TOPICAL REVIEW — Statistical physics and complex systems

Percolation on networks with dependence links*

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As a classical model of statistical physics, the percolation theory provides a powerful approach to analyze the network structure and dynamics. Recently, to model the relations among interacting agents beyond the connection of the networked system, the concept of dependence link is proposed to represent the dependence relationship of agents. These studies suggest that the percolation properties of these networks differ greatly from those of the ordinary networks. In particular, unlike the well known continuous transition on the ordinary networks, the percolation transitions on these networks are discontinuous. Moreover, these networks are more fragile for a broader degree distribution, which is opposite to the famous results for the ordinary networks. In this article, we give a summary of the theoretical approaches to study the percolation process on networks with inter- and inner-dependence links, and review the recent advances in this field, focusing on the topology and robustness of such networks.

Keywords: percolation, network, dependence link, phase transition

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1. Introduction

With a view of network, interactions between agents in a system can be represented clearly by the pattern of connections, which provides an effective approach to study the structure and dynamics of a system.[1] However, the structure and function of a system are often affected by various disturbances. For this reason, the study of the robustness of network connections becomes inevitable.[2,3] Theoretically, the percolation theory is particularly useful in studying the robustness of networks against random failure and intentional attack.[4–7]

Generally speaking, the percolation theory explains the existence of a giant component of a network after a fraction $1-p$ of nodes are removed, where the giant component is the largest set of nodes that are connected to one another by at least one path. The percolation process on a network demonstrates a second order phase transition, that is, the size of the giant component continuously decreases to zero when p approaches from the critical point p_c above. The critical point p_c is often used to evaluate the robustness of the network. A small critical point p_c means that the network is robust against a large amount of damage.

The success of these studies is based on the simplicity of the interactions among agents. However, in reality, there are some interactions between two agents beyond the connection of the network. For example, considering a financial network, the trading and sale connections enable the companies to interact with each other, which can be understood as connectivity links. For two companies belonging to the same owner, if one company fails, the owner might not be able to finance the other company that is also going to fail. Such a dependence is often independent of the connection of the financial network, since there could be no trade between the two companies.[8] As another example, consider a power network and a communication network. Although the connection structures of the two networks are independent of each other, there is a bidirectional dependence between them: the power stations depend on the communication nodes for control, and the communication nodes depend on the power stations for their electricity supply. These two networks are often called interdependent networks.[9]

To model these correlations of nodes beyond the connection of networks, the dependence link is proposed to represent the dependence relationship of nodes in real networks (see Fig. 1).[9] Specifically, a dependence link between node i and node j means that if node i fails, node j also fails, and vice versa. Due to the independence between the dependence and the connectivity links, the failures caused by the dependence links are independent of the network structure. Thus, after a fraction of nodes are removed, the failures of nodes caused by the connectivity and the dependence links will occur alternately, leading to a cascading failure. Obviously, this cascading failure makes the network more fragile than that without dependence links.

Furthermore, percolation on networks with dependence links shows many phenomena that are different from those on

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the ordinary networks, such as the discontinuous percolation transition. In this article, we will review the recent advances in this field, focusing on the topology and the robustness of networks with dependence links. In the next section, we will summarize the theoretical approaches to study the percolation processes on these networks. In Section 3, recent advances of percolation on these networks will be reviewed. The last section is the summary and the prospect of this field.

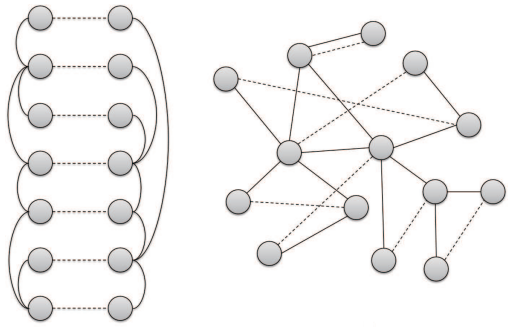


Fig. 1. Networks with dependence links. The solid lines represent the connectivity links, while the dash lines represent the dependence links. (a) Interdependent networks. Each network is composed of a set of nodes and connectivity links, and the inter-dependence between the two networks is represented by dependence links. (b) Single network with inner-dependence. The network is formed by nodes and connectivity links, and the dependence links represent the inner-dependence of the network.

2. Theoretical approach

In this section, we will review the generating function approach[10–12] to study the percolation on the ordinary networks first. Then, based on this technique, a theoretical approach[8,9,13,14] to study the percolation on the networks containing dependence links will be illustrated.

