A feedback control method for the stabilization of a nonlinear diffusion system on a graph*

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In this paper, we consider the internal stabilization problems of FitzHugh–Nagumo (FHN) systems on the locally finite connected weighted graphs, which describe the process of signal transmission across axons in neurobiology. We will establish the proper condition on the weighted Dirichlet–Laplace operator on a graph such that the nonlinear FHN system can be stabilized exponentially and globally only using internal actuation over a sub-domain with a linear feedback form.

Keywords: FitzHugh–Nagumo system, graph, stabilization, internal controller

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1. Introduction

Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological, social, and information systems. For example, information and mass flows (such as gossip, infectious diseases, computer viruses, etc.) propagate from one individual to another through mutual interactions. All individual participants involved in the propagation of information and mass form a complex social-network (such as in Refs. [1]–[3], just to name a few). In addition, in the system control and control community, the multi-agent system is now a quite popular research topic to study a group of smart agents which have protocols to its neighbors in the sense of connectivity.[4,5] Collective behaviors, such as consensus and flocking, can be explained using the theoretical results in the control of multi-agent systems. Therefore, it is quite interesting to study spatial-temporal evolutionary dynamics over graphs.

In this paper, we will design a feedback controller to stabilize a nonlinear diffusion system on the locally finite connected weighted graph. Let $G=(V,E,\nu)$ be a connected undirected weighted graph, where V is the set of vertices or nodes, E is the set of undirected edges, and ν is the weight function. The graph G is called locally finite if each vertex has a finite number of edges. In this paper, we assume that G is a locally finite connected weighted graph.

Let Ω be a given connected subset of V with a finite number of vertices and ω be a given subset of Ω.

In this paper, we consider the following FitzHugh–Nagumo (FHN) system:

$$\begin{cases} u_t - d\Delta_\Omega u - cu(1-u)(u-a) + \gamma v = \chi_\omega z, \\ (x,t) \in \Omega \times \mathbb{R}^+, \\ v_t - \sigma u + \beta v = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are unknown functions, $z = z(x,t)$ is the control input, positive constants a, c, d, γ, σ, and β are given, χ_ω is the characteristic function of ω, Δ_Ω is the weighted Dirichlet Laplace operator on graph, and

$$\partial\Omega = \{\, y \in V \setminus \Omega \,|\, \{x,y\} \in E \text{ and } x \in \Omega \,\}.$$

The FHN system arises as a simplification of the Hodgkin–Huxley model describing the signal transmission across axons in neurobiology (see, for example, Refs. [6] and [7] and the references therein). Recently, various properties of FHN systems, such as existences of the attractors, traveling solutions, and chaotic properties, have been studied by many authors.[8–14] Since experiments have been performed on nerve axons of vertebrates and it has been shown that, electrically, nerve fibers behave as spatially discrete structures in vertebrates (see Ref. [15]), it is not only appropriate but correct to model signal transmission across axons on graphs.

In this paper, we will establish conditions on the feedback control lying on a subset for the exponential decay of the diffusion on the whole graph. Note that in general, some vertices in graphs are unknown. Also note that due to the limit of people's ability, it is often very difficulty to control all the vertices. Thus, it is very important to design a control law that is only acting on a subset, which can be chosen in a convenient way.

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The stabilization problems of diffusion systems, governed by the linear or nonlinear partial differential equations defined on the continuous spatial domains, have been studied by many researchers in the past. In Ref. [16], the author has shown that the steady-state solutions to Navier–Stokes equations with internal controllers are locally exponentially stabilizable by linear feedback controllers provided by an LQ control problem associated with the linearized equation. The work of Ref. [17] is concerned with the internal stabilization of the steady-state solutions to semilinear parabolic systems via finite-dimensional feedback controllers. In Ref. [18], the authors studied the periodic feedback stabilization for a class of linear T-periodic evolution equations. Several equivalent conditions on the linear periodic feedback stabilization are obtained. For the stabilization problems with boundary feedback controllers, we also mention the works of Refs. [19]–[22].

