TOPICAL REVIEW — Statistical physics and complex systems

Sub-diffusive scaling with power-law trapping times*

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Thermally driven diffusive motion of a particle underlies many physical and biological processes. In the presence of traps and obstacles, the spread of the particle is substantially impeded, leading to subdiffusive scaling at long times. The statistical mechanical treatment of diffusion in a disordered environment is often quite involved. In this short review, we present a simple and unified view of the many quantitative results on anomalous diffusion in the literature, including the scaling of the diffusion front and the mean first-passage time. Various analytic calculations and physical arguments are examined to highlight the role of dimensionality, energy landscape, and rare events in affecting the particle trajectory statistics. The general understanding that emerges will aid the interpretation of relevant experimental and simulation results.

Keywords: anomalous diffusion, subdiffusive scaling, random walk in a disordered medium, glass transition

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1. Introduction

The Brownian motion of a particle in a fluid, driven by random molecular collisions, is an archetypical example of a large class of fundamental physical processes in nature. In the absence of a confining potential, the size L of the region explored by the particle in a time t increases as $L \sim t^{1/2}$, independent of the dimensionality d of the medium. The first statistical mechanical theory of Brownian motion was developed by Einstein in 1905. Since then, the study of Brownian motion and related problems has become an important branch of statistical mechanics that deals with the kinetics of thermal fluctuations and transport, among others.

In recent years, the development of single-molecule fluorescence microscopy has opened the door to direct observation of the spatiotemporal trajectory of nanometer-sized biomolecules in a living cell.[1–3] The time series data encodes many secrets of life which are of great interest to biologists including, e.g., kinetic trapping of a protein in a certain part of the cell, the frequency with which it meets its binding partners or other targets, the speed of migration to other parts of the cell, through random diffusion or guided motion, etc. The statistical mechanical studies carried out so far, as reviewed recently by Barkai *et al.*,[4] have generally indicated subdiffusive motion of the molecules investigated. One suggested explanation for the subdiffusive behavior is the crowded and heterogeneous environment encountered by a diffusing molecule in a living cell[5,6] that impedes its motion. The precise nature of such hinderance, however, is under active debate, as the many different types of proteins and other organelles in the cell often have their own dynamics which are complex and ill-characterized, so it is not clear if the barriers to diffusion should be treated as quenched or annealed disorder in theoretical descriptions. This is particularly so for proteins imbedded in a membrane as discussed in, for example, Refs. [7] and [8].

There have been a number of reviews on random walks in a random environment over the years. A pedagogical review on random walks on ordered and disordered lattices was given by Haus and Kehr[9] in 1987. The authors discussed in detail several important models including the continuous time random walk (CTRW), the quenched trap model (QTM), the random barrier model, etc. However, their treatment was restricted to the normal diffusive regime. Anomalous diffusion in disordered media was reviewed by Bouchaud and Georges,[10] especially from the point of view of the breaking of the central limit theorem, localization and Arrhenius behavior at low temperatures. The 2005 review by Iglói and Monthus[11] focuses on the renormalization group approach to the strongly disordered systems. Very recently, Barkai *et al.*[4] summarized the key theoretical results on different types of subdiffusive behavior in connection with the interpretation of single-molecular experiments.

The purpose of this short review is to present a simple and unified view of anomalous diffusion so as to make the somewhat specialized mathematical literature accessible to a broader audience. We will focus on elucidating the relation between the broadly-distributed sojourn time at a given site and the total time t needed to explore a region of size L. Systems with a more or less constant sojourn time t_0 exhibits nor-

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mal diffusion with $t \simeq t_0 L^2$. For power-law distributed sojourn times which lack a specific time scale, one might still expect a power-law scaling

$$t \simeq t_0 L^z, \tag{1}$$

where z is known as the dynamic exponent. The question is then under what conditions will z differ from 2, and when and how will the dimensionality of the system enters the discussion. In addition, the broad distribution of the sojourn times may also lead to a broad distribution of the particle displacement in different realizations of the stochastic process. Methods to unambiguously characterize the ensemble of trajectories are often needed in the analysis of experimental data on, e.g., the motion of proteins inside a cell.

The paper is organized as follows. In Section 2 we review the Scher–Montroll–Weiss theory of continuous time random walk which also serves as the annealed approximation to the models discussed in later sections. Section 3 presents a discussion of random walk in a quenched random environment, focusing mostly on a model where the random site energies follow the Gumbel distribution. In Section 4 we examine the first-passage time statistics of a random walker in a logarithmically correlated potential. A short summary is presented in Section 5.

2. Continuous time random walk

2.1. Diffusion on a lattice

To set the stage and introduce the basic notation, let us briefly revisit the random walk problem on a d-dimensional hypercubic lattice with lattice constant a. The trajectory of a particle in a random walk is generated with a stochastic rule. In each time step, the particle hops from its current site to one of its $2d$ neighbors with equal probability. Let $\boldsymbol{x}_N$ be the position of the particle after N steps, starting from the origin at $t = 0$. The mean-square displacement of the particle, averaged over many repetitions of the process, is given by

$$\xi_N^2 \equiv \left\langle |\boldsymbol{x}_N|^2 \right\rangle = Na^2. \tag{2}$$

The linear growth of ξ_N^2 with the step number N simply follows from the statistical independence of individual steps along the path, which in this case has zero mean and variance a^2.

