Space–time fractional KdV–Burgers equation for dust acoustic shock waves in dusty plasma with non-thermal ions

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The KdV–Burgers equation for dust acoustic waves in unmagnetized plasma having electrons, singly charged non-thermal ions, and hot and cold dust species is derived using the reductive perturbation method. The Boltzmann distribution is used for electrons in the presence of the cold (hot) dust viscosity coefficients. The semi-inverse method and Agrawal variational technique are applied to formulate the space–time fractional KdV–Burgers equation which is solved using the fractional sub-equation method. The effect of the fractional parameter on the behavior of the dust acoustic shock waves in the dusty plasma is investigated.

Keywords: dust-acoustic waves, reductive perturbation method, modified Riemann–Liouville fractional derivative, space–time fractional KdV–Burgers equation

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1. Introduction

Dusty plasma is ubiquitous in most astrophysical space environments, such as lower and upper mesosphere, cometary's tails, planetary rings, planetary magnetosphere, interplanetary spaces, interstellar media, etc. Dusty plasma applications range from astrophysics to strongly coupled dusty plasmas and dusty plasma crystals.[1–6] Zhang and Xue[2] investigated the propagation of linear dust acoustic waves in inhomogeneous dusty plasmas due to spatial gradients of dust charge. They found that, the inhomogeneity and non-adiabatic dust charge fluctuation have strong influence on the frequency and the damping rate of waves. The nonlinear dust acoustic waves in collisional self-gravitating dusty plasma using the damped KdV equation have been explained.[3] Furthermore, the effects of the dust size distribution on the ultracold and multicomponent dust grains in cold quantum dusty plasma have been studied.[4,5] Ren *et al.*[6] studied the effects of impurity on eigenmodes in one-dimensional dusty plasma lattices. Akhtar *et al.*[7] investigated the Sagdeev's pseudo potential in two types of unmagnetized dust fluids (one cold and one hot) in the presence of Boltzmann ions and electrons. Nonthermal ions from the Earth's bow shock and foreshock have been detected by satellites.[8,9] Elwakil *et al.*[10] studied the effect of nonthermality of ions on the nature of dust-acoustic waves (DAWs) for two types of dust fluid in unmagnetized, collisionless dusty plasma consisting of electrons, charged non-thermal ions, and hot and cold dust grains. The dusty plasma medium with dissipative properties supports the existence of shock waves instead of solitons. The dust fluid dissipation can be caused by dust–dust collision, Landau damping, dust charge fluctuation, and dust fluid viscosity, which would modify the wave profile structures.[11–13] Nakamura *et al.*[13] observed shock solitary waves in unmagnetized dusty plasma. They showed that the development of the wave shock is due to the KdV–Burgers (KdV–B) equation. Mamun and Tasnim[14] have studied the dust-ion-acoustic shock and solitary waves in dusty electronegative plasma and discussed the basic properties of the shock and solitary waves, which are associated with the presence of positive ion dynamics and dust charge fluctuation. The effects of both the dissipation due to kinematic viscosity and the dispersion caused by the charge separation as well as the quantum tunneling associated with the Bohm potential on the formation of dust ion-acoustic shock waves has been studied by Misra.[15] The behavior of the dust acoustic shock waves caused by dust fluid viscosity in the dusty plasma has been investigated in two types of dust fluid in unmagnetized, collisionless dusty plasma consisting of electrons, singly charged ions, and hot and cold dust grains.[16]

The dusty plasma has dispersive and/or dissipative forces, which are non-conservative. The classical description of the forces with regular differential equations is not convenient to treat the effect of these forces. The most successful mechanism used to represent dispersion and energy dissipation is that of fractional order derivative, which can describe the non-conservative forces. During the last few years, several interesting models have been developed by using fractional calculus to study the physical processes in almost every field of science, engineering and mathematics. Some of the areas where fractional calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory.[17–20] There are many definitions of fractional derivatives such as

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Riemann–Liouville, Grünwald–Letnikov, and Riesz fractional derivatives, which are mostly used in fractional systems.[21,22] Recently, Jumaire derived a fractional derivative called modified Riemann–Liouville,[23,24] which was used to describe many different problems.[25–27] Based on Riewe's work,[28,29] Agrawal[30,31] used a variational technique to formulate the fractional equations of motion in terms of Riemann–Liouville fractional derivatives. Some other researchers[32–38] used the Euler–Lagrange equations to investigate different real problems.

Several methods have been used to solve fractional differential equations such as, the Laplace transform method, the Fourier transform method, the iteration method, and the operational method.[21,22] Recently, there have been various papers dealing with the existence and multiplicity of solutions of the nonlinear fractional differential equation using techniques of nonlinear analysis.[39–44] The semi-inverse method was proposed by He to establish generalized variational principles for physical problems.[45,46] It has been successfully applied to many nonlinear equations arising in mathematical physics and nonlinear sciences.[47–49]

Our motive here is to derive the space–time fractional KdV–Burgers (STFKdV–B) equation using semi-inverse method[45,46] and Agrawal's variational technique.[30,31] The derived equation is solved by using the fractional sub-equation method.[40,41] Moreover, the effects of the space–time fractional parameter on the electrostatic structures for a system of unmagnetized collisionless dusty plasma with nonthermal ions are studied.