Control problems for linear systems on graphs have been studied by many researchers.[23–25] However, for nonlinear diffusion systems on graphs, few results about the stabilization problems have been obtained. In this paper, we will derive conditions on the weighted Dirichlet–Laplace operators on graphs such that the nonlinear FHN system (1) is globally exponentially stabilizable by a linear feedback controller. Moreover, the internal controller only needs to act on a subset of the domain that the FHN system is defined on.

2. Definitions and preliminaries

We first extend the weight function ν from E to $V\times V$. More specifically, for each $(x,y)\in V\times V$,

$$\nu_{xy}=\begin{cases}\nu_{xy}, & \text{if } \{x,y\}\in E,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, for each pair $(x,y)\in V\times V$, $\nu_{xy}=\nu_{yx}$. Furthermore, based on the weight function ν on the edges, we can also define the weight $\mu: V\to\mathbb{R}$ on each vertex as follows:

$$\mu(x)=\sum_{y\in V}\nu_{xy},\quad x\in V.$$

For any subset $U\subseteq V$, define

$$E_U=\{\,\{x,y\}\in E\,|\,x,y\in U\,\}.$$

We can obtain an induced subgraph, denoted by $G_U=(U,E_U)$, from the subset U. Furthermore, the vertex boundary of the subgraph G_U can be defined as follows:

$$\partial U=\{\,y\in V\setminus U\,|\,\{x,y\}\in E \text{ and } x\in U\,\}.$$

It is clear that if $U\subsetneq V$, then $\partial U\neq\emptyset$ since G is connected.

For any subset $U\subseteq V$, let $\bar{U}=U\bigcup\partial U$ and let $\mathcal{F}_U$ denote the set of functions from $\bar{U}$ to $\mathbb{R}$ with homogeneous Dirichlet boundary conditions, namely,

$$\mathcal{F}_U=\{\,f:\bar{U}\longrightarrow\mathbb{R}\,|\,f(x)=0 \text{ for all } x\in\partial U\,\}.$$

It is easy to see that $\mathcal{F}_U$ is a linear space of dimension N, where N is the number of vertices in U.

We now define the weighted Dirichlet–Laplace operator Δ_U on $\mathcal{F}_U$ (see Refs. [26] and [27]). For each $f\in\mathcal{F}_U$, let

$$\Delta_U f(x)=\frac{1}{\mu(x)}\sum_{y\in U}f(y)\nu_{xy}-f(x),\quad x\in U. \tag{2}$$

Obviously, the operator Δ_U is linear.

For any $U\subseteq V$, define the following inner product for $\mathcal{F}_U$:

$$(f,g)_U=\sum_{x\in U}f(x)g(x)\mu(x),\quad (f,g)\in\mathcal{F}_U\times\mathcal{F}_U.$$

This inner product can be viewed as the integral of fg on U against measure μ. The norm of $\mathcal{F}_U$ can be defined by

$$\|f\|_U=\sqrt{(f,f)_U},\quad f\in\mathcal{F}_U.$$

Lemma 1 For any $U\subseteq V$, the operator $-\Delta_U$ defined by Eq. (2) satisfies

$$(-\Delta_U f,g)_U=\sum_{x\in U}f(x)g(x)\mu(x)-\sum_{x,y\in U}f(x)g(y)\nu_{xy},\quad (f,g)\in\mathcal{F}_U\times\mathcal{F}_U.$$

Proof According to the definition (2), we have

$$\begin{aligned}(-\Delta_U f,g)_U&=\sum_{x\in U}\left[\left(f(x)-\frac{1}{\mu(x)}\sum_{y\in U}f(y)\nu_{xy}\right)g(x)\right]\mu(x),\\&=\sum_{x\in U}f(x)g(x)\mu(x)-\sum_{x,y\in U}f(y)g(x)\nu_{xy}\\&=\sum_{x\in U}f(x)g(x)\mu(x)-\sum_{x,y\in U}f(y)g(x)\nu_{yx}\\&=\sum_{x\in U}f(x)g(x)\mu(x)-\sum_{x,y\in U}f(x)g(y)\nu_{xy},\end{aligned}$$

as required.