A full characterization of the random walk process requires knowledge of the probability density function (PDF) $P(\boldsymbol{x},N)$, i.e., the probability of finding the particle at $\boldsymbol{x}$ after N steps. In the present case, $P(\boldsymbol{x},N)$ satisfies the master equation or the lattice diffusion equation in discrete time

$$P(\boldsymbol{x},N+1) = \frac{1}{2d} \sum_{\boldsymbol{y}\in \text{n.n. of } \boldsymbol{x}} P(\boldsymbol{y},N). \tag{3}$$

While the above equation can be solved exactly for arbitrary initial distribution $P(\boldsymbol{x},0)$ using the lattice Green's function and generating function techniques,[12] its long-time, large distance properties are more conveniently discussed by taking the continuum limit of Eq. (3), which yields the familiar diffusion equation

$$\frac{\partial P}{\partial t} = D\boldsymbol{\nabla}^2 P. \tag{4}$$

Here, $t = Nt_0$ with t_0 being the time interval between successive hops. The diffusion constant $D = a^2/(2dt_0)$. It is then easy to show that, launching the particle from the origin at $t = 0$, solution to Eq. (4) takes the scaling form

$$P(\boldsymbol{x},t) \simeq \frac{1}{(2Dt)^{d/2}} f\left(|\boldsymbol{x}|/(2Dt)^{1/2}\right). \tag{5}$$

The scaling function $f(u)$ describes the asymptotic shape of the PDF and in particular the diffusion front. For the lattice walk, $f(u) = (2\pi)^{-d/2}\exp(-u^2/2)$.

Another characteristic of the diffusion process is the first-passage time (FPT) $t_{\rm F}$. In the description of reaction kinetics, for example, $t_{\rm F}$ is the time required for the diffusing particle to reach its reaction partner or target site. Depending on the system to be modeled, the target may be a single site, or a collection of sites that form a set $\mathcal{B}$.

Due to the stochastic nature of the diffusion process, the FPT $t_{\rm F}$ is a random variable whose distribution can be calculated by following the diffusion of the particle in the presence of absorbing sites $\mathcal{B}$. Mathematically, the PDF of the particle satisfies Eq. (3) or Eq. (4) with the sink set $\mathcal{B}$. In the case of the lattice walk, the mean first-passage time $\langle t_{\rm F}\rangle$ to reach an enclosing surface at distance L from the launching site scales as $\langle t_{\rm F}\rangle \simeq t_0(L/a)^2$, in agreement with the scaling of the diffusion front Eq. (5).[12] This is also expected to be true in the subdiffusive case. However, caution needs to be taken when $t_{\rm F}$ itself has a broad distribution, a situation we encounter in Section 4.

We note in passing that, for $d > 2$, the mean first-passage time to a given site (or a finite set of sites, but not an enclosing boundary) diverges. This is due to the fact that, in such a situation, there is a finite probability for the particle to avoid the sink site(s) and escape to infinity.[12] The $d = 2$ case is marginal with various power-laws replaced by logarithms.

2.2. Continuous time random walk and anomalous diffusion

The continuous time random walk was introduced by Montroll and Weiss[13] in 1965 and then proposed by Scher and Montroll[14] in the 1970's as a generic model for subdiffusive scaling (i.e., $z > 2$) seen in many experiments and simulations. Instead of assigning a fixed time t_0 to the time interval between successive hops as in Section 2.1, one draws the waiting time or sojourn time t_n from a distribution $\psi(t)$. The total time for the walker to complete a journey of N steps is given by $t = \sum_{n=1}^{N} t_n$.

In the case where the sojourn times t_n are chosen statistically independently from each other, the trajectory of the particle on the lattice is decoupled from the time it spends to complete the journey. This property, known as subordination,[15–19] greatly simplifies the mathematical treatment of the CTRW. In particular, the PDF of a particle launched from the origin at $t = 0$ can be expressed as

$$P(\boldsymbol{x},t) = \sum_n P_L(\boldsymbol{x},n)\Psi_n(t), \tag{6}$$

where $P_L(\boldsymbol{x},n)$ is the probability to reach site $\boldsymbol{x}$ in n steps in the lattice walk introduced in Section 2.1, and $\Psi_n(t)$ is the probability of making n jumps in a time interval t. It is straightforward to show that the Laplace transform of $\Psi_n(t)$ is given by

$$\hat{\Psi}_n(s) \equiv \int_0^\infty \Psi_n(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = [\hat{\psi}(s)]^n, \tag{7}$$

where $\hat{\psi}(s) = \int_0^\infty \psi(t)\exp(-st)\,\mathrm{d}t$ is the Laplace transform of the sojourn time distribution $\psi(t)$.