2. Basic equations

We consider homogeneous unmagnetized plasma having electrons, singly charged ions, and hot and cold dust species. The electrons and ions are assumed to follow Boltzmann's distributions with $T_e, T_i \gg T_h$, where T_e and T_i are respectively the temperatures of the electrons and ions, and T_h is the temperature of the hot dust. The hot dust is assumed to be heated adiabatically. The one-dimensional continuity and momentum equations for cold and hot dust are respectively given by

$$\frac{\partial n_c}{\partial t} + \frac{\partial}{\partial x}(n_c u_c) = 0, \tag{1a}$$

$$\frac{\partial u_c}{\partial t} + u_c \frac{\partial u_c}{\partial x} - \frac{e\,Z_c}{m_c}\frac{\partial \phi}{\partial x} + \eta_1 \frac{\partial^2 u_c}{\partial x^2} = 0; \tag{1b}$$

$$\frac{\partial n_h}{\partial t} + \frac{\partial}{\partial x}(n_h u_h) = 0, \tag{2a}$$

$$\frac{\partial u_h}{\partial t} + u_h \frac{\partial u_h}{\partial x} - \frac{eZ_h}{m_h}\frac{\partial \phi}{\partial x} + \frac{1}{m_h n_h}\frac{\partial P_h}{\partial x} + \eta_2 \frac{\partial^2 u_h}{\partial x^2} = 0. \tag{2b}$$

For adiabatic hot dust, we have

$$P_h = P_{h0}(n_h/N_{h0})^{\gamma}, \tag{3}$$

where $\gamma = (N+2)/N$ and N is the number of degrees of freedom. For the present work, $N = 1$, and hence $\gamma = 3$ and $P_{h0} = N_{h0}T_h$. The Poisson equation can be written as

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(n_e - n_i + Z_c n_c + Z_h n_h). \tag{4a}$$

In equilibrium we have

$$n_{i0} = n_{e0} + Z_c\, N_{c0} + Z_h\, N_{h0}. \tag{4b}$$

The Boltzmann distributed electrons follow the density equation

$$n_e = n_{e0}\, \exp(e\phi/T_e), \tag{5}$$

where ϕ is the electrostatic potential and Z_c (Z_h) is the cold (hot) charge number for negatively charged dust, u_c (u_h) is the cold (hot) dusty plasma velocity, n_c, n_h, n_e, n_i, N_{c0}, N_{h0}, n_{e0}, and n_{i0} are the perturbed and equilibrium number densities of the species, and η_1 (η_2) is the cold (hot) dust viscosity coefficient. To study the effect of non-thermal ions on the nonlinear dust acoustic waves, we choose a more general class of ion distributions which accounts for non-thermal ions as well, that is,[31–33]

$$f_i(v) = [(1 + v^4\delta)/(1+3\delta)]\exp(-v^2/2)/\sqrt{2\pi}, \tag{6}$$

where the velocity v is normalized by the mean ion thermal speed v_{ti}, and δ is a parameter governing the population of non-thermal ions in our plasma model. The effect of electrostatic disturbances of the equilibrium ion distribution can be easily introduced by replacing v^2 in the exponent of Eq. (6) by $v^2 + 2\phi$. Integrating the resulting distribution function over the entire velocity space yields

$$n_i = n_{i0}[1 + \mu\,(e\phi/T_i) + \mu\,(e\phi/T_i)^2]\exp(-e\phi/T_i), \tag{7a}$$

$$\mu = 4\delta/(1+3\delta). \tag{7b}$$

According to the general method of reductive perturbation theory,[50] we introduce the slowly stretched coordinates as

$$\tau = \varepsilon^{3/2}t, \quad \zeta = \varepsilon^{1/2}(x - \lambda t), \tag{8a}$$

where ε is a small dimensionless expansion parameter measuring the strength of nonlinearity, and λ is the wave speed. As the viscosity coefficients of the dust species are medium characteristics, they can be stretched into the following form:

$$\eta_1 = \varepsilon^{1/2}\eta_c, \quad \eta_2 = \varepsilon^{1/2}\eta_h. \tag{8b}$$

All physical quantities in Eqs. (1)–(7) are expanded as power series in ε about their equilibrium values as[50,51]

$$n_c = N_{c0} + \varepsilon n_{c1} + \varepsilon^2 n_{c2} + \varepsilon^3 n_{c3} + \cdots, \tag{9a}$$

$$u_c = \varepsilon u_{c1} + \varepsilon^2 u_{c2} + \varepsilon^3 u_{c3} + \cdots, \tag{9b}$$

$$n_h = N_{h0} + \varepsilon n_{h1} + \varepsilon^2 n_{h2} + \varepsilon^3 n_{h3} + \cdots, \tag{9c}$$

$$u_h = \varepsilon u_{h1} + \varepsilon^2 u_{h2} + \varepsilon^3 u_{h3} + \cdots, \tag{9d}$$