Define the difference operator ∇_{xy} as follows:

$$\nabla_{xy}f=f(y)-f(x).$$

Then, we have the following lemma.

Lemma 2 Let $U\subseteq V$. Then, for each $(f,g)\in\mathcal{F}_U\times\mathcal{F}_U$,

$$(-\Delta_U f,g)_U=\frac{1}{2}\sum_{x,y\in\bar{U}}(\nabla_{xy}f)(\nabla_{xy}g)\nu_{xy}.$$

Proof A similar result has been proved in Ref. [27]. For completeness, we give the proof of this lemma. Noting that for any $x\in U$,

$$\mu(x)=\sum_{y\in V}\nu_{xy}=\sum_{y\in\bar{U}}\nu_{xy},$$

and that $f(x)=g(x)=0$ for all $x\in\partial U$, we have

$$\sum_{x,y\in\bar{U}}(\nabla_{xy}f)(\nabla_{xy}g)\nu_{xy}$$

$$
\begin{aligned}
&= \sum_{x,y\in\bar{U}} [f(y)-f(x)][g(y)-g(x)]\nu_{xy} \\
&= \sum_{x,y\in\bar{U}} f(y)g(y)\nu_{xy} + \sum_{x,y\in\bar{U}} f(x)g(x)\nu_{xy} \\
&\quad - \sum_{x,y\in\bar{U}} f(y)g(x)\nu_{xy} - \sum_{x,y\in\bar{U}} f(x)g(y)\nu_{xy} \\
&= \left(\sum_{x\in\bar{U},y\in U} f(y)g(y)\nu_{xy} + \sum_{x\in\bar{U},y\in\partial U} f(y)g(y)\nu_{xy} \right) \\
&\quad + \left(\sum_{x\in U,y\in\bar{U}} f(x)g(x)\nu_{xy} + \sum_{x\in\partial U,y\in\bar{U}} f(x)g(x)\nu_{xy} \right) \\
&\quad - \sum_{x,y\in\bar{U}} f(y)g(x)\nu_{xy} - \sum_{x,y\in\bar{U}} f(x)g(y)\nu_{xy} \\
&= \sum_{y\in U} f(y)g(y)\mu(y) + \sum_{x\in U} f(x)g(x)\mu(x) \\
&\quad - \sum_{x,y\in U} f(y)g(x)\nu_{xy} - \sum_{x,y\in U} f(x)g(y)\nu_{xy} \\
&= 2\sum_{x\in U} f(x)g(x)\mu(x) - 2\sum_{x,y\in U} f(x)g(y)\nu_{xy}.
\end{aligned}
$$

Thus, it follows from the above equation and Lemma 1 that

$$\sum_{x,y\in\bar{U}} (\nabla_{xy}f)(\nabla_{xy}g)\nu_{xy} = 2(-\Delta_U f, g)_U,$$

as required. The proof is complete.

By Lemma 2, the operator $-\Delta_U$ is symmetric. Thus, it follows from Rayleigh's principle that

$$\lambda^1(U) = \inf\{\,(-\Delta_U f, f)_U \,|\, f\in\mathcal{F}_U, \|f\|_U = 1\,\}. \tag{3}$$

Furthermore, we have the following result.

Lemma 3 Let $U\subseteq V$. Then $\lambda^1(U)$ monotonically increases when the region U decreases, where $\lambda^1(U)$ is the first eigenvalue of $-\Delta_U$.