For convenience, let S be the order parameter usually used in the percolation theory, that is, the probability that a randomly chosen node belongs to the giant component of the network, i.e., the size of the giant component. In addition, let R be the probability that the node arriving by following a randomly chosen link belongs to the giant component.

2.1. Networks containing only connectivity links

To obtain the equation for the order parameter S, let us check the conditions of a node belonging to the giant component. If a node belongs to the giant component of a network, at least one of its links should connect to the giant component. Therefore, the order parameter S satisfies

$$\begin{aligned} S &= \sum_{k=1} p_k \sum_{i=1} \binom{k}{i} R^i (1-R)^{k-i} \\ &= 1 - \sum_{k=0} p_k (1-R)^k, \end{aligned} \tag{1}$$

where p_k is the degree distribution of the network. Using the generating function of the degree distribution $G_0(x) = \sum_k p_k x^k$, this equation can be rewritten as

$$S = 1 - G_0(1-R). \tag{2}$$

Similarly, one can write down the equation for R

$$\begin{aligned} R &= \sum_{k=2} \frac{p_k k}{\langle k \rangle} \sum_{i=1} \binom{k-1}{i} R^i (1-R)^{k-1-i} \\ &= 1 - G_1(1-R). \end{aligned} \tag{3}$$

Here, the brackets $\langle \cdots \rangle$ denote the average over the degree distribution p_k. The expression $p_k k/\langle k \rangle$ is the excess degree distribution of the network, and $G_1(x) = G_0'(x)/G_0'(1)$ is the corresponding generating function. Note that this equation holds only for the tree-like networks, since for a network with clustering, the excess degree distribution cannot be expressed as $p_k k/\langle k \rangle$.[15,16]

For an arbitrary degree distribution, we can solve Eq. (3) to obtain R, and then insert it into Eq. (2) to obtain the order parameter S.

Next, let us consider the percolation process, that is, a fraction $1-p$ of the nodes in the network have been removed. The degree distribution of the network formed by the preserved nodes can be written as

$$\tilde{p}_k = \sum_{k'=k} p_{k'} \binom{k'}{k} p^k (1-p)^{k'-k}, \tag{4}$$

where $p_{k'}$ is the degree distribution of the original network. Thus, we can write down the generating functions of the degree distribution and the excess degree distribution for this network

$$\sum_{k=0} \tilde{p}_k x^k = G_0(1-p+px), \tag{5}$$

$$\sum_{k=1} \frac{\tilde{p}_k k}{p\langle k \rangle} x^{k-1} = G_1(1-p+px). \tag{6}$$

Using the two functions, we can rewrite Eqs. (2) and (3) as

$$\psi = 1 - G_0(1-p\phi), \tag{7}$$

$$\phi = 1 - G_1(1-p\phi). \tag{8}$$

Here, ψ is the size of the giant component of the network after nodes removal. Note that ψ is not equal to the order parameter S, which is the fraction of nodes in the giant component of the original network. However, the product $p\psi$ is equal to the order parameter S. Similarly, ϕ has the same meaning as R, but for the network after nodes removal.

Substituting $S = p\psi$ and $R = p\phi$ into Eqs. (7) and (8), we obtain the equations for the order parameter S

$$S = p\left[1 - G_0(1-R)\right], \tag{9}$$

$$R = p\left[1 - G_1(1-R)\right]. \tag{10}$$

These are the equations for the percolation process on a network containing only connectivity links. In fact, instead of

using Eqs. (7) and (8), these two equations can be obtained directly by simple analogy with Eqs. (2) and (3).

The percolation process demonstrates a second order phase transition (see Fig. 2), for which the critical point corresponds to the solution of Eq. (10) in the condition of $R \to 0$. This yields

$$1 = p_c G_1'(1). \tag{11}$$

It is easy to rewrite this equation in a more common and famous form $p_c = \langle k \rangle / (\langle k^2 \rangle - \langle k \rangle)$. For Erdös–Rényi (ER) networks, $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$, thus the critical point is $p_c = 1/\langle k \rangle$.

In addition, expanding Eq. (10) at the critical point, in the leading order, we find

$$S \propto R \propto (p - p_c), \tag{12}$$

i.e., for tree-like networks, the percolation transition is second order with critical exponent 1.