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots. \tag{9e}$$

We impose the boundary conditions that as $|\zeta| \to \infty$, $n_c = N_{c0}$, $n_h = N_{h0}$, $u_c = u_h = 0$, and $\phi = 0$. Substituting Eqs. (8) and (9) into Eqs. (1)–(7) and equating coefficients of like powers of ε, then from the lowest-order equations in ε, we obtain

$$n_{c1} = -[e\,N_{c0}\,Z_c/(m_c\lambda^2)]\phi_1, \tag{10a}$$
$$u_{c1} = -[e\,Z_c/(m_c\lambda)]\phi_1, \tag{10b}$$
$$n_{h1} = \{e[m_c\lambda^2(n_{i0}T_e(\mu-1) - n_{e0}T_i) + n_{c0}T_eT_iZ_c^2] / (m_c\lambda^2T_eT_iZ_h)\}\phi_1, \tag{10c}$$
$$u_{h1} = \{e\lambda[m_c\lambda^2(n_{i0}T_e(\mu-1) - n_{e0}T_i) + n_{c0}T_eT_iZ_c^2] / (m_c\lambda^2T_eT_iZ_hN_{h0})\}\phi_1. \tag{10d}$$

Poisson's equation gives the linear dispersion relation for the wave speed λ as

$$m_c\lambda^2N_{h0}^2T_eT_iZ_h^2 + (m_h\lambda^2N_{h0} - 3p_0) \times \{m_c\lambda^2[n_{i0}T_e(\mu-1) - n_{e0}T_i] + n_{c0}T_eT_iZ_c^2\} = 0. \tag{11}$$

Proceeding to order of ε^2, we obtain

$$\frac{\partial n_{c1}}{\partial\tau} + u_{c1}\frac{\partial n_{c1}}{\partial\zeta} + n_{c1}\frac{\partial u_{c1}}{\partial\zeta} - \lambda\frac{\partial n_{c2}}{\partial\zeta} + N_{c0}\frac{\partial u_{c2}}{\partial\zeta} = 0, \tag{12a}$$
$$\frac{\partial n_{h1}}{\partial\tau} + u_{h1}\frac{\partial n_{h1}}{\partial\zeta} + n_{h1}\frac{\partial u_{h1}}{\partial\zeta} - \lambda\frac{\partial n_{h2}}{\partial\zeta} + N_{h0}\frac{\partial u_{h2}}{\partial\zeta} = 0, \tag{12b}$$
$$\frac{\partial u_{c1}}{\partial\tau} + u_{c1}\frac{\partial u_{c1}}{\partial\zeta} - \lambda\frac{\partial u_{c2}}{\partial\zeta} - \frac{Z_ce}{m_c}\frac{\partial}{\partial\zeta}\phi_2 + \eta_c\frac{\partial^2}{\partial\zeta^2}u_{c1} = 0, \tag{12c}$$
$$\frac{\partial u_{h1}}{\partial\tau} + \frac{3P_{h0}}{m_hN_{h0}^3}n_{h1}\frac{\partial n_{h1}}{\partial\zeta} + \frac{3P_{h0}}{m_hN_{h0}^2}\frac{\partial n_{h2}}{\partial\zeta} + u_{h1}\frac{\partial u_{h1}}{\partial\zeta} - \lambda\frac{\partial u_{h2}}{\partial\zeta} - \frac{eZ_h}{m_h}\frac{\partial\phi_2}{\partial\zeta} + \eta_h\frac{\partial^2u_{h1}}{\partial\zeta^2} = 0, \tag{12d}$$
$$\frac{\partial^2\phi_1}{\partial\zeta^2} - 4\pi e(Z_cn_{c2} - Z_hn_{h2}) - 2\pi e^3\left(\frac{n_{e0}}{T_e^2} - \frac{n_{i0}}{T_i^2}\right)\phi_1^2 + 4\pi e^2\left[\frac{n_{e0}}{T_e} - \frac{n_{i0}}{T_i}(1-\mu)\,\phi_2\right] = 0. \tag{12e}$$

Eliminating the second-order perturbed quantities n_{c2}, n_{h2}, u_{c2}, u_{h2}, and ϕ_2, we obtain the desired KdV–Burgers equation for the first-order perturbed potential as

$$\frac{\partial}{\partial\tau}\phi(\zeta,\tau) + A\,\phi(\zeta,\tau)\frac{\partial}{\partial\zeta}\phi(\zeta,\tau) + B\frac{\partial^2}{\partial\zeta^2}\phi(\zeta,\tau) + C\,\frac{\partial^3}{\partial\zeta^3}\phi(\zeta,\tau) = 0, \tag{13}$$

where $\phi(\zeta,\tau) = \phi_1(\zeta,\tau)$, and the nonlinear coefficient is given in the form

$$A = 4\pi e^3C\{3(m_h\lambda^2N_{h0} + p_0)[m_c\lambda^2(n_{i0}T_e(\mu-1) - n_{e0}T_i) + n_{c0}T_eT_iZ_c^2]^2 - N_{h0}(m_h\lambda^2N_{h0} - 3p_0) \times [m_c^2\lambda^4(n_{i0}T_e^2 - n_{e0}T_i^2) - 3n_{c0}T_e^2T_i^2Z_c^3]Z_h\} / [m_c^2\lambda^4N_{h0}(m_h\lambda^2N_{h0} - 3p_0)T_e^2T_i^2Z_h], \tag{14a}$$

while the dissipative coefficient is represented by

$$B = -4\pi e^2C\{m_h\lambda^2N_{h0}[m_c\lambda^2(n_{i0}T_e(\mu-1) - n_{e0}T_i)\eta_h + n_{c0}T_eT_iZ_c^2(\eta_h - \eta_c)] + 3p_0n_{c0}T_eT_iZ_c^2\eta_c\} / [m_c\lambda^3(m_h\lambda^2N_{h0} - 3p_0)\,T_eT_i], \tag{14b}$$

and the dispersion coefficient is

$$C = [-m_c\lambda^3(m_h\lambda^2N_{h0} - 3p_0)\,T_eT_i]/\{8e^2\pi\{m_cm_h\lambda^4N_{h0} \times [n_{i0}T_e(\mu-1) - n_{e0}T_i] + 3p_0n_{c0}T_eT_iZ_c^2\}\}. \tag{14c}$$

The nonlinear KdV–B equation (13) can be converted into space–time fractional form using the semi-inverse method[45,46] and the variational technique of Agrawal[30,31] in terms of the modified Riemann–Liouville fractional derivative[23,24] (see below).