The scaling property of $P(\boldsymbol{x},t)$ depends on whether the mean sojourn time $t_\mathrm{w} = \int_0^\infty t\,\psi(t)\,\mathrm{d}t$ is finite or infinite. When t_w is finite, in the long time limit $t \to \infty$, the central limit theorem guarantees that the total time $t \simeq Nt_\mathrm{w}$. In this case the scaling (5) holds, i.e., the particle behaves as a normal random walker at long times. On the other hand, in the case $t_\mathrm{w} = \infty$, anomalous diffusion is expected. In addition, the random walk process is non-stationary and exhibits a phenomenon known as weak ergodicity breaking,[4,10,20,21] as explained in Section 2.3.

In the special case of a power-law sojourn time distribution,[10]

$$\psi(t) \simeq t_0^\mu t^{-(1+\mu)}, \quad 0 < \mu < 1, \tag{8}$$

the first moment indeed diverges. It is easy to show that the longest time in N draws from the distribution scales with N as $t_\mathrm{max} \simeq t_0 N^{1/\mu}$, which grows faster than linear in N. Thus, the total time $t = \sum_{n=1}^N t_n$ spent by the particle in N steps is dominated by the longest time t_max in the sum, leading to the scaling

$$t \simeq t_\mathrm{max} \simeq t_0 N^{1/\mu}. \tag{9}$$

On the other hand, the mean-square displacement of the particle in N steps is still proportional to N. Hence

$$\left\langle |\boldsymbol{x}|^2 \right\rangle \sim \begin{cases} (t/t_0)^\mu, & \text{for } 0 < \mu < 1; \\ t/[t_0 \ln(t/t_0)], & \text{for } \mu = 1. \end{cases} \tag{10}$$

By making use of the subordination property equation (6), one can show[10] that the PDF satisfies scaling

$$P(\boldsymbol{x},t) = t^{-d/z} F_\mu(\boldsymbol{x}/t^{1/z}), \tag{11}$$

with the dynamic exponent

$$z = 2/\mu. \tag{12}$$

The scaling function decays for large u as $F_\mu(u) \sim \mathrm{e}^{-A_\mu u^\delta}$, where $\delta = 2/(2-\mu)$ and A_μ is a constant that depends on μ.

2.3. Weak ergodicity breaking

For the CTRW with an infinite mean sojourn time, dominance of the longest sojourn time in completing a given walk introduces unusual features in the statistics.[10] For example, one may consider the mean-square displacement of the particle over a time interval Δ and average it over the total duration t of the walk

$$\overline{\delta^2(\Delta)} = \frac{1}{t-\Delta}\int_0^{t-\Delta} \mathrm{d}t'\,[\boldsymbol{x}(t'+\Delta) - \boldsymbol{x}(t')]^2. \tag{13}$$

For a longer walk, there is a larger percentage of the total elapsed time where the displacement vanishes, i.e., the particle stays put. Hence, we expect Eq. (13) to decrease with increasing t for a fixed Δ. This phenomenon was analyzed in detail by He *et al.* in Ref. [22]. Their analytic and numerical calculations show that, for $\mu < 1$,

$$\overline{\delta^2(\Delta)} \sim D_\mu t^{\mu-1}\Delta. \tag{14}$$

The amplitude of the time-averaged mean square displacement varies greatly from trajectory to trajectory, but the linear scaling with the time interval Δ holds for all trajectories with $\Delta \ll t$.

It is interesting to compare Eq. (14) with the expected scaling,

$$\langle [\boldsymbol{x}(\Delta) - \boldsymbol{x}(0)]^2 \rangle \simeq D_\mu \Delta^\mu, \tag{15}$$

where the angular brackets denote average over different realizations of the random walk process, i.e., the ensemble average. Here we have an explicit example where the time average equation (14) over a single trajectory is reduced from the ensemble average equation (15) by a factor $(\Delta/t)^{1-\mu}$ that tends to zero in the limit $t \gg \Delta$. This behavior is termed "weak ergodicity breaking" by Bouchaud[20] in comparison with aging phenomenon in glassy systems. As time goes on, the particle spends longer and longer time at a given site, thereby reducing the value of the mean-square displacement over a fixed time interval. The result has important implications on the analysis and modeling of single-molecule experiments, as discussed by Barkai and collaborators.[4] Note however that, when the walker is confined to a region of finite volume, it will be able to visit all sites with equal probability given sufficient time. Therefore the breakdown of ergodicity here is a subtle feature that depends on how the "thermodynamic limit" is taken.

3. Random walk in a random environment

3.1. Trap versus Langevin dynamics

The CTRW is a nice model to analyze mathematically due to the subordination property, but its physical realization requires the assumption of an internal process that controls the sojourn time. The broad distribution of sojourn times for anomalous diffusion is realized when the particle instead diffuses in a disordered medium specified by a set of site-specific random energies V_i.[23] At equilibrium, the probability of finding the particle at site i is given by the Boltzmann distribution $P_i^{\rm B} \propto {\rm e}^{-V_i/T}$, where T is temperature. The dynamics of a particle diffusing in such a medium has been studied in a class of Markov models defined by the hopping rate from site i to a nearby site j,

$$w_{i\to j} = \omega_0 \exp[(\alpha V_j - (1-\alpha)V_i)/T]. \tag{16}$$

Here, ω_0 is a characteristic hopping frequency and α is a number between 0 and 1. It is easy to check that, for any value of α in this range, the detailed balance condition $w_{i\to j}P_i^{\rm B} = w_{j\to i}P_j^{\rm B}$ is fulfilled.