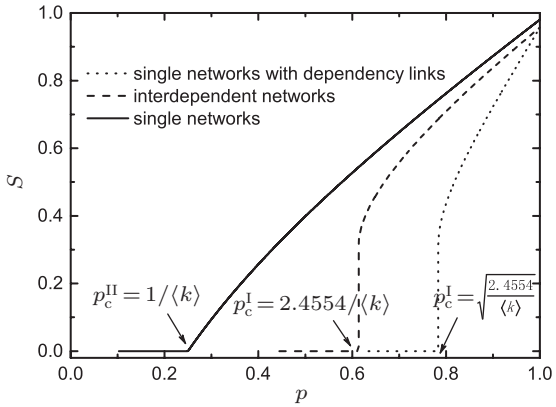


Fig. 2. The giant component of a network as a function of the fraction p of the preserved nodes after initial nodes removal. All the networks used here are ER networks with average degree $\langle k \rangle = 4$. For an ER network, the degree distribution satisfies $p_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$. This yields $G_0(x) = G_1(x) = e^{\langle k \rangle (x-1)}$. As a result, the auxiliary parameter R is equivalent to the order parameter S. For single networks, equation (9) takes a simple form $S = p(1 - e^{-\langle k \rangle S})$. For interdependent ER networks, equation (28) takes a simple form $S = p(1 - e^{-\langle k \rangle S})^2$. For single ER networks with dependence links, equation (31) takes a simple form $S = p^2(1 - e^{-\langle k \rangle S})^2$. Using the method presented in the text, the critical points for these cases are easily obtained: $1/\langle k \rangle$, $2.4554/\langle k \rangle$, and $\sqrt{2.4554/\langle k \rangle}$.

2.2. Networks containing both connectivity and dependence links

2.2.1. Interdependent networks

Most studies of percolation on networks containing dependence links focus on the so-called interdependent networks model.[9] In this model, there are two networks A and B with degree distributions p_k^A and p_k^B, respectively. In general, it is assumed that the two networks have the same size. To present the inter-dependence of the nodes between the two networks, the mutual dependence links are used to connect nodes from different networks randomly, and each node has exactly one dependence link (see Fig. 1). A dependence link between node A_i in network A and node B_i in network B means that if node A_i fails, node B_i also fails, and vice versa.

The removal of nodes will lead to the failures of their links, including the connectivity and the dependence links. The failures of the connectivity links may cause some other nodes in the same network to disconnect from the network, and the failures of the dependence links lead to the failures of their partners in the other network. Therefore, after the initial removal of nodes, the two failure processes will occur alternately, until no nodes removal can be done. In the steady state, the sizes of the giant components of networks A and B formed by the preserved nodes (if any) must be the same, called the mutual giant component of the interdependent networks.

Next, let us consider the cascading process step-by-step to solve this model. The cascading process begins by randomly removing a fraction $1 - p$ of nodes of network A. For better understanding, instead of considering the giant component formed by the preserved nodes, we only consider the fraction T of the preserved nodes in each cascading stage. Once the fraction T of the nodes preserved in the final network is obtained, it is easy to obtain the size of the mutual giant component by Eqs. (7) and (8).

After the initial removal, the fraction T_1^A of the preserved nodes in network A is

$$T_1^A = p. \tag{13}$$

This is not the size of the preserved nodes in the steady state, because the failures of the dependence links will lead to the failures of the nodes in network B. As the model setting,[9] the nodes in network B depending on the nodes that do not belong to the giant component of network A will also fail. Therefore, in this stage, the fraction of the preserved nodes in network B is

$$T_1^B = p g_A(T_1^A), \tag{14}$$

where $g_A(T_1^A)$ is the size of the giant component of network A formed by the preserved nodes. Here $g_A(x) = 1 - G_0(1 - xy)$ and $y = 1 - G_1(1 - xy)$, which are just Eqs. (7) and (8). The parameter p gives the probability that a random chosen node is preserved in the initial attack.

In stage 2, the preserved nodes in network A will decrease due to the dependence between the two networks. In network A, all the nodes preserved after stage 2 must satisfy two conditions. First, they cannot be removed in the initial attack. Second, their dependence nodes must belong to the giant component of network B formed by the preserved nodes in stage 1. This yields (see the supplementary information of Ref. [9] for details)

$$T_2^A = p g_B(T_1^B). \tag{15}$$

Similarly, we can write down the fraction of the preserved nodes for each stage

$$\begin{aligned} T_1^A &= p, & T_1^B &= pg_A(T_1^A), \\ T_2^A &= pg_B(T_1^B), & T_2^B &= pg_A(T_2^A), \\ T_3^A &= pg_B(T_2^B), & T_3^B &= pg_A(T_3^A), \\ &\vdots \\ T_n^A &= pg_B(T_{n-1}^B), & T_n^B &= pg_A(T_n^A). \end{aligned} \tag{16}$$