3. Semi-inverse formulation of STFKdV–B equation

Using a potential function $U(\zeta,\tau)$, where $\phi(\zeta,\tau) = U_\zeta(\zeta,\tau)$, gives the potential equation of the regular KdV–B equation in the form

$$U_{\tau\zeta}(\zeta,\tau) + AU_\zeta(\zeta,\tau)U_{\zeta\zeta}(\zeta,\tau) + B\phi_{\zeta\zeta}(\zeta,\tau) + CU_{\zeta\zeta\zeta\zeta}(\zeta,\tau) = 0, \tag{15}$$

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Euler–Lagrange equation of the regular KdV–B equation can be derived using the semi-inverse method[45,46] as follows.

The functional of the potential equation can be represented by

$$J(U) = \int_R \mathrm{d}\zeta\int_T \mathrm{d}\tau U(\zeta,\tau)\{c_1U_{\tau\zeta}(\zeta,\tau) + c_2AU_\zeta(\zeta,\tau)U_{\zeta\zeta}(\zeta,\tau) + c_3B\phi_{\zeta\zeta}(\zeta,\tau) + c_4CU_{\zeta\zeta\zeta\zeta}(\zeta,\tau)\}, \tag{16}$$

where c_1, c_2, c_3, and c_4 are constant Lagrangian multipliers. Integrating this equation by parts where $U_\zeta|_R = U_\tau|_T = U_{\zeta\zeta}|_R = 0$ and $\phi_{\zeta\zeta}(\zeta,\tau)$ is considered as a fixed function and applying the variation of this functional with respect to $U(\zeta,\tau)$ lead to

$$\delta J(U) = \int_R \mathrm{d}\zeta\int_T \mathrm{d}\tau\Big\{-c_1[U_\tau(\zeta,\tau)\delta U_\zeta(\zeta,\tau) + U_\zeta(\zeta,\tau)\delta U_\tau(\zeta,\tau)] - \frac{3}{2}c_2A[U_\zeta(\zeta,\tau)]^2\delta U_\zeta(\zeta,\tau) + c_3B\phi_{\zeta\zeta}(\zeta,\tau)\delta U(\zeta,\tau) + 2c_4C[U_{\zeta\zeta}(\zeta,\tau)\delta U_{\zeta\zeta}(\zeta,\tau)]\Big\}.$$

Integrating by parts using $U_\zeta|_R = U_\tau|_T = U_{\zeta\zeta}|_R = 0$ and optimizing this variation, $\delta J(U) = 0$, give

$$2c_1 U_{\tau\zeta}(\zeta,\tau) + 3c_2 A U_\zeta(\zeta,\tau) U_{\zeta\zeta}(\zeta,\tau) + c_3 B \phi_{\zeta\zeta}(\zeta,\tau) + 2c_4 C U_{\zeta\zeta\zeta\zeta}(\zeta,\tau) = 0.$$

Comparing the above equation with Eq. (A1) gives the constant multipliers as

$$c_1 = \frac{1}{2},\ \ c_2 = \frac{1}{3},\ \ c_3 = 1,\ \ c_4 = \frac{1}{2}.$$

The functional relation yields directly the Lagrangian of the potential equation as

$$\begin{aligned} &L(U, U_\tau, U_\zeta, U_{\zeta\zeta}) \\ &= \frac{1}{2} U_\tau(\zeta,\tau) U_\zeta(\zeta,\tau) + \frac{1}{6} A [U_\zeta(\zeta,\tau)]^3 \\ &\quad - B U(\zeta,\tau) \phi_{\zeta\zeta}(\zeta,\tau) - \frac{1}{2} C [U_{\zeta\zeta}(\zeta,\tau)]^2. \end{aligned}$$

Similarly, the Lagrangian of the time–space fractional version of the KdV–B equation could be written in the form

$$\begin{aligned} &F(U, D_\tau^\alpha U, D_\zeta^\beta U, D_\zeta^{\beta\beta} U) \\ &= \frac{1}{2} D_\tau^\alpha U(\zeta,\tau) D_\zeta^\beta U(\zeta,\tau) + \frac{1}{6} A [D_\zeta^\beta U(\zeta,\tau)]^3 \\ &\quad - B U(\zeta,\tau) D_\zeta^{\beta\beta} \phi(\zeta,\tau) - \frac{1}{2} C [D_\zeta^{\beta\beta} U(\zeta,\tau)]^2, \end{aligned} \quad (17)$$

where $D_z^{\gamma\gamma} f(z) = D_z^\gamma [D_z^\gamma f(z)]$, while the fractional derivative $D_z^\gamma f(z)$ is the modified Riemann–Liouville (mRL) fractional derivative[23,24] defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \left\{ \int_a^x \mathrm{d}\xi [f(\zeta) - f(a)]/(x-\xi)^\alpha \right\}, \quad 0 < \alpha < 1, \quad (18a)$$

$$D_x^\alpha f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \{ D_x^{\alpha-n} f(x) \},\ n \le \alpha < n+1,\ n \ge 1. \quad (18b)$$

In Eq. (17), as $\phi(\zeta,\tau) = D_\zeta^\beta U(\zeta,\tau)$, the last term $(-C[D_\zeta^{\beta\beta} U(\zeta,\tau)]^2/2)$ represents the fractional form of the kinetic energy of the system while the remaining terms represent the negative of the fractional potential of the system.