The case $\alpha = 0$ is known as trap dynamics (see Fig. 1(a)). Here the hopping rate out of a site i, $w_i = \sum_j w_{i\to j} = 2d\omega_0 \exp(V_i/T)$, depends only on the energy V_i at the site. All target sites have equal probability to be reached in one hop. This situation is the closest to the CTRW model and hence is more amenable to exact mathematical treatment. The sojourn time t of a particle at site i has an exponential distribution

$$\psi_i(t) = \tau_i^{-1} \exp(-t/\tau_i), \tag{17}$$

where

$$\tau_i = w_i^{-1} = (2d\omega_0)^{-1} \exp(-V_i/T) \tag{18}$$

is the mean sojourn time at site i. Since V_i enters on the exponential, small variations in V_i may lead to large variations in τ_i at low temperatures. The particle on average spends considerably longer time at sites with lower energies. Of particular interest is when the random energies are drawn independently from the exponential tail of the Gumbel distribution,

$$P(V) = \begin{cases} T_{\rm g}^{-1}\, {\rm e}^{V/T_{\rm g}}, & V < 0; \\ 0, & V \geq 0. \end{cases} \tag{19}$$

Consequently, the distribution of the τ_i is a power-law, $P(\tau) \sim \tau^{-(1+\mu)}$ with $\mu = T/T_{\rm g}$. We shall refer to this case as the quenched trap model (QTM).

The $\alpha = 1/2$ case is known as the Langevin dynamics (see Fig. 1(b)). Here $w_{i\to j} = \omega_0 \exp[\beta(V_j - V_i)/2]$ is set by the energy difference between the current and target sites. Such a rule can be viewed as the lattice version of the Langevin description of a Brownian particle driven by the gradient of the potential V_i. It may also be considered as a variant of the commonly adopted Metropolis *et al*. rule in Monte Carlo simulations of a particle in a potential.

Even though both dynamics satisfy the detailed balance condition and yields the equilibrium distribution $P_i^{\rm B}$ at long times when the particle is confined to a finite volume, the precise relation between the statistics of the trajectories followed by the particle under the two types of hopping rules is still an open issue. At short times, one expects quite different behavior as the hopping rates can be drastically different in the two cases. On the other hand, if some sort of local equilibrium is established after some time before the particle escapes to other parts of the system, then the difference in the diffusion profile $P(x,t)$ is expected to be much less significant. Whether this actually happens depends on many detailed properties of the system, not least its dimensionality. For the spatially uncorrelated random energies V_i drawn from Eq. (19), Monthus has shown that, in the limit $T \sim 0$, the Langevin dynamics can be mapped to the trap dynamics on certain renormalized landscape.[23]

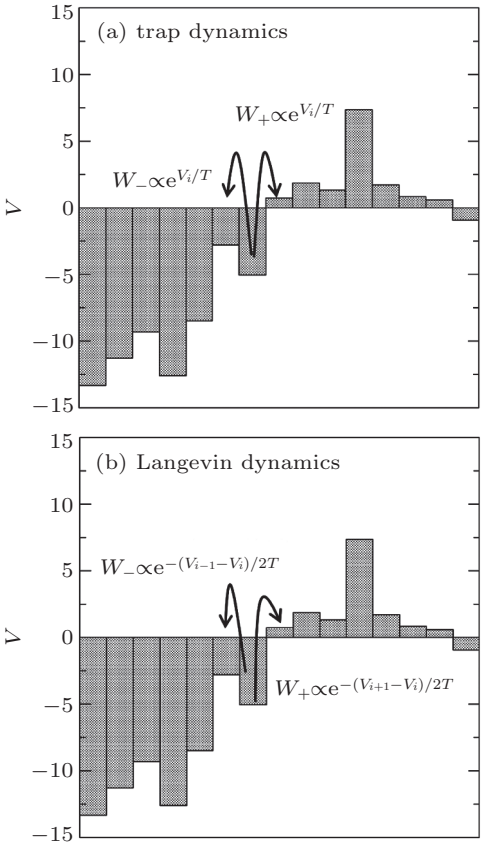


Fig. 1. Diffusion of a particle in a random energy landscape under (a) trap dynamics, and (b) Langevin dynamics.

3.2. Annealed approximation to the quenched trap model

The trap dynamics is the closest to the solvable CTRW model and hence if certain correspondence between the two can be established, then various properties of the former are readily understood. For sure, since the trap dynamics yields

the site-dependent Boltzmann distribution $P_i^{\rm B}$ when the particle is restricted to move in a finite volume for a long time, the equilibrium behavior under the two dynamics must differ. On the other hand, when the particle is left to explore an open geometry with an ever expanding horizon, its nonequilibrium diffusion front may exhibit the same scaling with time in the two cases.