Proof Let $U_1\subseteq U_2\subseteq V$ be arbitrary but fixed. For each $f\in\mathcal{F}_{U_1}$, define $g_f:\bar{U}_2\to\mathbb{R}$ as follows:

$$g_f(x) = \begin{cases} f(x), & x\in U_1, \\ 0, & x\in\bar{U}_2\setminus U_1. \end{cases}$$

Then, it is easy to see that $g_f\in\mathcal{F}_{U_2}$ and $(g_f,g_f)_{U_2} = (f,f)_{U_1}$. Furthermore, by Lemma 1,

$$
\begin{aligned}
&(-\Delta_{U_2}g_f, g_f)_{U_2} \\
&= \sum_{x\in U_2} g_f(x)g_f(x)\mu(x) - \sum_{x,y\in U_2} g_f(x)g_f(y)\nu_{xy} \\
&= \sum_{x\in U_1} f(x)f(x)\mu(x) - \sum_{x,y\in U_1} f(x)f(y)\nu_{xy} \\
&= (-\Delta_{U_1}f, f)_{U_1}.
\end{aligned}
$$

It thus follows from Eq. (3) that

$$
\begin{aligned}
\lambda^1(U_2) &= \inf\{\,(-\Delta_{U_2}f,f)_{U_2} \,|\, f\in\mathcal{F}_{U_2}, \|f\|_{U_2}=1\,\} \\
&\le \inf\{\,(-\Delta_{U_2}g_f,g_f)_{U_2} \,|\, f\in\mathcal{F}_{U_1}, \|f\|_{U_1}=1\,\} \\
&= \inf\{\,(-\Delta_{U_1}f,f)_{U_1} \,|\, f\in\mathcal{F}_{U_1}, \|f\|_{U_1}=1\,\} \\
&= \lambda^1(U_1).
\end{aligned}
$$

This completes the proof.

Recall that $\omega\subseteq\Omega\subseteq V$. Define the operator $\mathcal{L}_k$ on $\mathcal{F}_\Omega$ with a Dirichlet homogeneous boundary value condition as follows:

$$\mathcal{L}_k f = (-\Delta_\Omega + k\chi_\omega)f, \quad f\in\mathcal{F}_\Omega,$$

where Δ_Ω is defined as in Eq. (2). Then, we have the following result.

Theorem 1 Let $\lambda_k^1(\Omega)$ be the first eigenvalue of the operator $\mathcal{L}_k$. Then,

$$\lim_{k\to\infty} \lambda_k^1(\Omega) = \lambda^1(\Omega_\omega), \tag{4}$$

where $\Omega_\omega = \Omega\setminus\omega$.

Proof Obviously, for any $k>0$, $\mathcal{L}_k$ is a positive operator. By the Rayleigh's principle,

$$
\begin{aligned}
\lambda_k^1(\Omega) &= \inf\{\,(\mathcal{L}_k f, f)_\Omega \,|\, f\in\mathcal{F}_\Omega, \|f\|_\Omega = 1\,\} \\
&= \inf\{\,((-\Delta_\Omega + k\chi_\omega)f, f)_\Omega \,|\, f\in\mathcal{F}_\Omega, \|f\|_\Omega = 1\,\}. \qquad (5)
\end{aligned}
$$

Note that by Lemma 2, for each $f\in\mathcal{F}_\Omega$ with $\|f\|_\Omega = 1$,

$$
\begin{aligned}
((-\Delta_\Omega + k\chi_\omega)f, f)_\Omega &= (-\Delta_\Omega f, f)_\Omega + (k\chi_\omega f, f)_\Omega \\
&= (-\Delta_\Omega f, f)_\Omega + k(f,f)_\omega \\
&= \frac{1}{2}\sum_{x,y\in\bar{\Omega}} (\nabla_{xy}f)^2\nu_{xy} + k\|f\|_\omega^2, \qquad (6)
\end{aligned}
$$

where

$$\|f\|_\omega = \sqrt{(f,f)_\omega} = \sqrt{\sum_{x\in\omega}(f(x))^2\mu(x)}.$$

Hence, combining Eq. (6) with Eq. (5), we can see easily that $\lambda_k^1(\Omega)$ is increasing as k increases.