The fractional functional of the STFKdV–B equation takes the form

$$J_{\mathrm{F}}(U) = \int_T (\mathrm{d}\tau)^\alpha \int_R (\mathrm{d}\zeta)^\beta F(U, D_\tau^\alpha U, D_\zeta^\beta U, D_\zeta^{\beta\beta} U), \quad (19)$$

where $\int_a^z (\mathrm{d}\zeta)^\gamma f(\xi) = \gamma \int_a^z \mathrm{d}\zeta (z-\xi)^{\gamma-1} f(\xi)$, and $0 < \gamma \le 1$.[23,24]

The variation of this functional with respect to $U(\zeta,\tau)$ leads to

$$\begin{aligned} \delta J_{\mathrm{F}}(U) = \int_T (\mathrm{d}\tau)^\alpha \int_R (\mathrm{d}\zeta)^\beta \Bigg\{ &\left(\frac{\partial F}{\partial U}\right) \delta U + \left(\frac{\partial F}{\partial D_\tau^\alpha U}\right) \delta D_\tau^\alpha U + \left(\frac{\partial F}{\partial D_\zeta^\beta U}\right) \delta D_\zeta^\beta U \\ &+ \left(\frac{\partial F}{\partial D_\zeta^{\beta\beta} U}\right) \delta D_\zeta^{\beta\beta} U \Bigg\}. \end{aligned} \quad (20)$$

Therefore, integrating this equation by parts using the definition given by[23,24]

$$\begin{aligned} &\int_a^b (\mathrm{d}x)^\gamma f(x) D_x^\gamma g(x) \\ &= \Gamma(1+\gamma) [g(x) f(x)|_a^b \\ &\quad - \int_a^b (\mathrm{d}x)^\gamma g(x) D_x^\gamma f(x),\ \ f(z),\ \ g(z) \in [a,b], \end{aligned} \quad (21)$$

and optimizing this relation, $\delta J_{\mathrm{F}}(U) = 0$, the fractional Euler–Lagrange equation for the STFKdV–B equation has the form

$$\left(\frac{\partial F}{\partial U}\right) - D_\tau^\alpha \left(\frac{\partial F}{\partial D_\tau^\alpha U}\right) - D_\zeta^\beta \left(\frac{\partial F}{\partial D_\zeta^\beta U}\right) + D_\zeta^{\beta\beta} \left(\frac{\partial F}{\partial D_\zeta^{\beta\beta} U}\right) = 0. \quad (22)$$

Substituting the Lagrange of the STFKdV–B (17) into this Euler–Lagrange formula gives

$$-B D_\zeta^{\beta\beta} \phi(\zeta,\tau) - D_\tau^\alpha [D_\zeta^\beta U(\zeta,\tau)] - \frac{1}{2} A D_\zeta^\beta [D_\zeta^\beta U(\zeta,\tau)]^2 - C D_\zeta^{\beta\beta} [D_\zeta^{\beta\beta} U(\zeta,\tau)] = 0. \quad (23)$$

Substituting $\phi(\zeta,\tau) = D_\zeta^\beta U(\zeta,\tau)$, using the fractional differentiation of multiplication of two functions given by[23,24]

$$\begin{aligned} &D_x^\gamma [u(x)\, v(x)] \\ &= D_x^\gamma [u(x)] v(x) + u(x) D_x^\gamma [v(x)],\ \ 0 < \gamma \le 1, \end{aligned} \quad (24)$$

lead to the space–time fractional Koreweg–de Vries–Burgers equation for $\beta = \alpha$ as

$$D_\tau^\alpha \phi(\zeta,\tau) + A \phi(\zeta,\tau) D_\zeta^\alpha \phi(\zeta,\tau) + B D_\zeta^{\alpha\alpha} \phi(\zeta,\tau) + C D_\zeta^{\alpha\alpha\alpha} \phi(\zeta,\tau) = 0, \quad (25)$$

where $\phi(\zeta,\tau)$ is the electrostatic potential, $D_{\tau(\zeta)}^\alpha$ is the modified Riemann–Liouville fractional derivative with respect to time (τ) or space (ζ), and α is the space–time fractional order.

The STFKdV–B equation will be solved using the fractional sub-equation method.[40,41]

4. Fractional sub-equation solution of STFKdV–B equation

In this section, the STFKdV–B equation will be solved using the fractional sub-equation method[40,41] as follows.