In the steady state, no more nodes can be removed. Thus, $T_n^A = T_{n-1}^A$, and $T_n^B = T_{n-1}^B$. Then, we have

$$T^A = pg_B(T^B), \tag{17}$$

$$T^B = pg_A(T^A). \tag{18}$$

To obtain T^A and T^B, we can use the self-consistent equations

$$T^A = pg_B(pg_A(T^A)), \tag{19}$$

$$T^B = pg_A(pg_B(T^B)). \tag{20}$$

For the final networks, the giant components of networks A and B must be the same, that is the mutual giant component

$$\begin{aligned} S &= T^A g_A(T^A) = T^B g_B(T^B) \\ &= pg_A(T^A)g_B(T^B). \end{aligned} \tag{21}$$

For arbitrary degree distributions p_k^A, p_k^B and a fraction of initial removal $1-p$, we can solve Eqs. (19) and (20) to obtain T^A and T^B, and then insert them into Eq. (21) to obtain the order parameter S.

The critical points are determined by Eqs. (19) and (20). Take the derivatives of Eqs. (19) and (20) with respect to T^A and T^B, respectively

$$1 = p_c^2 g_B'(p_c g_A(T^A))g_A'(T^A), \tag{22}$$

$$1 = p_c^2 g_A'(p_c g_B(T^B))g_B'(T^B). \tag{23}$$

Obviously, these two equations do not have solutions in the condition of $T^A \to 0$ and $T^B \to 0$. This means that the percolation process on the interdependent networks demonstrates a discontinuous phase transition (see Fig. 2), which is different to the one on the network containing only connectivity links. As pointed out in Ref. [17], this discontinuous phase transition is a hybrid phase transition, which satisfies

$$S - S_c \propto (p - p_c)^{\frac{1}{2}}. \tag{24}$$

That is the combination of a jump and the square root critical singularity.

Next, substituting Eqs. (7) and (8) into Eq. (21) yields

$$S = p\left[1 - G_0^A(1 - T^A y^A)\right]\left[1 - G_0^B(1 - T^B y^B)\right], \tag{25}$$

$$y^A = 1 - G_1^A(1 - T^A y^A), \tag{26}$$

$$y^B = 1 - G_1^B(1 - T^B y^B), \tag{27}$$

where $G_0^A(x)$, $G_0^B(x)$, $G_1^A(x)$, and $G_1^B(x)$ are the corresponding generating functions of networks A and B. Let $R^A = T^A y^A$ and $R^B = T^B y^B$. Using Eqs. (17) and (18), we find

$$S = p\left[1 - G_0^A(1 - R^A)\right]\left[1 - G_0^B(1 - R^B)\right], \tag{28}$$

$$R^A = p\left[1 - G_1^A(1 - R^A)\right]\left[1 - G_0^B(1 - R^B)\right], \tag{29}$$

$$R^B = p\left[1 - G_0^A(1 - R^A)\right]\left[1 - G_1^B(1 - R^B)\right]. \tag{30}$$

Here, R^A (R^B) represents the probability that the node arriving by following a randomly chosen link in network A (B) belongs to the mutual giant component. By simple analogy with the generating function technique for the ordinary networks, we can know that these equations describe the final state after the cascades, which are easier to understand than Eqs. (19)–(21). This simple way (Eqs. (28)–(30)) was first used to analyse the percolation on the networks with dependence links by Son *et al.*[13] (see Ref. [14] for details).

2.2.2. Single networks with dependence links

Next, we discuss the model for a single network containing both connectivity and dependence links.[8] In this model, the dependence pairs are chosen in one network. For a network with N nodes, there are $N/2$ dependence links.

Let us consider the steady state after removing a fraction $1-p$ of nodes. The order parameter S is equal to the probability that a randomly chosen node and its dependence node belong to the giant component simultaneously. This yields

$$S = p^2\left[1 - G_0(1 - R)\right]^2, \tag{31}$$

where p^2 expresses the dependence partners that are preserved after the initial removal. Note that there is only one network, so we only need one auxiliary parameter R and one generating function. Thus, we can write down the equation for R as

$$R = p^2\left[1 - G_1(1 - R)\right]\left[1 - G_0(1 - R)\right]. \tag{32}$$

It should be pointed that this model is not equal to the model of two interdependent networks with the same degree distribution. In this model, the probability of a pair of dependence nodes preserved after the initial removal is p^2, while it is p for the interdependent networks. Similar to the interdependent networks, this model also demonstrates a discontinuous phase transition (see Fig. 2).