Similar to the proof of Lemma 3, for each $f\in\mathcal{F}_{\Omega_\omega}$, let

$$g_f(x) = \begin{cases} f(x), & x\in\Omega_\omega, \\ 0, & x\in\omega\cup\partial\Omega. \end{cases}$$

Then $g_f\in\mathcal{F}_\Omega$ and $\|g_f\|_\Omega = \|f\|_{\Omega_\omega}$. Furthermore,

$$
\begin{aligned}
&((-\Delta_\Omega + k\chi_\omega)g_f, g_f)_\Omega \\
&= (-\Delta_\Omega g_f, g_f)_\Omega + (k\chi_\omega g_f, g_f)_\Omega \\
&= (-\Delta_\Omega g_f, g_f)_\Omega = (-\Delta_{\Omega_\omega}f, f)_{\Omega_\omega}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lambda_k^1(\Omega) \\
&= \inf\{\,((-\Delta_\Omega + k\chi_\omega)f, f)_\Omega \,|\, f\in\mathcal{F}_\Omega, \|f\|_\Omega = 1\,\} \\
&\le \inf\{\,((-\Delta_\Omega + k\chi_\omega)g_f, g_f)_\Omega \,|\, f\in\mathcal{F}_{\Omega_\omega}, \|f\|_{\mathcal{F}_{\Omega_\omega}} = 1\,\} \\
&= \inf\{\,(-\Delta_{\Omega_\omega}f, f)_{\Omega_\omega} \,|\, f\in\mathcal{F}_{\Omega_\omega}, \|f\|_{\mathcal{F}_{\Omega_\omega}} = 1\,\} \\
&= \lambda^1(\Omega_\omega). \qquad (7)
\end{aligned}
$$

Noting that $\lambda_k^1(\Omega)$ is increasing as $k \to \infty$, the limit of the sequence $\{\lambda_k^1(\Omega)\}$ exists and we have

$$\lim_{k\to\infty} \lambda_k^1(\Omega) \le \lambda^1(\Omega_\omega). \tag{8}$$

Let ϕ_k^1 be the eigenfunction corresponding to $\lambda_k^1(\Omega)$ with

$$\|\phi_k^1\|_\Omega = 1. \tag{9}$$

Using once again Rayleigh's principle, we have

$$\begin{aligned}\lambda_k^1(\Omega) &= (\mathcal{L}_k\phi_k^1, \phi_k^1)\\ &= ((-\Delta_\Omega + k\chi_\omega)\phi_k^1, \phi_k^1)_\Omega\\ &= (-\Delta_\Omega\phi_k^1, \phi_k^1)_\Omega + k(\phi_k^1, \phi_k^1)_\omega\\ &= \frac{1}{2}\sum_{x,y\in\bar{\Omega}} (\nabla_{xy}\phi_k^1)^2 \nu_{xy} + k\|\phi_k^1\|_\omega^2, \quad k \in \mathbb{N}.\end{aligned}$$

Thus together with Eqs. (7) and (9), the above equation implies that there exists a subsequence, which is still denoted by $\{\phi_k^1\}_{k=1}^\infty$, and $\phi^1 \in \mathcal{F}_\Omega$ such that

$$\begin{aligned}&\phi_k^1(x) \longrightarrow 0, \quad x \in \omega,\\ &\phi_k^1(x) \longrightarrow \phi^1(x), \quad x \in \Omega,\end{aligned}$$

as $k \longrightarrow \infty$. Thus,

$$\phi^1(x) = 0, \quad x \in \omega,$$

and

$$1 = \lim_{k\to\infty} \|\phi_k^1\|_\Omega = \|\phi^1\|_\Omega = \|\phi^1\|_{\Omega_\omega}.$$

Hence, by Lemma 1 and Eq. (3),

$$\begin{aligned}&\sum_{x\in\Omega} (\phi^1(x))^2\mu(x) - \sum_{x,y\in\Omega} \phi^1(x)\phi^1(y)\nu_{xy}\\ &= \sum_{x\in\Omega_\omega} (\phi^1(x))^2\mu(x) - \sum_{x,y\in\Omega_\omega} \phi^1(x)\phi^1(y)\nu_{xy}\\ &= (-\Delta_{\Omega_\omega}\phi^1, \phi^1)_{\Omega_\omega} \ge \lambda^1(\Omega_\omega).\end{aligned} \tag{10}$$