Considering traveling wave transformation $\phi(\zeta,\tau) = \Phi(\xi)$, $\xi = \zeta + v\tau$, and the relation[23,24]

$$D_x^\gamma f(u(x)) = D_x^\gamma u(x) \left(\frac{\mathrm{d}f}{\mathrm{d}u}\right) = D_u^\gamma f(u) \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^\gamma, \quad (26)$$

equation (25) can be reduced to the fractional nonlinear ordinary differential equation

$$v^{\alpha} D_{\xi}^{\alpha}\Phi(\xi) + A\Phi(\xi) D_{\xi}^{\alpha}\Phi(\xi) + B D_{\xi}^{\alpha\alpha}\Phi(\xi) + C D_{\xi}^{\alpha\alpha\alpha}\Phi(\xi) = 0. \quad (27)$$

The fractional sub-equation method[40,41] assumes the solution of this equation as

$$\Phi(\xi) = \sum_{k=0}^{n} a_k \varphi^k(\xi), \quad (28)$$

where $\varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_{\xi}^{\alpha}\varphi(\xi) = \sum_{j=0}^{m} b_j \varphi^j(\xi), \quad (29)$$

where a_k $(k = 0, ..., n)$ are constant coefficients to be determined later and b_j $(j = 0, ..., m)$ are arbitrary coefficients.

Balancing the highest order derivative term and nonlinear term in Eq. (27), the value of n can be determined, which has, in this problem, the value $n = 2$. We suppose that equation (27) has the following formal solution:

$$\Phi(\xi) = a_0 + a_1\varphi(\xi) + a_2[\varphi(\xi)]^2, \quad (30)$$

where a_0, a_1, and a_2 are constant coefficients to be determined and $\varphi(\xi)$ satisfies the fractional Riccati equation

$$D_{\xi}^{\alpha}\varphi(\xi) = b_0 + b_2\varphi^2(\xi), \quad (31)$$

where b_0 and b_2 are arbitrary coefficients.

Using the generalized Exp-function method via Mittag–Leffler functions, Zhang *et al.*[39] obtained the following solution of fractional Riccati equation (31):

$$\varphi_1(\xi) = -\sqrt{-b_0}\tanh_{\alpha}(\sqrt{-b_0}\xi), \ b_0 < 0, \ b_2 = 1, \quad (32a)$$
$$\varphi_2(\xi) = -\sqrt{-b_0}\coth_{\alpha}(\sqrt{-b_0}\xi), \ b_0 < 0, \ b_2 = 1, \quad (32b)$$
$$\varphi_3(\xi) = \sqrt{b_0}\tan_{\alpha}(\sqrt{b_0}\xi), \ b_0 > 0, \ b_2 = 1, \quad (32c)$$
$$\varphi_4(\xi) = -\sqrt{b_0}\cot_{\alpha}(\sqrt{b_0}\xi), \ b_0 > 0, \ b_2 = 1, \quad (32d)$$
$$\varphi_5(\xi) = -\Gamma(1+\alpha)/(\xi^{\alpha} + \omega), \quad b_0 = 0, \ b_2 = 1, \ \omega = \text{constant}, \quad (32e)$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_{\alpha}(x) = \sinh_{\alpha}(x)/\cosh_{\alpha}(x), \quad \coth_{\alpha}(x) = \cosh_{\alpha}(x)/\sinh_{\alpha}(x), \quad (33a)$$
$$\sinh_{\alpha}(x) = [E_{\alpha}(x) - E_{\alpha}(-x)]/2, \quad \cosh_{\alpha}(x) = [E_{\alpha}(x) + E_{\alpha}(-x)]/2, \quad (33b)$$
$$\tan_{\alpha}(x) = \sin_{\alpha}(x)/\cos_{\alpha}(x), \quad \cot_{\alpha}(x) = \cos_{\alpha}(x)/\sin_{\alpha}(x), \quad (33c)$$
$$\sin_{\alpha}(x) = [E_{\alpha}(\mathrm{i}x) - E_{\alpha}(-\mathrm{i}x)]/(2\mathrm{i}), \quad \cos_{\alpha}(x) = [E_{\alpha}(\mathrm{i}x) + E_{\alpha}(-\mathrm{i}x)]/2, \quad (33d)$$

where $\mathrm{i} = \sqrt{-1}$, and $E_{\alpha}(x)$ is the Mittag–Leffler function defined by

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} x^k/\Gamma(1+k\alpha). \quad (33e)$$

Substituting Eq. (30) along with Eq. (31) into Eq. (27) and setting the coefficients of $\varphi^k(\xi)$ as zero lead to a set of algebraic equations in terms of the coefficients a_0, a_1, a_2, b_0, and b_2. Solving the algebraic set of equations by Maple gives the set of coefficients of the solution of STFKdV–B equation is given by

$$b_0 = -\frac{B^2}{100C^2}, \ b_2 = 1, \ a_0 = \frac{3B^2 - 25v^{\alpha}C}{25AC}, \quad a_1 = -\frac{12B}{5A}, \ a_2 = -\frac{12C}{A}, \quad (34)$$

where $AC \neq 0$. As the coefficient b_0 is negative, therefore the solutions corresponding to this set of coefficients are two solutions only represented by forms 32(a) and 32(b). The fractional sub-equation method[40,41] gives the solutions of this equation in the form

$$\Phi_1(\xi) = \frac{3B^2 - 25v^{\alpha}C}{25AC} + \frac{6B^2}{25AC}\tanh_{\alpha}\left(\frac{B}{10C}\xi\right) - \frac{3B^2}{25AC}\left[\tanh_{\alpha}\left(\frac{B}{10C}\xi\right)\right]^2, \quad (35a)$$

$$\Phi_2(\xi) = \frac{3B^2 - 25v^{\alpha}C}{25AC} + \frac{6B^2}{25AC}\coth_{\alpha}\left(\frac{B}{10C}\xi\right) - \frac{3B^2}{25AC}\left[\coth_{\alpha}\left(\frac{B}{10C}\xi\right)\right]^2. \quad (35b)$$

These two forms represent the electrostatic potential of the plasma system described in this work.