3. Networks with dependence links

In this section, we will review the recent advances in percolation on the networks with dependence links, focusing on the effects of some actual factors on the robustness of these networks, such as network structure, dependence strength, multiple dependence, targeted attack, and inter-similarity.

3.1. Network structure

As we know, the structure plays a key role in the robustness of the ordinary network.[3] For networks with the same average degree, the broader the degree distribution is, the more robust the network is. That is to say, the critical point of the percolation on a scale-free (SF) network is smaller than that on the ER network with the same average degree.[4] In addition, the degree correlation and the clustering both have great influences on the percolation process on the networks. For example, increasing the assortativity of a network, which relates to the degree correlation and the clustering of the network, makes the network more robust against nodes removal.[18,19]

Recent studies found that the effects of the network structure on the robustness of the network are very different when the dependence links are considered. In Refs. [8] and [9], researchers found that a network with a broader degree distribution results in a larger p_c, which is opposite to that found in the ordinary networks. As we know, the hub (extremely large degree) nodes have a dominant role in the robustness of a network. However, when the dependence links are considered, the hub node could depend on a small degree node, which is quite vulnerable during the cascading process. Moreover, a broader degree distribution with the same average degree implies that there are more small degree nodes. Therefore, the advantage of a broad degree distribution for the ordinary network becomes a disadvantage for the network with dependence links.

Zhou *et al.* studied the effect of internal node correlations in each of the two interdependent networks on the robustness of the system,[20] and found that the assortativity within a single network decreases the robustness of the entire system. Furthermore, using the model developed for the single clustered network by Newman,[21] Huang *et al.* developed an analytical method for studying a system of the interdependent clustering networks.[22] They found that the interdependent networks with high clustering are more fragile under random failure than those with the same degree distribution but lower clustering.

The studies on the interdependent square lattice networks suggested that the percolation transitions are not always sharpened by making networks interdependent.[13,23] They found that the type of the percolation transition is dependent on the distance r of two dependence nodes in the square lattice. For $r < r_c \approx 8$, the percolation transitions are of the second order, while for $r > r_c$, the percolation transitions are of the first order. The critical point p_c increases linearly with increasing r in the range of $r < r_c$ and gradually decreases to 0.863 for $r \to \infty$. Their studies indicated that interdependent systems embedded in the Euclidean space become most vulnerable when the distance between interdependent nodes is in the intermediate range, which is much smaller than the size of the system.[24]

3.2. Correlation of dependent partners

For real-world situations, there are some correlations between two dependence nodes, such as degree, neighbor, connection, clustering, etc. For interdependent networks, the correlation can be understood as the inter-similarity of the two networks.

Buldyrev *et al.* studied a simple case of the interdependent networks with inter-similarity, that is, the dependence links always connect two nodes with an identical degree.[25] This implies that the two networks have the same degree distribution. They found that the system undergoes a first order percolation transition, for which the critical point is always smaller than that for randomly interdependent networks with the same degree distribution.

Furthermore, to describe the inter-similarity of two interdependent networks, Parshani *et al.* introduced two quantities for evaluating the level of inter-similarity between the networks: the inter degree–degree correlation and the inter-clustering coefficient.[26] The simulation results demonstrated that the inter-similar interdependent networks are significantly more robust to random failure than a randomly interdependent system.

Theoretically, Hu *et al.* developed an approach to analyze the interdependent networks with inter-similarity.[27] In their model, there are some probabilities that the dependence nodes of two adjacent nodes in one network are also connected. The studies showed that this inter-similarity can improve the robustness of the interdependent networks and change the critical behavior of the percolation transition. Instead of the first order phase transition, the percolation on the interdependent networks with high inter-similarity is similar to that on a single network and is a second order phase transition. Therefore, for different inter-similarities, the crossover between the first and the second order percolation transitions can be found in their model. In addition, a similar work has been carried out on the so-called multiplex networks.[28]

For a single network containing dependence links, our recent studies considered the connection of two dependence partners, and studied the cascading dynamics of the network with an overlapping of dependence and connectivity links.[29] In this model, a fraction of the dependence links overlap with the connectivity links, which means that these nodes are adjacent to their dependence partners. The simulation and the analytical results both demonstrated that a high density of dependence links does not always make the percolation transition sharpened. For a small fraction of overlapping, the dependence links could easily propagate the failures to the whole network, and the network is easy to be destroyed in the form of the first order phase transition. For a large fraction of overlapping, most nodes are adjacent to their dependence partners, so

the cascading failures caused by the dependence links are confined to some small areas. Therefore, the networks are more robust than that with a small fraction of overlapping, and the system demonstrates a second order phase transition.