To identify the conditions for the scaling equivalence, let us consider a typical path Γ of the normal random walk on the lattice. To complete the path, the time spent on each step t_i is distributed according to Eq. (17) in the QTM, but is drawn from the same distribution $\psi(t)$ in the CTRW model. If each site is visited only once, and the τ_i's for different sites of Γ are statistically independent, then the total journey time in the QTM follows the same distribution as that of the CTRW if we choose

$$\psi(t) = \langle \psi_i(t) \rangle_{\tau_i}. \tag{20}$$

In particular, if τ_i satisfies a power-law distribution, the sojourn time t in the annealed approximation is also power-law distributed with the same exponent.

Bouchaud and collaborators[10] proposed a simple argument on the critical dimension of the QTM. For $d > 2$, the lattice walk is non-space-filling in the sense that the number of distinct sites visited by an N-step walk, S_N, is much less than the total number of sites $N^{d/2}$ in a volume defined by the diffusion length $L \sim N^{1/2}$. In addition, $S_N \sim N$ so that any given site on the path is visited not more than a few times. In such a situation, the CTRW model is expected to capture the asymptotic behavior of the QTM at long times. On the other hand, when $d < 2$, $S_N \simeq N^{d/2} \ll N$ for large N. The frequent revisits to the same site with the same mean sojourn time invalidate the renewal assumption in the CTRW approximation.

The above qualitative consideration is in agreement with the real-space renormalization group calculation by Machta[24,25] on the QTM. His results suggest that, for $T < T_{\rm g}$, the fixed point indeed locks to that of the CTRW model at $\mu = T/T_{\rm g}$ for $d > 2$, while correlations in the sojourn times leads to a different behavior for $d < 2$.

3.3. Exact results in one dimension

The annealed approximation to the QTM as discussed in the previous section fails in one dimension. Nevertheless, a number of exact results have been derived for the exponentially distributed random energies equation (19) with the help of exact calculations and renormalization group analysis, as we review briefly below.

For the QTM at $d = 1$ and $\mu = T/T_{\rm g} \leq 1$, the scaling of the root-mean-square displacement with time has been well established (See, e.g. Ref. [10]),

$$\xi(t) \sim \begin{cases} t^{\mu/(1+\mu)}, & \text{for } 0 < \mu < 1; \\ t^{1/2}(\ln t)^{-1/2}, & \text{for } \mu = 1. \end{cases} \tag{21}$$

Compared to Eq. (12) for the annealed case, the dynamic exponent

$$z = 1 + \mu^{-1} \tag{22}$$

takes a smaller value, i.e., the diffusion length increases faster with t. A full description of the diffusion profile that characterizes the statistics of the particle trajectories was achieved in 2003 by Bertin and Bouchaud[26] and by Monthus.[23]

Generally, in the long time limit, the disorder-averaged diffusion profile satisfies scaling

$$\langle P(x,t) \rangle = \xi^{-1}(t) f_\mu(x/\xi), \tag{23}$$

where $\xi(t)$ is given by Eq. (21). The scaling function $f_\mu(y)$ was presented explicitly by Bertin and Bouchaud[26] for $\mu \simeq 1$ and by Monthus[23] for $\mu \simeq 0$. In the case $\mu \simeq 1$ which is close to the transition between normal and anomalous diffusion, the frontal part of the profile is given by

$$f_{\mu\simeq 1}(y) \propto |y|^{-(1-\mu)/2} \, {\rm e}^{-b|y|^{1+\mu}}, \quad y \to \infty. \tag{24}$$

Here, b is a parameter that depends on μ. Equation (24) reduces to the gaussian distribution at $\mu = 1$. Close to the origin,

$$f_{\mu\simeq 1}(y) \propto f_0 - f_1(1-\mu)|y|^\mu - f_2|y|^2, \quad y \to 0, \tag{25}$$

where f_0, f_1, and f_2 are parameters that depend on μ. Using a real-space renormalization approach, Monthus estimated the diffusion front in the low temperature limit $\mu \simeq 0$,

$$f_{\mu\simeq 0}(y) \propto |y|^{-1} \, {\rm e}^{-|y|}, \quad y \to \infty, \tag{26}$$

$$f_{\mu\simeq 0}(y) \propto 1 - |y| \left[\ln \frac{1}{|y|} - \gamma_{\rm E} \right], \quad y \to 0, \tag{27}$$

where $\gamma_{\rm E}$ is the Euler constant. More recently, by further generalizing the subordination scheme to the QTM with multiple visits to sites with long sojourn times, Burov and Barkai[27,28] analytically and numerically estimated the diffusion front in the whole subdiffusive regime. Quite generally, due to the broad distribution of the sojourn times, the diffusion front has a slower than gaussian decay at far front and a cusp singularity at the origin.

Monthus[23] also studied the general kinetic model defined by Eq. (16) with an arbitrary $\alpha \in [0,1]$. Here again the random potentials follow the exponential distribution equation (19). With the help of a real-space renormalization procedure, she showed that the dynamic exponent is given by Eq. (22), independent of α.