From Lemma 1 that

$$\begin{aligned}\lambda_k^1(\Omega) &= ((-\Delta_\Omega + k\chi_\omega)\phi_k^1, \phi_k^1)_\Omega\\ &\ge (-\Delta_\Omega\phi_k^1, \phi_k^1)_\Omega\\ &= \sum_{x\in\Omega} (\phi_k^1(x))^2\mu(x) - \sum_{x,y\in\Omega} \phi_k^1(x)\phi_k^1(y)\nu_{xy}.\end{aligned}$$

Taking the limit as $k \to \infty$, we have

$$\begin{aligned}&\lim_{k\to\infty} \lambda_k^1(\Omega)\\ &\ge \lim_{k\to\infty} \left(\sum_{x\in\Omega} (\phi_k^1(x))^2\mu(x) - \sum_{x,y\in\Omega} \phi_k^1(x)\phi_k^1(y)\nu_{xy} \right)\\ &= \sum_{x\in\Omega} (\phi^1(x))^2\mu(x) - \sum_{x,y\in\Omega} \phi^1(x)\phi^1(y)\nu_{xy} \ge \lambda^1(\Omega_\omega),\end{aligned}$$

where the last inequality follows from formula (10). Hence, together with Eq. (8), the above inequality shows that

$$\lim_{k\to\infty} \lambda_k^1(\Omega) = \lambda^1(\Omega_\omega).$$

The proof is complete.

3. The main result

We now give our main result.

Theorem 2 Suppose that

$$\lambda^1(\Omega_\omega) - \frac{c(a-1)^2}{4d} > 0. \tag{11}$$

Then there exist positive constants K, M, and α such that if $k \ge K$, then for any $(u_0, v_0) \in \mathcal{F}_\Omega \times \mathcal{F}_\Omega$, the solution (u, v) of system (1) with $z = -ku$ satisfies

$$\|u(\cdot,t)\|_\Omega^2 + \|v(\cdot,t)\|_\Omega^2 \le Me^{-\alpha t}\left(\|u_0\|_\Omega^2 + \|v_0\|_\Omega^2\right), \quad t > 0.$$

Remark 1 By Lemma 3, $\lambda^1(\Omega_\omega)$ is a monotonically increasing sequence when ω increases. Thus it is possible to select a large enough ω, where the controls act on, such that inequality (11) holds. In this case, the FHN system (1) can be stabilized via the linear feedback control $z = -ku$.

Proof of Theorem 2 Multiplying the first equation of system (1) by σu, taking the sum over Ω, and applying Lemma 2, we can obtain

$$\begin{aligned}&\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Omega} u^2\mu(x) + \frac{\sigma d}{2}\sum_{x,y\in\bar{\Omega}} |\nabla_{xy}u|^2\nu_{xy} + \sigma c\sum_{x\in\Omega} u^4\mu(x)\\ &-\sigma(a+1)c\sum_{x\in\Omega} u^3\mu(x) + \sigma ca\sum_{x\in\Omega} u^2\mu(x) + \sigma\gamma\sum_{x\in\Omega} uv\mu(x)\\ &= -\sigma k\sum_{x\in\omega} u^2\mu(x).\end{aligned} \tag{12}$$

Multiplying the second equation of system (1) by γv and taking the sum over Ω, we have

$$\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Omega} v^2\mu(x) - \gamma\sigma\sum_{x\in\Omega} uv\mu(x) + \gamma\beta\sum_{x\in\Omega} v^2\mu(x) = 0. \tag{13}$$

From Eqs. (12) and (13),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{2}\sum_{x\in\Omega} u^2\mu(x) + \frac{\gamma}{2}\sum_{x\in\Omega} v^2\mu(x)\right) + \sigma c\sum_{x\in\Omega} u^4\mu(x)\\ &= -\frac{\sigma d}{2}\sum_{x,y\in\bar{\Omega}} |\nabla_{xy}u|^2\nu_{xy} - \sigma k\sum_{x\in\omega} u^2\mu(x)\\ &\quad + \sigma(a+1)c\sum_{x\in\Omega} u^3\mu(x) - \sigma ac\sum_{x\in\Omega} u^2\mu(x)\\ &\quad - \beta\gamma\sum_{x\in\Omega} v^2\mu(x).\end{aligned} \tag{14}$$

