \mathrm{e}^{ga^{\dagger 2}} \\ = & \mathrm{e}^{fa^2} a^m \int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| a^{\dagger n} \mathrm{e}^{ga^{\dagger 2}} \\ = & \int \frac{\mathrm{d}^2 z}{\pi} z^m z^{*n} : \mathrm{e}^{-(z^* - a^\dagger)(z-a) + fz^2 + gz^{*2}} : \\ = & A'^{-1/2} \left(\mathrm{i}\sqrt{\frac{g}{A'}}\right)^n \left(\mathrm{i}\sqrt{\frac{f}{A'}}\right)^m \sum_{k=0}^{\min[n,m]} \left(\frac{-1}{\sqrt{fg}}\right)^k \\ & \times \frac{n!m!}{k!(n-k)!(m-k)!} \\ & \times : \mathrm{H}_{n-k}\left(\frac{2a^\dagger g + a}{2\mathrm{i}\sqrt{gA'}}\right) \mathrm{H}_{m-k}\left(\frac{2fa + a^\dagger}{2\mathrm{i}\sqrt{fA'}}\right) \\ & \times \exp\left(\frac{4fga^\dagger a + ga^{\dagger 2} + fa^2}{A'}\right) : , \end{aligned} \tag{37}$$

where $A' = 1 - 4fg$. This is useful for normalizing the following photon-added squeezed states defined in Eq. (1):

$$\begin{aligned} 1 = & \langle \psi_1 | \psi_1 \rangle \\ = & |C_1|^2 \, {}_n\langle \lambda | \lambda \rangle_n \\ = & |C_1|^2 \operatorname{sech}\lambda \, \langle 0 | \exp\left(\frac{a^2}{2}\tanh\lambda\right) a^n a^{\dagger n} \\ & \times \exp\left(\frac{1}{2}a^{\dagger 2}\tanh\lambda\right) |0\rangle . \end{aligned} \tag{38}$$

In Eq. (37), when we take $f = g = \tanh\lambda/2$, $A' = \operatorname{sech}^2\lambda$, $m = n$, it follows that

$$\begin{aligned} 1 = & |C_1|^2 \operatorname{sech}\lambda \, \langle 0 | \exp\left(\frac{a^2}{2}\tanh\lambda\right) a^n a^{\dagger n} \\ & \times \exp\left(\frac{a^{\dagger 2}}{2}\tanh\lambda\right) |0\rangle \\ = & |C_1|^2 \left(-\frac{1}{4}\sinh 2\lambda\right)^n \sum_{k=0}^{n} \left(-\frac{2}{\tanh\lambda}\right)^k \\ & \times \frac{n!m!}{k!(n-k)!(m-k)!} \left[\mathrm{H}_{n-k}(0)\right]^2, \end{aligned} \tag{39}$$

where

$$\mathrm{H}_{2n}(0) = (-1)^n 2^n (2n-1)!!, \quad \mathrm{H}_{2n+1}(0) = 0. \tag{40}$$

On the other hand, in Ref. [16] we have shown that $|\lambda\rangle_n$ is normalized as

$${}_n\langle \lambda | \lambda \rangle_n = n! (\cosh\lambda)^n \mathrm{P}_n(\cosh\lambda) = |C_1|^{-2}, \tag{41}$$

where $\mathrm{P}_n(\cosh\lambda)$ is a Legendre polynomial. Comparing Eqs. (38) and (41) with Eq. (39) we obtain

$$\sum_{k=0}^{n}\left(-\frac{2}{\tanh\lambda}\right)^k\frac{(n!)^2}{k!\left[(n-k)!\right]^2}\left[\mathrm{H}_{n-k}(0)\right]^2 = \frac{n!(-2)^n}{\sinh^n\lambda}\mathrm{P}_n(\cosh\lambda), \tag{42}$$

which is a new relation.

4. Concise operator identity for normally ordering $\mathrm{e}^{ga^2}a^{\dagger n}a^m\mathrm{e}^{fa^{\dagger 2}}$

For normalizing photon subtracted-squeezed states, we need another operator identity for transforming $\mathrm{e}^{ga^2}a^{\dagger n}a^m\mathrm{e}^{fa^{\dagger 2}}$ to its normal ordering. For the possibility of using the IWOP method to solve this problem, we need to derive the following new integration formula

$$\begin{aligned} I_3 &\equiv \int\frac{\mathrm{d}^2z}{\pi}\mathrm{H}_{n,m}(z,z^*)\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)\\ &= \frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\int\frac{\mathrm{d}^2z}{\pi}\exp\left[\zeta|z|^2+(\xi+t)z+(\eta+\tau)z^*+fz^2+gz^{*2}-t\tau\right]|_{t,\tau=0}\\ &= \frac{1}{\sqrt{A}}\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\exp\left\{\frac{1}{A}\left[-\zeta(\xi+t)(\eta+\tau)+f(\eta+\tau)^2+g(\xi+t)^2\right]-t\tau\right\}|_{t,\tau=0}\\ &= \frac{1}{\sqrt{A}}\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\exp\left(-\frac{\zeta+A}{A}t\tau+t\frac{2g\xi-\zeta\eta}{A}+\frac{gt^2}{A}+\tau\frac{2f\eta-\zeta\xi}{A}+\frac{f\tau^2}{A}\right)|_{t,\tau=0}\mathrm{e}^W, \end{aligned} \tag{43}$$

where the generating function formula Eq. (20) of $\mathrm{H}_{n,m}$ has been used, $A=\zeta^2-4fg$ and $W=(-\zeta\xi\eta+g\xi^2+f\eta^2)/A$. Then we use the generating function formula Eq. (29) of H_n

