Characteristics of phase transitions via intervention in random networks*

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We present a percolation process in which the classical Erdös–Rényi (ER) random evolutionary network is intervened by the product rule (PR) from some moment t_0. The parameter t_0 is continuously tunable over the real interval [0, 1]. This model becomes the random network under the Achlioptas process at $t_0 = 0$ and the ER network at $t_0 = 1$. For the percolation process at $t_0 \leq 1$, we introduce a relatively slow-growing point, after which the largest cluster begins growing faster than that in the ER model. A weakly discontinuous transition is generated in the percolation process at $t_0 \leq 0.5$. We take the relatively slow-growing point as the lower pseudotransition point and the maximum gap point of the order parameter as the upper pseudotransition point. The critical point can be approximately predicted by each fitting function of the two points about t_0. This contributes to understanding the rapid mergence of the large clusters at the critical point. The numerical simulations indicate that the lower pseudotransition point and the upper pseudotransition point are equal in the thermodynamic limit. When $t_0 > 0.5$, the percolation processes generate a continuous transition. The scaling analyses of several quantities are presented, including the relatively slow-growing point, the duration of the relatively slow-growing process, as well as the relatively maximum strength between the percolation percolation at $t_0 < 1$ and the ER network about different t_0. The presented mechanism can be viewed as a two-stage percolation process that has many potential applications in the growth processes of real networks.

Keywords: percolation, phase transitions, networks

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1. Introduction

The percolation phase transition of a network through the gradual addition of links is used in many fields to model dynamics, such as social networks, physical systems, epidemics and spread of infectious diseases, and porous media.[1–6] A lot of papers have paid attention to whether or not the explosive percolation transition with the other growth rules[7–15] is continuous[16–21] since certain Achlioptas processes exhibit this phenomenon.[17] The model is considered as adding edges from N isolated vertices, where N is very large. At each step, two unoccupied edges are chosen totally randomly and only one of the edges is added to the growing network, according to some rules. In the whole process, only a rule plays a part. It was proposed that a hybrid model with an additional parameter q could interpolate between the classical percolation regime and the discontinuous explosive percolation.[14] It was also presented that the edge with a minimum product of two connecting cluster sizes is taken with a probability p from the two randomly chosen edges.[15]

In the real world, the evolution rule of a network can be changed. For example, the initial evolutionary rule of the epidemic's spread is replaced by another rule because of a gene mutation. In this paper, we put forward a two-stage percolation process. As opposed to the previous single rule case, the second stage evolves according to another rule following the result of the first stage. Such phase transitions contribute to our understanding of the critical behavior on the basis of the cluster size distribution of the first stage. Studying the phase transitions with a geometrical or topological confinement occurring in nature is an important and active area in network research.[9] In an epidemic, the emergence of a giant component corresponds to the outbreak.[8] Hence, the study of the first emergence of a giant component has attracted a lot of attention in percolation theory.[1,22,23] Predicting where the macroscopic components emerge abruptly is significant for the prevention of diseases. It is worthwhile to learn where to control the growth under the condition of the appearance of the largest component in many social and technological networked systems.[24,25] The numerical simulations can give approximate answers by introducing a relatively slow-growing point.

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2. Model formulation

As in other models, in our model the number of occupied links L increases one by one at each step and the occupied link density (or time), $t = L/N$, is a control parameter.[20] The model we considered is as follows. Start with N unconnected vertices. At each step, add an edge between two uniformly randomly chosen vertices. Repeat the procedure for a spell of t_0. This is the first stage of the model. The second stage follows the first stage. At moment t_0, the product rule (PR) replacing the rule in the first stage is introduced and the network proceeds the evolution until the N vertices become connected. The parameter t_0 is continuously tunable over the real number interval [0, 1]. Here, the PR rule is used to select the edge with the smaller product of the sizes of components containing its end vertices from the two random edges. In the whole process, it is noted that we consider those edges which connect different clusters. The order parameter is defined as $C(t_0,t)/N$, where $C(t_0,t)$ indicates the number of nodes belonging to the largest connected component in the network for our intervention model at t_0.[20] At the first stage, it is without competition in the process of adding edges, which is the same as that in the Erdös–Rényi (ER) model that adds edges in a completely random manner.[1–4] When it comes to the second stage, in which the PR rule is functional in the process, the added edges suppress the appearance of large clusters. At $t_0 = 1$, the model recovers to the classical ER model. It is also easy to know that the model reduces to the PR model with two competitive edges when $t_0 = 0$. The typical evolution of the order parameter $C(t_0,t)/N$ under different t_0 is shown in Fig. 1. In essence, t_0 represents the action time under the ER model. With the increase of t_0 from 0 to 0.5, the curve shifts to a smaller t, indicating accelerated transitions. This is due to the encouragement of the large components in the first stage. Once the PR rule is introduced, the large components are hindered to increase until the critical point, where nearly equal size clusters are linked together by additional few edges yielding weakly discontinuous transitions. When $t_0 > 0.5$, the order parameter undergoes a continuous transitions before t_0. It then experiences a relatively slow-growing process and a fast-growing process in comparison with the ER model.