5. Results and discussion

The nonlinear propagation of DAWs in homogeneous unmagnetized plasma having electrons with Boltzmann distribution, singly charged nonthermal ions, hot and cold viscous dust grains have been studied. The effect of gravity in the system is assumed to be neglected ($r_{\mathrm{d}} \ll 1$) in the absence of neutrals. The Saturn F-rings are one of the space plasma observations that satisfy the conditions: (i) there are no neutrals, (ii) the ratio between the inter-grain distances between dust particles to plasma Debye radius is less than one, (iii) coupling parameter Γ is less than one, and (iv) r_{d} is smaller than 1 μm. Hence, numerical calculations have been carried out using plasma parameters close to those values corresponding to the Saturn F-ring's, i.e., electron and hot dust equilibrium densities are respectively $n_{\mathrm{e0}} = 10\ \mathrm{cm}^{-3}$ and $N_{\mathrm{h0}} = 10\ \mathrm{cm}^{-3}$, and hot dust charge and mass are taken as $Z_{\mathrm{h}} = 10^2$–10^3 and $m_{\mathrm{h}} = m_{\mathrm{c}} = 10^{12} m_{\mathrm{i}}$, respectively. The STFKdV–B equation is derived using the semi-inverse method[45,46] and is solved using the fractional sub-equation method.[40,41] The fractional

sub-equation method gives two solutions. The solution 35(a) is a monotonic-type shock pulse. Solution 35(b) leads to the propagation of explosive shock pulses. Since one of our motivations is to study the space–time fractional effects on the formation of shock waves, we have studied the effect of parameters like the energetic population parameter of ions δ, the cold and hot dust viscosity coefficients η_c and η_h on the existence of shock waves. For example, figure 1 considers the effect of space–time fractional parameters (α) on the shock wave height. It is shown that as α increases the height of shock wave increases. To examine how the energetic population parameter of ions (δ) modify the properties of the dust acoustic shock waves, figure 2 shows that, positive and negative shock wave are formed due to the variation of δ. Figures 3 and 4 show the effects of the kinematic viscosities of the cold (η_c) and hot (η_h) dust species on the structure of monotonic shock waves. Steepness of shock wave decreases while the height increases by increasing the kinematic viscosities of the cold and hot dust species.

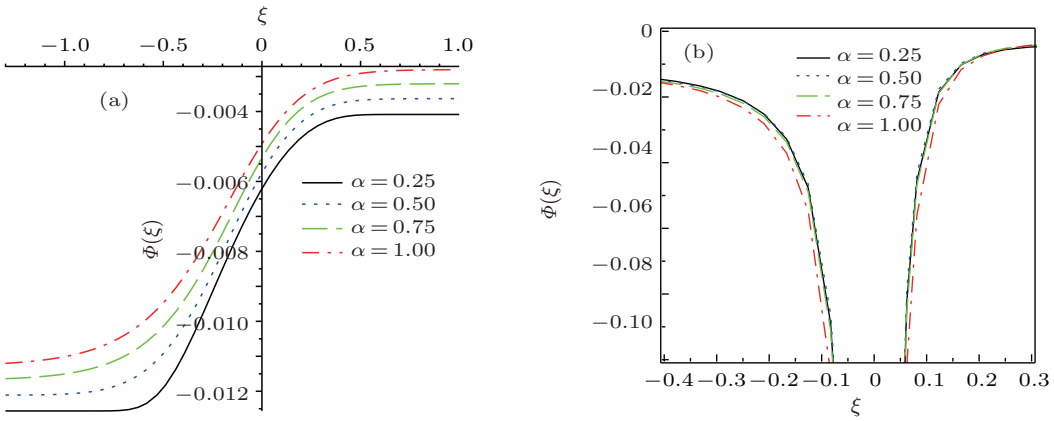


Fig. 1. (color online) Electrostatic potential (a) $\Phi_1(\xi)$ and (b) $\Phi_2(\xi)$ for $v = 0.8$, $Z_c = 500$, $Z_h = 500$, $N_{c0} = 8$, $N_{h0} = 5$, $\eta_c = 0.2$, $\eta_h = 0.2$, $\delta = 0.2$, and different values of α.