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{A}}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\partial^{2l}}{\partial\left(\frac{2g\xi-\zeta\eta}{A}\right)^l\partial\left(\frac{2f\eta-\zeta\xi}{A}\right)^l}\left(-\frac{\zeta+A}{A}\right)^l\frac{\partial^{n+m}}{\partial t^n\partial\tau^m}\exp\left(t\frac{2g\xi-\zeta\eta}{A}+t^2\frac{g}{A}+\tau\frac{2f\eta-\zeta\xi}{A}+\tau^2\frac{f}{A}\right)\Big|_{t,\tau=0}\mathrm{e}^W\\ &= \frac{1}{\sqrt{A}}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\partial^{2l}}{\partial\left(\frac{2g\xi-\zeta\eta}{A}\right)^l\partial\left(\frac{2f\eta-\zeta\xi}{A}\right)^l}\left(-\frac{\zeta+A}{A}\right)^l\left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^n\mathrm{H}_n\left(\frac{2g\xi-\zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)\left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m\mathrm{H}_m\left(\frac{2f\eta-\zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)\mathrm{e}^W. \end{aligned} \tag{44}$$

Again using Eq. (33), we obtain the new associative integration formula

$$\begin{aligned} &\int\frac{\mathrm{d}^2z}{\pi}\mathrm{H}_{n,m}(z,z^*)\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)\\ &= \frac{1}{\sqrt{A}}\sum_{l=0}^{\min[n,m]}\left(-\frac{\zeta+A}{A}\right)^l\left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^{n-l}\left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^{m-l}\\ &\quad\times\frac{n!m!}{l!(n-l)!(m-l)!}\mathrm{H}_{n-l}\left(\frac{2g\xi-\zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)\\ &\quad\times\mathrm{H}_{m-l}\left(\frac{2f\eta-\zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)\mathrm{e}^W. \end{aligned} \tag{45}$$

Now we use Eqs. (5), (22), and (45) to derive the following operator identity:

$$\begin{aligned} &\mathrm{e}^{fa^2}a^{\dagger m}a^n\mathrm{e}^{ga^{\dagger 2}}\\ &= \mathrm{e}^{ga^2}{:}\mathrm{H}_{m,n}(a^\dagger,a){:}\int\frac{\mathrm{d}^2z}{\pi}|z\rangle\langle z|\mathrm{e}^{ga^{\dagger 2}}\\ &= \int\frac{\mathrm{d}^2z}{\pi}\mathrm{H}_{n,m}(z,z^*){:}\,\mathrm{e}^{-(z^*-a^\dagger)(z-a)+fz^2+gz^{*2}}{:}\\ &= \frac{1}{\sqrt{A'}}\sum_{l=0}^{\min[n,m]}\left[\frac{1}{A'}-1\right]^l\left(\mathrm{i}\sqrt{\frac{g}{A'}}\right)^{n-l}\left(\mathrm{i}\sqrt{\frac{f}{A'}}\right)^{m-l}\\ &\quad\times\frac{n!m!}{l!(n-l)!(m-l)!}{:}\,\mathrm{H}_{n-l}\left(\frac{2ga^\dagger+a}{2\mathrm{i}\sqrt{gA'}}\right)\\ &\quad\times\mathrm{H}_{m-l}\left(\frac{2fa+a^\dagger}{2\mathrm{i}\sqrt{fA'}}\right)\exp\left(\frac{4fga^\dagger a+ga^{\dagger 2}+fa^2}{A'}\right){:}\,, \end{aligned} \tag{46}$$

which, in the case of $f=g=\tanh\lambda/2$, $A'=\mathrm{sech}^2\lambda$, $m=n$, can be used for normalizing the photon subtracted-squeezed state $|\psi_2\rangle$ in Eq. (2),

$$\begin{aligned} 1 &= \langle\psi_2|\psi_2\rangle = |C_2|^2\,\mathrm{sech}\,\lambda\exp\left(\frac{1}{2}a^2\tanh\lambda\right)a^{\dagger n}a^n\\ &\quad\times\exp\left(\frac{1}{2}a^{\dagger 2}\tanh\lambda\right)|0\rangle\\ &= |C_2|^2\sum_{k=0}^{n}\left(-\frac{\sinh^2\lambda}{4}\right)^{n-k}\\ &\quad\times\frac{n!m!}{k!(n-k)!(m-k)!}\left[\mathrm{H}_{n-k}(0)\right]^2. \end{aligned} \tag{47}$$

On the other hand, in Ref. [17], we have shown

$$|C_2|^{-2} = n!(-\mathrm{i}\sinh\lambda)^n\mathrm{P}_n(\mathrm{i}\sinh\lambda), \tag{48}$$

thus we see another new relation

$$\sum_{k=0}^{n}\left(-\frac{\sinh^2\lambda}{4}\right)^{n-k}\frac{n!m!}{k!\,(n-k)!\,(m-k)!}\left[\mathrm{H}_{n-k}\,(0)\right]^2 = n!\,(-\mathrm{i}\sinh\lambda)^n\,\mathrm{P}_n\,(\mathrm{i}\sinh\lambda)\,. \tag{49}$$

5. Summary

In summary, for constructing optical non-Gaussian states[14] in optical engineering, we need to use the IWOP method to derive some new operator identities. We obtained some new associative integration transformation formulas about one- and two-variable Hermite polynomials in complex function space. These operator identities can also be used to calculate a non-Gaussian state Wigner function, P-representation, etc.

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