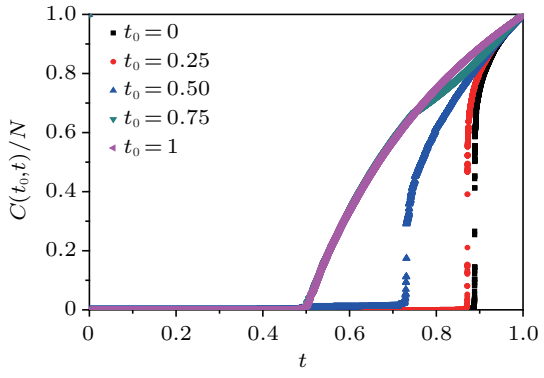


Fig. 1. (color online) Typical evolution of $C(t_0,t)/N$ vs. t for different t_0 under $N = 10^6$. The horizontal axis t represents the occupied edge density L/N. A single realization is shown for each percolation process.

3. Results and discussion

Many methods can be used to identify the critical point. One uses the scaling of the pseudotransition point.[26] Another method finds the critical value t_c where the percolation strength yields the best power-law fit about N.[26] The third method consists of studying the behavior of the width of the transition window, which describes the nature of the transition.[17] Based on the third method, they take the instant when the cluster size heterogeneity becomes maximum as the lower pseudotransition point and define the moment when the second-largest cluster size becomes maximum as the upper pseudotransition point.[20] In this paper, we define a relatively slow-growing point after which the order parameter will grow faster than that in the ER model. The relatively slow-growing point t_s is defined as

$$t_s = \left\{ t : h = \max_{t\in[0,1]} \frac{1}{N}(C(1,t) - C(t_0,t)) \right\}, \tag{1}$$

where h indicates the relatively maximum percolation strength between the intervention model at t_0 and the ER model. The relatively slow-growing point t_s indicates the t when h obtains its maximum. For the percolation process at $t_0 \leq 0.5$, the relatively slow-growing point is taken as the lower pseudotransition point. Starting from the intervention time t_0, h becomes larger and larger with the increase of t until the point where the order parameter begins increasing faster than that in the ER model. In the thermodynamic limit, the order parameter is zero below the critical point t_c and becomes finite for $t > t_c$.[20] Thus, it is reasonable to consider the relatively slow-growing point as a preceding symptom of the percolating onset. The upper pseudotransition t_f is defined as the moment when the maximum gap of the order parameter just closes. To predict the scope of the critical point in finite systems, we give a fitting curve for the lower and the upper pseudotransition points, respectively, in Fig. 2(a). The upper pseudotransition point is given by

$$\begin{aligned} t_f = & -0.00398\exp(t_0/0.14548) \\ & - 3.13305\times 10^{-16}\exp(t_0/0.0155) + 0.89323. \end{aligned} \tag{2}$$

Accordingly, the lower pseudotransition point is expressed as

$$\begin{aligned} t_s = & -0.00403\exp(t_0/0.14584) \\ & - 8.46953\times 10^{-16}\exp(t_0/0.01597) + 0.89076. \end{aligned} \tag{3}$$

The gap between t_s and t_f is also ploted in Fig. 2(b). This indicates the critical width of the phase transition. The critical width is basically increasing with the increase of t_0. In

Fig. 2(c), we find t_s and t_f for different system sizes. It is clear that the critical width between t_s and t_f becomes smaller and smaller with the increase of the system size.