3.3. Dependence strength

The dependence links reflect some dependence relationships in reality, but not all the nodes in a networked system have this dependence. In this way, the percolation on networks with weak dependence has been studied.[30] In this model, instead of each node having a dependence link, only a fraction q of the nodes have dependence links. When $q = 0$, the network reduces to the ordinary one.

Take the interdependent networks as an example, equations (17) and (18) can be rewritten as

$$T^A = p\left[1-q+qg_B(T^B)\right], \tag{33}$$
$$T^B = 1-q+qpg_A(T^A). \tag{34}$$

Note that the initial node removal is on network A. Thus, in this model, the giant components of the two interdependent networks after the cascades are different generally. As a result, two order parameters are needed for the two networks, respectively. According to Eqs. (28)–(30), the equations for the two order parameters S^A and S^B can be written as

$$S^A = p\left\{q\left[1-G_0^A(1-R^A)\right]\left[1-G_0^B(1-R^B)\right] + (1-q)\left[1-G_0^A(1-R^A)\right]\right\}, \tag{35}$$
$$S^B = qp\left[1-G_0^A(1-R^A)\right]\left[1-G_0^B(1-R^B)\right] + (1-q)\left[1-G_0^B(1-R^B)\right], \tag{36}$$
$$R^A = p\left\{q\left[1-G_1^A(1-R^A)\right]\left[1-G_0^B(1-R^B)\right] + (1-q)\left[1-G_1^A(1-R^A)\right]\right\}, \tag{37}$$
$$R^B = qp\left[1-G_0^A(1-R^A)\right]\left[1-G_1^B(1-R^B)\right] + (1-q)\left[1-G_1^B(1-R^B)\right]. \tag{38}$$

Let $q = 1$, these equations reduce to the full dependence model discussed in the last section, which demonstrates a discontinuous phase transition. While $q = 0$, these equations reduce to that for networks containing only connectivity links, which demonstrates a continuous phase transition. Therefore, the crossover between the discontinuous and the continuous percolation transitions can be found in this model (see Fig. 3).

After that, single networks with partial dependence,[8] partially interdependent scale-free networks,[31] and degree–degree correlated partially interdependent networks[32] were also studied. All these works revealed the existence of the crossover between the discontinuous and the continuous percolation transitions for different dependence strengths.

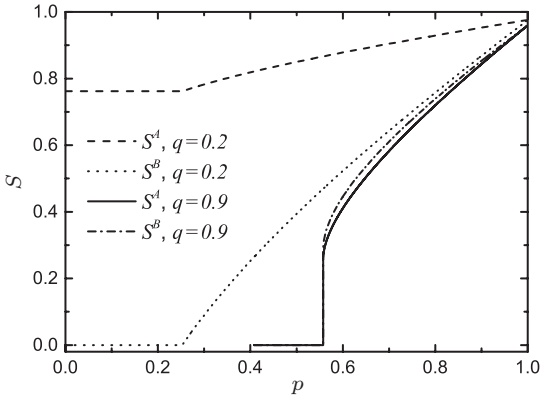


Fig. 3. The giant component of partially interdependent networks as a function of the fraction p of the preserved nodes after the initial node removal. The two interdependent networks used here are both ER networks with the same average degree $\langle k\rangle = 4$. For the ER network, the degree distribution satisfies $p_k = \mathrm{e}^{-\langle k\rangle}\langle k\rangle^k/k!$. This yields $G_0(x) = G_1(x) = \mathrm{e}^{\langle k\rangle(x-1)}$. As a result, the auxiliary parameter R is equivalent to the order parameter S. Then, equations (35)–(38) reduce to two equations $S^A = p[q(1-\mathrm{e}^{-\langle k\rangle S^A})(1-\mathrm{e}^{-\langle k\rangle S^B}) + (1-q)(1-\mathrm{e}^{-\langle k\rangle S^A})]$ and $S^B = pq(1-\mathrm{e}^{-\langle k\rangle S^A})(1-\mathrm{e}^{-\langle k\rangle S^B}) + (1-q)(1-\mathrm{e}^{-\langle k\rangle S^B})$. The critical points are 0.2528 and 0.5576 for $q = 0.2$ and 0.9, respectively. Note that for a small q, there is always a giant component in network B, since the initial node removal is on network A.