4. First-passage times in correlated landscapes

4.1. General results on diffusion in a correlated landscape

The previous section focused on diffusion in a random environment where the site energies V_i are independently selected from a given distribution. There is another class of problems where the site energies follow a gaussian distribution but with spatial correlations of the form,

$$\left\langle |V(\boldsymbol{x}) - V(\boldsymbol{x}')|^2 \right\rangle \simeq |\boldsymbol{x} - \boldsymbol{x}'|^{2H}. \tag{28}$$

The energy landscape $V(\boldsymbol{x})$ is self-affine with an Hurst exponent H, and can be generated from independent, gaussian distributed Fourier amplitudes at different wavevectors. A well-known example is the famous Sinaï model[29] where each lattice site is assigned a random force. The potential $V(\boldsymbol{x})$, which is the integral of the force, has a correlation with $H = 1/2$. Scaling arguments[10,30,31] suggest that the diffusion length of the particle scales with t as

$$\xi \sim (\ln t)^{1/H}. \tag{29}$$

The scaling has shown to be exact for the Sinaï model in $d = 1$ and is verified in the $d = 2$ case by Monthus and Garel[32] using numerical renormalization methods.[33]

The logarithmic scaling equation (29) is quite natural in view of the fact that, for $H > 0$, the typical energy barrier $\Delta V \sim r^H$ increase as a power-law of the distance r. According to the Arrhenius law, the rate to escape from a local minimum of the potential is given by $k \sim \exp(-\Delta V/T)$. Consequently the time t to travel a distance r is estimated to be $t \simeq 1/k \sim \exp(\Delta V/T)$, leading to $r \sim (\Delta V)^{1/H} \sim (\ln t)^{1/H}$, i.e., Eq. (29).

The marginal case of logarithmically diverging landscape was analyzed by Carpentier and Le Doussal.[37] This model has a freezing transition in the equilibrium case.[38,39] The diffusion is anomalous at all temperatures, but it also exhibits a dynamic transition to the phase dominated by the deepest minimum at the same temperature. These properties are reviewed briefly below.

4.2. The random energy model

The logarithmically diverging potential can be defined via the correlator

$$\left\langle |V(\boldsymbol{x}) - V(\boldsymbol{x}')|^2 \right\rangle = 4\sigma \ln \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{a}, \quad \text{for} \quad |\boldsymbol{x} - \boldsymbol{x}'| \geq a. \tag{30}$$

As shown in Ref. [39], such a correlated landscape can be coded by a branching tree. Each generation of the tree is associated with a particular length scale. The potential $V(\boldsymbol{x})$ is a sum of contributions from logarithmically binned regions. Through this mapping, the task of computing the partition function of a particle moving in $V(\boldsymbol{x})$ is made equivalent to that of the directed polymer on the Cayley tree, which was solved by Derrida and Spohn in 1988.[40] Essential features of the solution are well captured by the random energy model (REM)[41] which is much easier to describe.

Consider a system of N states with energies x_i drawn independently from a given distribution $\psi(x)$. The partition function of the system is given by

$$Z_N = \sum_{i=1}^{N} \mathrm{e}^{-x_i/T}. \tag{31}$$

Let $x_{\min}$ denote the smallest of the N energies, and $\mathcal{N}(x)$ the number of energy levels less than x. For large N, the energy levels are densely distributed except very close to $x_{\min}$ (and perhaps also on the high energy side, which does not contribute significantly to the sum). In such a situation, one may write $\mathrm{d}\mathcal{N} \simeq N\psi(x)\,\mathrm{d}x$. The lowest energy $x_{\min}$ itself is a random variable. Its value x_0 in a typical realization can be estimated from the condition

$$N \int_{-\infty}^{x_0} \psi(x)\,\mathrm{d}x = 1. \tag{32}$$

Equation (31) can then be written as

$$Z_N = \mathrm{e}^{-x_{\min}/T} + \int_{x_0}^{\infty} \exp(-x/T)\,\mathrm{d}\mathcal{N}. \tag{33}$$

At high temperatures, the partition sum is typically dominated by the second term in Eq. (33). In this case, Z_N is well approximated by its "annealed average" $\langle Z_N \rangle = N\langle \mathrm{e}^{-x/T} \rangle$ with negligible sample-to-sample variations. This behavior changes below the freezing transition temperature $T_{\mathrm{g}}(N)$ where the ground state contribution, i.e., the largest term in the sum, dominates.

As an example, let us consider the gaussian random energies x_i with variance Δ,

$$\psi(x) = \frac{1}{\sqrt{2\pi\Delta}}\,\mathrm{e}^{-x^2/2\Delta}. \tag{34}$$

Equation (32) yields $x_0 \simeq -\sqrt{2\Delta \ln N}$. Switching of the two terms in Eq. (33) takes place at $T_{\mathrm{g}}(N) = \sqrt{\Delta/(2\ln N)}$. At this temperature, both terms are of the order of N^2.

4.3. Exact result for the mean first-passage time in one dimension

Castillo and Le Doussal (CLD)[43] analyzed the subdiffusive scaling of a particle in a logarithmically correlated potential $V(x)$ following the Langevin dynamics. In one dimension, the particle is released from the origin at $t = 0$ with the reflecting boundary condition at $x = 0$ and absorbing boundary condition at $x = L$. Previously, it has been shown that the mean FPT can be exactly evaluated as[44]

$$\tau(\{V\}) = \frac{1}{T} \int_x^L \mathrm{d}y \int_0^L \mathrm{d}x\, \mathrm{e}^{[V(y)-V(x)]/T}. \tag{35}$$

By introducing the REM partition function $Z_L(\{V\}) = \int_0^L \mathrm{d}x\, \exp[-V(x)/T]$, the upper and lower bound of τ can be estimated as

$$Z_{L/2}(\{-V\})Z_{L/2}(\{V\})/T \le \tau \le Z_L(\{-V\})Z_L(\{V\})/T. \quad (36)$$

The scaling of $\tau \sim L^z$ can then be obtained from the scaling of the partition function.