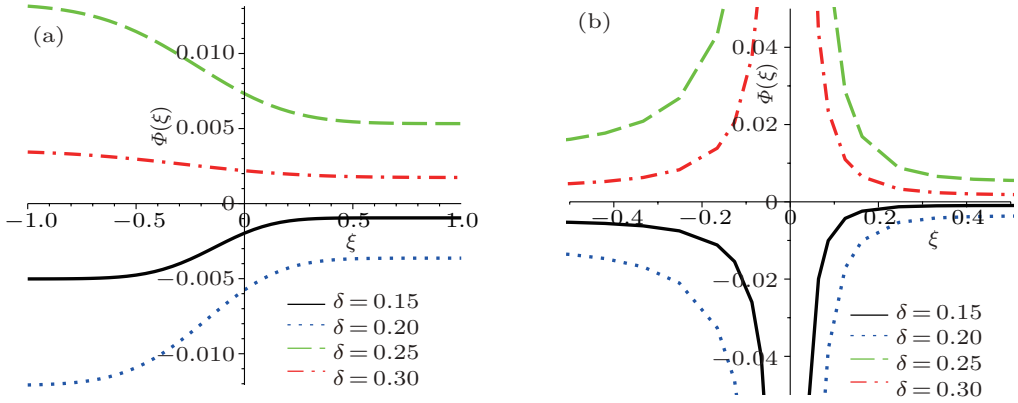


Fig. 2. (color online) Electrostatic potential (a) $\Phi_1(\xi)$ and (b) $\Phi_2(\xi)$ for $v = 0.8$, $Z_c = 500$, $Z_h = 500$, $N_{c0} = 8$, $N_{h0} = 5$, $\eta_c = 0.2$, $\eta_h = 0.2$, $\alpha = 0.5$, and different values of δ.

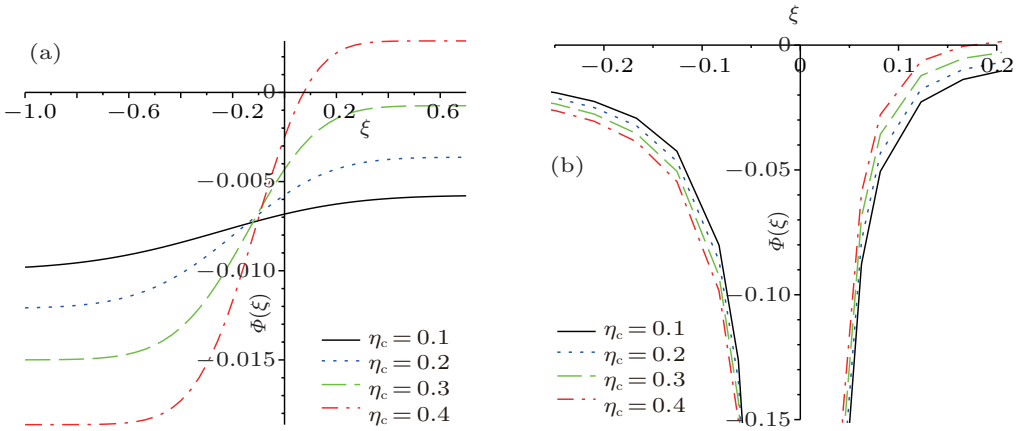


Fig. 3. (color online) Electrostatic potential (a) $\Phi_1(\xi)$ and (b) $\Phi_2(\xi)$ for $v = 0.8$, $Z_c = 500$, $Z_h = 500$, $N_{c0} = 8$, $N_{h0} = 5$, $\eta_h = 0.2$, $\delta = 0.2$, $\alpha = 0.5$, and different values of η_c.

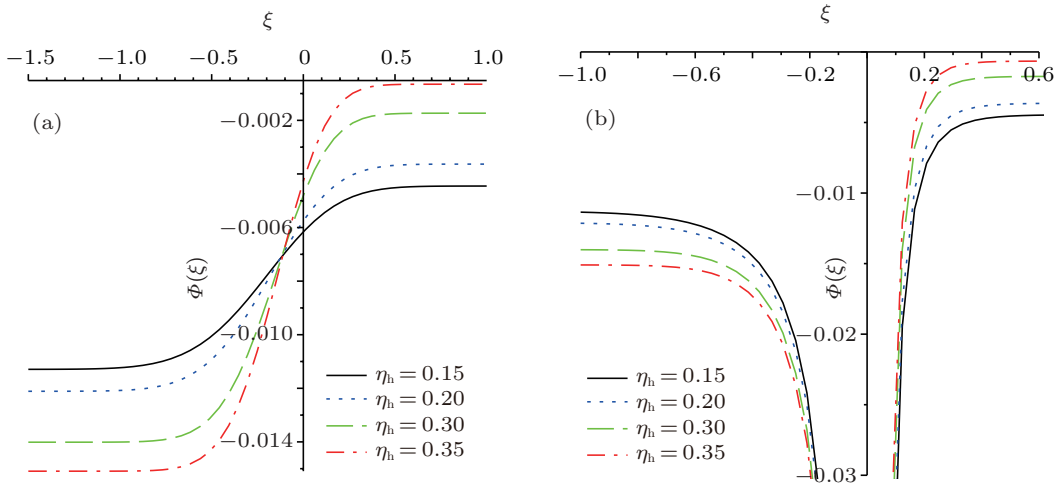


Fig. 4. (color online) Electrostatic potential (a) $\Phi_1(\xi)$ and (b) $\Phi_2(\xi)$ for $v = 0.8$, $Z_c = 500$, $Z_h = 500$, $N_{c0} = 8$, $N_{h0} = 5$, $\eta_c = 0.2$, $\delta = 0.2$, $\alpha = 0.5$, and different values of η_h.

6. Summary

In summary, it has been found that the presence of space–time fractional parameter and the population of non-thermal ions, and the cold and hot dust viscosity would modify the properties of the dust acoustic shock waves. The results presented here should be useful in understanding salient features of localized electrostatic perturbations in space and laboratory plasmas.

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