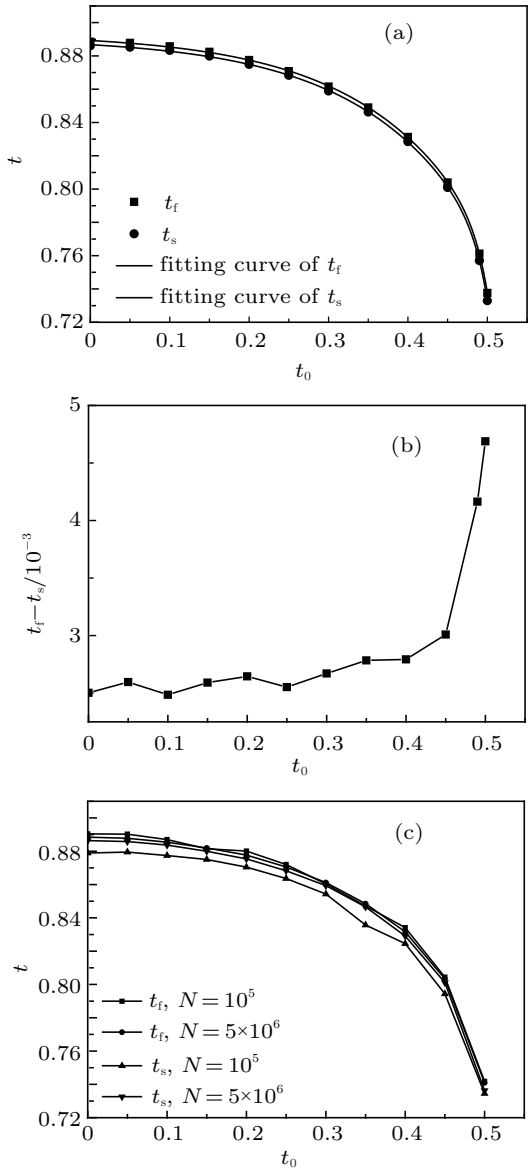


Fig. 2. (a) Two curve fittings to the lower and the upper pseudotransition points as a function of t_0. Each data point is an average over 100 realizations. The system size is fixed at $N = 2 \times 10^6$. (b) The gap between t_s and t_f vs. t_0 in panel (a). (c) The t_s and t_f vs. t_0 at $N = 10^5$ and $N = 5 \times 10^6$, respectively. A single simulation is presented.

In Fig. 3, two curve fittings about the average values of t_s and t_f against different system sizes are ploted and the corresponding fitting functions are also presented. For $t_0 = 0$, our model is equivalent to the PR model without adding intracluster edges. As clearly seen in Fig. 3(a), t_f declines very slowly and t_s ascends quickly to approach t_f with the increase of the system size. The upper pseudotransition point t_f and the lower pseudotransition point t_s converge to the same critical point at $N > 10^8$, which is beyond our numerical simulation's ability. Similarly, t_s and t_f converge to the same critical point at $N > 10^9$ in the numerical simulation shown in Fig. 3(b).

It should be pointed out that the system size, where t_s and t_f converge to the same critical point, becomes larger and larger with the increase of t_0. Therefore, this is the reason why the critical width is basically increasing with the increase of t_0 in Fig. 2(b). Furthermore, it indicates that the upper pseudotransition point t_f and the lower pseudotransition point t_s are equal in the thermodynamic limit for the percolation process with $t_0 \leq 0.5$. As can be seen in Fig. 3, with the increase of t_0, t_f declines more and more quickly and t_s begins to descend at $t_0 = 0.5$, which suggests that there exists some t_0 in [0, 0.5] with which t_s does not change with the increase of the system size.