3.4. Multiple dependence

In the original model described in Section 2, each node has only one dependence link. In reality, multiple dependence relations are often involved.

To study the multiple dependence, Shao *et al.* proposed a model of interdependent networks, in which each node can have more than one dependence link[33] (see Fig. 4(a)). They found that only in the limit of an infinite number of dependence links or in case of the existence of autonomous nodes which do not have any dependence links, the behavior of the mutual giant components can be described by the second order phase transition. While in all other cases, the system demonstrates a first order phase transition.

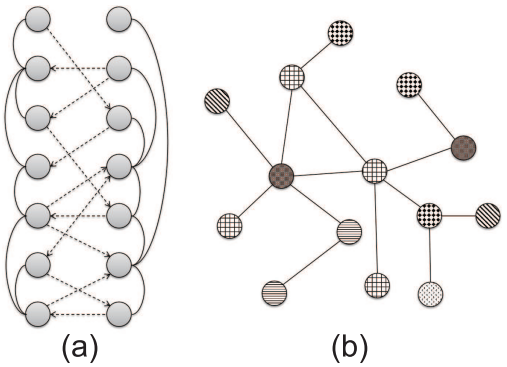


Fig. 4. Networks with multiple dependence. (a) Two interdependent networks with multiple dependence links. The solid lines represent connectivity links, while the dash lines represent dependence links. Each node can have more than one dependence link. The direction of the dependence link represents the support relation between two nodes. In the cascading process, a node will fail if it disconnects from the network or loses all its support links. (b) Single networks with dependence groups. The solid lines represent connectivity links. Different patterns are used to distinguish dependence groups, and nodes with the same pattern belong to the same group. In each group, if one node fails, all fail.

To study the multiple dependence on single networks, Bashan *et al.* developed a model with dependence groups[34,35] (see Fig. 4(b)). In a dependence group, if one node fails, all fail. They also developed the approach used for analyzing the percolation on the interdependent networks into a more general one. Next, we give a briefly introduction to this approach.

As discussed previously, after randomly removing a fraction $1-p$ of nodes, the percolation and the dependence processes will occur alternatively. In order to describe the effects of these two processes in the cascading process, $g_{\rm P}(x)$ is used to represent the giant component size of the network after the random removal of a fraction $1-x$ of nodes, and $g_{\rm D}(x)$ for the fraction of nodes that do not depend on the failed nodes after the node removal. As the discussion for the interdependent networks, using these two functions, we can write down the fraction of the preserved nodes in each stage

$$\begin{aligned} T_1^{\rm P} &= p, & T_1^{\rm D} &= pg_{\rm P}(T_1^{\rm P}),\\ T_2^{\rm P} &= pg_{\rm D}(T_1^{\rm D}), & T_2^{\rm D} &= pg_{\rm P}(T_2^{\rm P}),\\ T_3^{\rm P} &= pg_{\rm D}(T_2^{\rm D}), & T_3^{\rm D} &= pg_{\rm P}(T_3^{\rm P}),\\ &\vdots \\ T_n^{\rm P} &= pg_{\rm D}(T_{n-1}^{\rm D}), & T_n^{\rm D} &= pg_{\rm P}(T_n^{\rm P}). \end{aligned} \tag{39}$$

In the steady state, the fraction of the preserved nodes must satisfy

$$T^{\rm P} = pg_{\rm D}(T^{\rm D}), \tag{40}$$

$$T^{\rm D} = pg_{\rm P}(T^{\rm P}). \tag{41}$$

The order parameter S can be written as

$$S = T^{\rm P}g_{\rm P}(T^{\rm P}) = T^{\rm D}g_{\rm D}(T^{\rm D}) = pg_{\rm D}(T^{\rm D})g_{\rm P}(T^{\rm P}). \tag{42}$$

According to Eqs. (7) and (8), $g_{\rm P}(x)$ satisfies

$$g_{\rm P}(x) = 1 - G_0(1-xy), \tag{43}$$

where $y = 1 - G_1(1-xy)$. For an arbitrary dependence group size distribution p_s, $g_{\rm D}(x)$ takes the form

$$g_{\rm D}(x) = \sum_s q_s x^{s-1}. \tag{44}$$

Solving these equations, we will also find the crossover of the first and the second order phase transitions for different dependence group size distributions.[34]

In fact, $g_{\rm D}(x)$ and $g_{\rm P}(x)$ can express any two independent processes on a network. With a minor modification, this approach can be used to study other network dynamics with more than one interdependent process.