For the logarithmically growing potential equation (30), we have approximately $N = L$ and $\Delta = 2\sigma \ln L$ in the mapping to the REM. Therefore the freezing transition temperature is given by $T_g = \sigma^{1/2}$. The partition function of the REM has the scaling[41] $Z_L \sim L^{1+(T_g/T)^2}$ for $T > T_g$ and $Z_L \sim L^{2T_g/T}$ for $T < T_g$. Since $V(x)$ and $-V(x)$ have the same statistics in the present model, equation (36) then yields the dynamic exponent

$$z_{\mathrm{CLD}} = \begin{cases} 2 + 2\dfrac{T_g^2}{T^2}, & T > T_g; \\ 4\dfrac{T_g}{T}, & T < T_g. \end{cases} \quad (37)$$

Due to the logarithmically diverging potential, the diffusion is anomalous at all temperatures, but the exponent z increases less fast with T below the freezing transition.

Under the REM approximation, one can estimate the minimum value of the potential in a system of linear size L, $V_{\min} \simeq -2T_g \ln L$. The same procedure yields $V_{\max} \simeq 2T_g \ln L$. Thus the barrier to overcome to reach the system's boundary is $\Delta V \simeq V_{\max} - V_{\min} = 4T_g \ln L$. This yields an escape time $t \sim \exp(\Delta V/T) = L^{4T_g/T}$, in agreement with Eq. (37) at $T < T_g$. This argument shows that both the deepest trap and the highest barrier contribute to the slowing down of the diffusing particle under the Langevin dynamics.

We also note in passing that, for $T < T_g$, the partition function $Z_L(V)$ is dominated by $V_{\min}$ (and similarly $Z_L(-V)$ by $V_{\max}$) which varies from sample to sample. Therefore the mean FPT is also expected to vary from sample to sample, especially for $T < T_g$. The dynamic exponent given by Eq. (37) describes the scaling of the mean FPT with L for typical realizations of the disorder. Mathematically, it can be defined more rigorously through

$$z = \lim_{L\to\infty} \langle \ln \tau(L) \rangle / \ln L, \quad (38)$$

where the angular brackets denote averaging over the distribution of the random potential $V(x)$.

4.4. First-passage time in higher dimensions

Castillo and Le Doussal further performed numerical calculations of the mean first-passage time for the logarithmically growing potential in two dimensions. Their results are consistent with Eq. (37) except that the freezing temperature is replaced by $T_g = \sqrt{\sigma/2}$, the REM transition temperature at $d = 2$. Based on this result, they conjectured that Eq. (37) holds in all dimensions d with $T_g = \sqrt{\sigma/d}$.

Very recently, Luo and Tang[45] investigated diffusion in the same potential under the trap dynamics. For a particle released from the origin at $t = 0$, the mean first-passage time to an enclosing boundary at distance L can be expressed as a weighted average

$$\tau(\{V\}) = \sum_{i=1}^{N} w_i \tau_i. \quad (39)$$

Here $N \simeq L^d$ is the total number of sites in the enclosed volume, $\tau_i = t_0 \exp(-V_i/T)$ is the mean sojourn time at site i, and w_i is the mean number of visits to site i at $\boldsymbol{x}_i$, averaged over all first-passage trajectories. Due to the subordination property, w_i is independent of the disorder realization $\{V_i\}$ and can be obtained by solving the master equation (3) for the simple random walk. In the continuum limit and assuming spherical geometry, we have $w_i = w(|\boldsymbol{x}_i|)$ with

$$w(r) \simeq \begin{cases} \dfrac{1}{2-d}(L^{2-d} - r^{2-d}), & \text{for } d \neq 2; \\ \dfrac{1}{2\pi}\ln(L/r), & \text{for } d = 2. \end{cases} \quad (40)$$

With this weighting function, one can establish upper and lower bounds on $\tau(\{V\})$ by splitting the sum equation (39) into contributions from shells whose radii form a uniformly increasing series on the logarithmic scale.[30] Observing the fact that the largest shell makes the most contribution to $\tau(\{V\})$, one can show that $\tau \simeq t_0 L^{2-d} Z_N \sim L^z$ with

$$z = z_{\mathrm{mFPT}} = \begin{cases} 2 + d\left(\dfrac{T_g}{T}\right)^2, & T > T_g; \\ (2-d) + 2d\dfrac{T_g}{T}, & T < T_g. \end{cases} \quad (41)$$

Equation (41) differ from the CLD result equation (37) except at $d = 2$. The origin of the discrepancy is different for $d = 1$ and $d > 2$. For $d = 1$, the trap dynamics is insensitive to the energy barriers (i.e., those sites with $V_i > 0$) while the Langevin dynamics is. In fact, if we construct a new potential $\tilde{V}(x) = V(x)$ for sites with $V(x) \le 0$ and $\tilde{V}(x) = 0$ otherwise, then $Z_L(\{-\tilde{V}\}) \simeq L$ and hence, according to Eq. (36), $\tau_{\mathrm{CLD}} \sim L Z_L(\{V\}) \sim L^{z_{\mathrm{mFPT}}}$, the same as trap dynamics. Therefore in one dimension, the difference in scaling between the two types of dynamics can be attributed to the presence of energy barriers between traps.