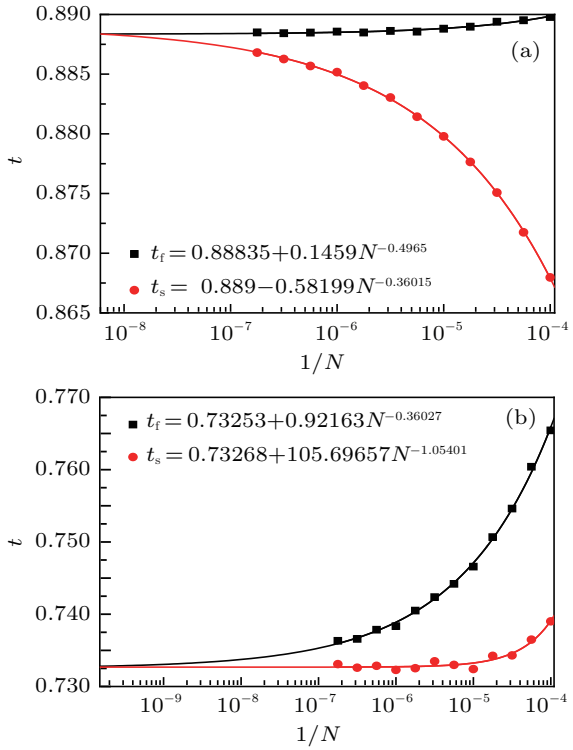


Fig. 3. (color online) (a) Two curve fittings about the lower and the upper pseudotransition points at $t_0 = 0$. The data points are obtained by averaging over 30–300 different realizations for each system size. As N increases, t_s (t_f) increases (decreases), approaching to the same critical point in the the thermodynamic limit. (b) The same plot as in panel (a) at $t_0 = 0.5$. As N increases, t_s and t_f decrease to approach to the same critical point in the the thermodynamic limit. The error bars for data points are within the symbol sizes.

The cluster size distribution for the percolation process at $t_0 = 0.5$ is presented in Fig. 4. Measurements have been performed at the lower and the upper pseudotransition points. As seen in the cluster size distribution, almost all those clusters which are larger than the other clusters merge to finish the percolation process from t_s to t_f. In Fig. 4(a), there is a remote cluster which corresponds to the giant cluster.

To further investigate the dynamics of the relatively maximum percolation strength and the relatively slow-growing point, the finite-size study for h and t_s is given in Fig. 5. The relatively slow-growing duration time defined by $t_s - t_0$

is also presented in Fig. 5. The relatively maximum percolation strength has a fast decline for the percolation process at $t_0 > 0.5$ in Fig. 5(a). This happens because the order parameter begins increasing fast at $t_0 = 0.5$ in the first stage and the PR rule with two candidate edges has not fully suppressed the growth of the largest cluster to yield a weakly discontinuous transition. The relatively slow-growing point has a fast decline for the percolation process at $t_0 = 0.5$. This hapens because those clusters which can be effectively suppressed become less and less under the PR rule with two candidate edges. At some t_0, it begins to ascend because t_s is always larger than t_0. The relatively slow-growing duration time is a very important parameter, especially in the network control, where it describes the effective suppression time. Figure 5(c) shows that $t_s - t_0$ has a linear drop area till the point t_0 which approximately equals to 0.4. It is easy to estimate the lower pseudotransition by the fitting function. The absolute value of t_0's coefficient in the right side is a little larger than 1, which suggests that the suppression effect on the large clusters between the PR model and our model at $t_0 < 0.4$ is nearly the same. On the evolution of the network in our model, figure 5 can approximately guide us when to take the intervention according to the actual requirements in network controlling.

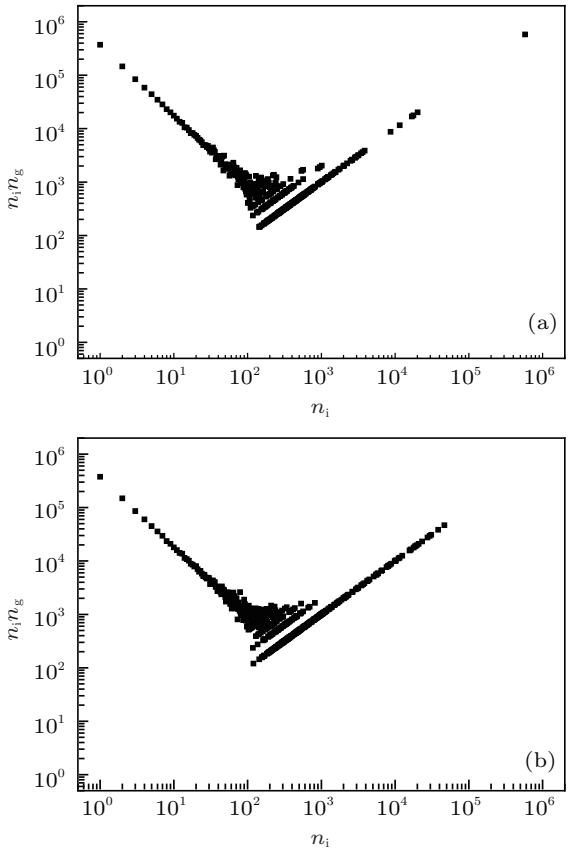


Fig. 4. (a) The cluster size distribution at the upper pseudotransition point. (b) The cluster size distribution at the lower pseudotransition point. The horizontal axis n_i represents the cluster size. The vertical axis $n_i n_g$ represents the product of the cluster size and the number corresponding to the cluster size. The system size is fixed at $N = 2 \times 10^6$ and the percolation process is at $t_0 = 0.5$.