3.5. Targeted attack

The robustness of networks under a targeted attack is also an important topic in the studies of networks. Huang *et al.* developed a theoretical framework for understanding the robustness of the interdependent networks under targeted attacks by mapping it to a random attack problem.[36]

In their model, the probability that a node i is initially attacked is $k_i^\alpha / \sum_i k_i^\alpha$. When $\alpha > 0$, nodes with a larger degree are more vulnerable in the initial attacks, while for $\alpha < 0$, nodes with a larger degree are defended and so have lower probabilities to fail. The case $\alpha = 0$ represents the random removal of nodes. In contrast to a single network, they found that the interdependent networks are difficult to defend by using strategies such as protecting the large degree nodes which have been found to be useful to significantly improve the robustness of single networks. After that, the percolation of partially interdependent networks under targeted attacks was studied too.[37,38]

3.6. Network of networks

Considering the real situation in which more than two networks are connected by the dependence links, Gao *et al.* developed the model of the interdependent networks into the model of a network of networks.[39–41] In this model, each network is composed of a set of nodes and connectivity links, and the dependence links connect each network to form a larger network (see Fig. 5). In addition, the network of networks is also studied in a model named multiplex networks.[17,42–48]

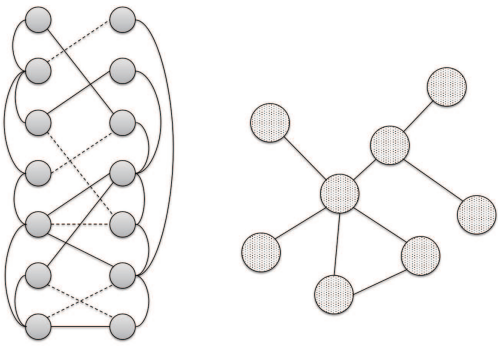


Fig. 5. (a) Two interdependent and interconnected networks. Each network is composed of a set of nodes and connectivity links. The interdependence between the two networks is represented by dependence links (dash lines), and the inter-connectivity between the two networks is represented by connectivity links (solid lines). (b) A network of networks. The circles represent networks, which are formed by nodes and connectivity links. The solid lines, which represent the dependence or connectivity relations between networks, connect the networks to a larger network.

Due to the cascading failures caused by the dependence of these networks, the percolation transition of a network of networks could be discontinuous and even a single node failure may lead to an abrupt collapse of the system. In real

life, infrastructural networks are governed and operated separately and the interactions are only allowed within well-defined boundaries. Therefore, for different situations, the percolation on the network of networks may be very different. In Ref. [49], a summary of recent studies on the theoretical approaches and applications of the network of networks is given.

Furthermore, inter-dependence is not the only relation between two networks, inter-connectivity exists too.[50] Therefore, the inter-connectivity links can also connect networks together to form a network of networks. The network of networks with only inter-connectivity links can be treated as a network with a community structure, which has been well studied and widely used.[51] However, by only considering a two-coupled-network system with both inter-dependence and inter-connectivity (see Fig. 5(a)), it has already shown rich phase transition phenomena.[52] Therefore, a network of networks with both inter-dependence and inter-connectivity (interacting networks) could be very complicated, and more research is needed.

4. Discussion and outlook

In this article, we have given a summary of the theoretical approaches to study the percolation processes on networks with dependence links. In general, there are two approaches that are usually used for different situations: one focuses on the cascading process, and the other focuses on the steady state after the cascade. We also give a simple proof of the equivalence of these two approaches in this article.

Recent advances on topology and robustness of networks with dependence links have also be reviewed in this article. Overall, the dependence links make the network much more vulnerable to random failures and targeted attacks, which offers a challenge to network protection and immunization strategies for disease.

On the other hand, the real networked systems are not always so fragile as the model with the dependence links presents. Recent investigations have already shown that the networks with dependence links are not so fragile when some realistic factors are considered, such as network structure, dependence strength, correlation of dependence nodes, etc. Our recent study demonstrated that the asymmetric dependence can also make these networks robust.[53] Hence, more realistic factors need to be introduced into this model to explain the structure and the robustness of the real networked system with dependence links.

Furthermore, the concepts of interdependent and interconnected networks should also be useful in the study of related subjects of recent interest, such as epidemic spreading,[54,55] diffusion dynamics,[56] and evolution of cooperation.[57,58] Examining these dynamics on networks with dependence links not only provides a new way of studying the dynamics in complex systems, but also it is meaningful for understanding the structures and the robustness of the networks with dependence links.

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