For $d \ge 2$, the general consensus is that the two types of dynamics should yield the same subdiffusive scaling as particles moving under the Langevin dynamics can avoid regions of high potential energy and find saddle point passages between traps. Therefore the discrepancy between Eqs. (37) and (41) is somewhat puzzling. In the section below we consider the annealed approximation which sheds light on this issue.

4.5. Annealed approximation

Consider a CTRW process where the sojourn times follow the distribution equation (20), with $\tau_i = t_0 \exp(-V_i/T)$ and the V_i's drawn from the gaussian distribution equation (34). The distribution $\psi(t)$ is not a simple power-law function. Nevertheless, the total time to complete an N step walk can be estimated to be

$$t_N \simeq \sum_{i=1}^{N} \tau_i = t_0 \sum_{i=1}^{N} \mathrm{e}^{-V_i/T} = t_0 Z_N. \tag{42}$$

The typical number of steps to reach the boundary at distance L from the launching site is $N_{\mathrm{typ}} = L^2$. Hence $t_N \simeq t_0 Z_{L^2} \simeq t_0 L^{z_{\mathrm{CLD}}}$.

Given that the simple lattice walk is non-space-filling in $d > 2$, one may regard Eq. (42) to represent the typical first-passage time to the boundary. On the other hand, equation (41) gives the scaling exponent for the average transit time over all first-passage trajectories. For $T < T_{\mathrm{g}}$, for example, the former is dominated by the deepest trap on a path of the order of L^2 distinct sites, while the latter is dominated by the deepest trap in the whole volume of L^d sites. The difference indicates that the first-passage time itself has a broad distribution, even for a given realization of the disorder potential. Further work is required to clarify the details of the FPT statistics.

5. Summary and conclusions

In this short review, we have considered subdiffusive behavior arising from three different types of disorder. The CTRW process can be treated exactly. Here, anomalous diffusion is encountered when the mean sojourn time t_{w} diverges. This is the case for power-law distributed sojourn times with a decay exponent $\mu < 1$. The dynamic exponent $z = 2/\mu$ in any dimension. The system also exhibits a phenomenon known as weak ergodicity breaking due to the non-stationarity of the diffusion process in open geometry.

The quenched trap model, with the mean sojourn time at each site determined by the random potential on the site, also exhibits anomalous diffusion below a certain temperature T_{g}. The exponentially distributed random potential received most attention in the literature. In this case, the mean sojourn time satisfies a power-law distribution with a decay exponent $\mu = T/T_{\mathrm{g}}$. For $d > 2$, it is generally accepted that the CTRW model offers a good approximation to the QTM due to the finite number of returns in the corresponding lattice walk, leading to the dynamic exponent $z = 2T_{\mathrm{g}}/T$ below T_{g}. This mapping breaks down partially in $d = 1$ due to repeated returns. In the latter case, $z = 1 + T_{\mathrm{g}}/T$ below the glass transition.

The third type of disorder considered is due to a logarithmically correlated potential $V(x)$ whose strength also increases logarithmically with the system size L. The work by Castillo and Le Doussal yielded an expression for the dynamic exponent which, when expressed as a function of T/T_{g}, is the same in all dimensions. Calculation of the mean first-passage time under trap dynamics by Luo and Tang yielded a dimension-dependent expression for the dynamic exponent which agrees with the result of Castillo and Le Doussal only at $d = 2$. For $d = 1$, the discrepancy can be attributed to the absence of barrier effect in the trap model. For $d > 2$, on the other hand, the discrepancy is suggested to arise from the difference in the scaling of average and typical mean first-passage times. This conjecture needs to be examined further.

Although rigorous mathematical treatment of quenched disorder is usually quite involved, in many cases the CTRW model captures the key features of the statistics. This offers great simplification in the derivation of quantitative estimates of various scaling properties. Another key idea from the mathematical point of view is the applicability of the central limit theorem to the sum of random variables which have a broad distribution. The freezing transition, for example, is associated with the emergence of a single sojourn time dominating the total journey time to the target site. The Random Energy Model offers a particularly clear representation of such freezing phenomenon.

In all the cases examined, the dynamic and equilibrium freezing transitions take place at the same T_{g}. This is perhaps not too surprising as the dynamics of diffusion satisfies the detailed balance condition. On the other hand, diffusion in open geometry is at best a close-to-equilibrium process. Due to the non-space-filling nature of lattice walk in $d > 2$ and the existence of saddle points in the energy landscape, one expects the nonequilibrium effects to be stronger in higher dimensions. The broader distribution of first-passage times in the logarithmically correlated potential in higher dimensions is in agreement with this expectation.

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