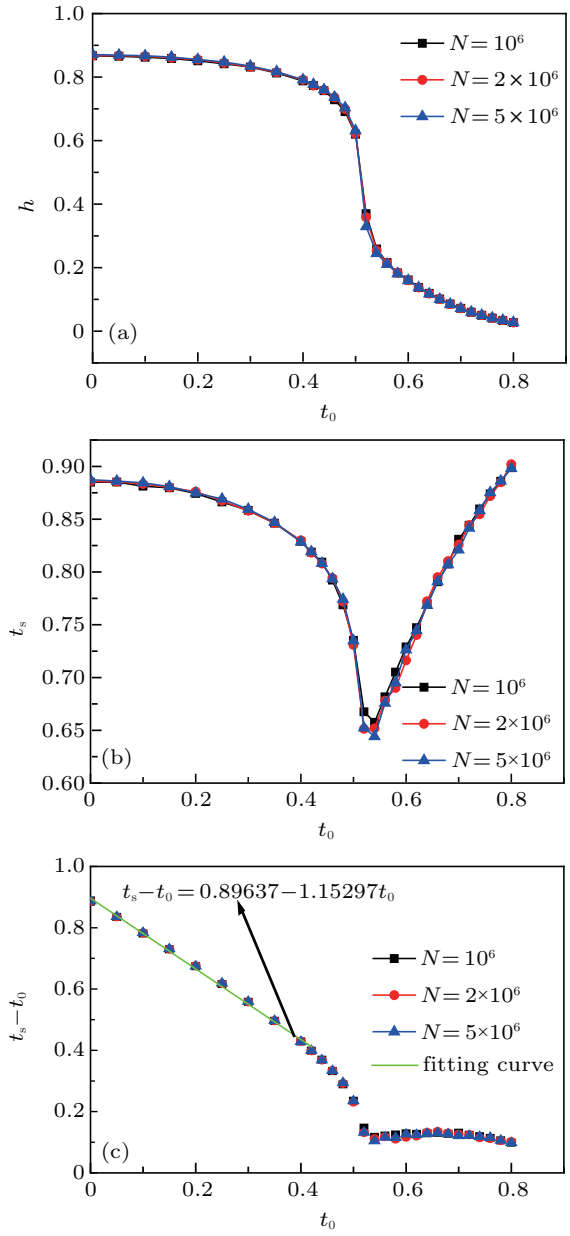


Fig. 5. (color online) (a) The relatively maximum percolation strength h vs. t_0. (b) The relatively slow-growing point t_s vs. t_0. (c) The relatively slow-growing duration time $t_s - t_0$ vs. t_0. A single realization is shown for each system size.

4. Summary

A two-stage percolation process is presented in this paper. This model becomes a random network under the Achlioptas process at $t_0 = 0$ and the ER network at $t_0 = 1$. Our model is helpful for desigining new random ER-type networks with desired characteristics where explosive transitions are desired in some situations and not in others.[18] When $t_0 \leq 0.5$, it generates a weakly discontinuous transition. We can predict the lower pseudotransition point by introducing a relatively slow-growing time point. Correspondingly, the upper pseudotransition point is also predicted. The numerical simulations indicate that they should be equal in the thermodynamic limit. Thus, we may have a good knowledge of the critical behavior

of the phase transition based on the cluster size distribution of the two points. For $t_0 > 0.5$, the percolation process undergoes a continuous transition. This is the same as that in the ER model before t_0. Starting from t_0, it experiences a relatively slow-growing process, after which the largest component begins to grow faster than that in the ER model. Several interesting characteristic quantities varying with t_0 are discribed. On the evolution of the networks, these parameters can help us to learn when the networks begin to grow faster than the ER network, they can also indicate the effective suppression time in our model. In addition, they also can tell us when to take interventions according to actual demand in network control.

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