Note that

$$(a+1)u \le u^2 + \frac{(a+1)^2}{4}.$$

Then, we have

$$\sigma(a+1)c\sum_{x\in\Omega} u^3\mu(x) \le \sigma c\sum_{x\in\Omega} u^4\mu(x)$$

$$+\frac{\sigma c(a+1)^2}{4}\sum_{x\in\Omega}u^2\mu(x),$$

since c, σ, and $\mu(x)$ are non-negative. It thus follows from Eq. (14) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{2}\sum_{x\in\Omega}u^2\mu(x)+\frac{\gamma}{2}\sum_{x\in\Omega}v^2\mu(x)\right)\\ \le&-\frac{\sigma d}{2}\sum_{x,y\in\bar{\Omega}}|\nabla_{xy}u|^2\nu_{xy}-\sigma k\sum_{x\in\omega}u^2\mu(x)\\ &+\frac{\sigma c(a-1)^2}{4}\sum_{x\in\Omega}u^2\mu(x)-\beta\gamma\sum_{x\in\Omega}v^2\mu(x).\end{aligned}\tag{15}$$

Taking

$$\varepsilon=\frac{1}{2}\left(\lambda^1(\Omega_\omega)-\frac{c(a-1)^2}{4d}\right)$$

and

$$\tau=\sigma d\varepsilon,\tag{16}$$

then it follows from inequality (11) that $\varepsilon>0$ and $\tau>0$. By Theorem 1, there exists $k_\varepsilon>0$ such that for all $k'\ge k_\varepsilon$,

$$\lambda^1_{k'}(\Omega)\ge\lambda^1(\Omega_\omega)-\varepsilon.$$

Thus, we have

$$\begin{aligned}&(\lambda^1(\Omega_\omega)-\varepsilon)\|u\|^2_\Omega\\ \le&(-\Delta_\Omega u+k'\chi_\omega u,u)_\Omega\\ =&\frac{1}{2}\sum_{x,y\in\bar{\Omega}}|\nabla_{xy}u|^2\nu_{xy}+k'\sum_{x\in\omega}u^2\mu(x),\end{aligned}\tag{17}$$

where the last equality follows from Lemma 2.

Take $K=dk_\varepsilon$. Then, for all $k\ge K$, we have $k/d\ge k_\varepsilon$. By expression (17),

$$\begin{aligned}&\frac{\sigma d}{2}\sum_{x,y\in\bar{\Omega}}|\nabla_{xy}u|^2\nu_{xy}+\sigma k\sum_{x\in\omega}u^2\mu(x)\\ =&\sigma d\left(\frac{1}{2}\sum_{x,y\in\bar{\Omega}}|\nabla_{xy}u|^2\nu_{xy}+\frac{k}{d}\sum_{x\in\omega}u^2\mu(x)\right)\\ \ge&\sigma d(\lambda^1(\Omega_\omega)-\varepsilon)\sum_{x\in\Omega}u^2\mu(x).\end{aligned}\tag{18}$$

Combining expressions (15) and (18), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{2}\sum_{x\in\Omega}u^2\mu(x)+\frac{\gamma}{2}\sum_{x\in\Omega}v^2\mu(x)\right)\\ \le&-\left(\sigma d(\lambda^1(\Omega_\omega)-\varepsilon)-\frac{\sigma c(a-1)^2}{4}\right)\sum_{x\in\Omega}u^2\mu(x)\\ &-\beta\gamma\sum_{x\in\Omega}v^2\mu(x)\\ =&-\tau\sum_{x\in\Omega}u^2\mu(x)-\beta\gamma\sum_{x\in\Omega}v^2\mu(x),\end{aligned}$$

where τ is defined by Eq. (16). Integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}&\frac{\sigma}{2}\sum_{x\in\Omega}u^2(x,t)\mu(x)+\frac{\gamma}{2}\sum_{x\in\Omega}v^2(x,t)\mu(x)\\ \le&-\tau\int_0^t\sum_{x\in\Omega}u^2(x,s)\mu(x)\mathrm{d}s-\beta\gamma\int_0^t\sum_{x\in\Omega}v^2(x,s)\mu(x)\mathrm{d}s\\ &+\frac{\sigma}{2}\sum_{x\in\Omega}u_0^2(x)\mu(x)+\frac{\gamma}{2}\sum_{x\in\Omega}v_0^2(x)\mu(x).\end{aligned}$$

Let $\gamma_1=\min\{\sigma/2,\ \gamma/2\}$, $\gamma_2=\max\{\sigma/2,\ \gamma/2\}$, $\alpha=1/\gamma_1\min\{\tau,\beta\gamma\}$, and $M=\gamma_2/\gamma_1$. Then

$$\begin{aligned}&\sum_{x\in\Omega}(u^2(x,t)+v^2(x,t))\mu(x)\\ \le&-\alpha\int_0^t\sum_{x\in\Omega}\big(u^2(x,s)+v^2(x,s)\big)\mu(x)\mathrm{d}s\\ &+M\sum_{x\in\Omega}(u_0^2(x)+v_0^2(x))\mu(x).\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}&\sum_{x\in\Omega}(u^2(x,t)+v^2(x,t))\mu(x)\\ \le&M\mathrm{e}^{-\alpha t}\sum_{x\in\Omega}(u_0^2(x)+v_0^2(x))\mu(x),\quad t>0,\end{aligned}$$

as required. This completes the proof.

Example 1 Given $N\in\mathbb{N}$, we define $h=1/(N+1)$. We consider the mesh

$$x_j=jh,\quad j=0,1,\ldots,N+1,$$

which divides $[0,1]$ into $N+1$ subintervals.

Let

$$\begin{aligned}V&=\{\,jh\,|\,j=0,1,\ldots,N+1\,\},\\ \Omega&=\{x_j|\ j=1,2,\ldots,N\},\end{aligned}$$

and

$$E=\{\{x_j,x_{j+1}\}|\ j=0,1,\ldots,N\}.$$

Then, $\partial\Omega=\{\,x_0,x_{N+1}\,\}$. For any $x,y\in V$,

$$\nu_{xy}=\begin{cases}1, & \text{if } \{x,y\}\in E,\\ 0, & \text{otherwise.}\end{cases}$$

Consider the Laplace operator Δ_Ω as follows:

$$\Delta_\Omega u(x_j)=\frac{u(x_{j+1})+u(x_{j-1})}{2}-u(x_j),\quad j=1,\ldots,N.$$

Let $d=2/h^2$. Then, it is easy to see that

$$\begin{aligned}d\Delta_\Omega u(x_j)=&\frac{1}{h^2}[u(x_{j+1})-2u(x_j)+u(x_{j-1})],\\ &j=1,\ldots,N,\end{aligned}$$

is the finite difference approximation of the second differential operator $\Delta=\partial^2/\partial x^2$.

Let $0<m<N$ and $\omega=\{m+1,m+2,\ldots,N\}$. Thus, $\Omega_\omega=\{1,2,\ldots,m\}$. Consider the following eigenvalue problem

$$-\Delta_{\Omega_\omega}u(x_j)=\lambda u(x_j),\quad j=1,2,\ldots,m,$$

$$u(x_0) = u(x_{m+1}) = 0.$$

In this case, the spectrum of Δ_{Ω_ω} can be computed explicitly (see Ref. [28] or Ref. [29]). The eigenvalues satisfy $0 < \lambda^1(\Omega_\omega) < \lambda^2(\Omega_\omega) < \cdots < \lambda^m(\Omega_\omega)$, where

$$\lambda^k(\Omega_\omega) = 2\sin^2\left(\frac{k\pi}{2(m+1)}\right), \quad k = 1, 2, \ldots, m.$$

Then if

$$4(N+1)^2 \sin^2\left(\frac{\pi}{2(m+1)}\right) > \frac{c(a-1)^2}{4},$$

condition (11) is satisfied.

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