# Optimal $1 \to M$ phase-covariant cloning in three dimensions*

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In this paper, we derive the explicit transformations of the optimal $1 \to 3$, 4, 5 phase-covariant cloning in three dimensions, and then generalize them to the cases of $1 \to M = 3n$, $3n+1$, $3n+2$ ($n \geq 1$ integer) cloning. The clone fidelities are coincident with the theoretical bounds found.

**Keywords:** quantum cloning, universal quantum cloning, phase-covariant cloning

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## 1. Introduction

The no-cloning theorem[1] states that any quantum state can not be cloned perfectly, and is the basis of the security of quantum cryptography.[2] Perfect copies are forbidden, however imperfect clones can be allowed. Therefore, a question arises: how well do the clones resemble the original states? This problem was first investigated in a seminal paper.[3] Since then, a vast of number of researches have been focused on quantum cloning (see Ref. [4] for a review and the references therein). In quantum cloning theory, there are many different cloning machines. To clone a completely unknown quantum state, the simplest case is to clone a pure input quantum state in the form of $|\Psi\rangle^{(\mathrm{in})} = \sum_{i=0}^{d-1} \alpha_i \mathrm{e}^{\mathrm{i}\varphi_i}|i\rangle$, where $\alpha_i \geq 0$ satisfying $\sum_{i=0}^{d-1} \alpha_i^2 = 1$ and $\varphi_i \in [0, 2\pi)$ are all unknown. If the copy fidelity is maximal and independent of the amplitudes and phase factors of the input state $|\Psi\rangle^{(\mathrm{in})}$, the cloner is the so-called optimal universal quantum cloner (UQC).[3,5–9] To clone the input state $|\Phi\rangle^{(\mathrm{in})} = d^{-1/2} \sum_{i=0}^{d-1} \mathrm{e}^{\mathrm{i}\varphi_i}|i\rangle$ without knowing $\varphi_i \in [0, 2\pi)$, the cloner is the optimal phase-covariant cloner (PCC).[10–16] Quantum cloning has an intimate relation to the state estimation,[17–20] and PCC has an important application in quantum cryptography.[2] Recently, these two cloners have been realized in experiments.[21–25] There are other cloners, such as entanglement state cloning[26,27] and orthogonal qubits cloning.[28,29] Many schemes for the realization of different cloning machines have been proposed.[30–34] Quantum cloning has more applications in quantum information science.[35–38]

In this paper, we start with some unitary transformations to derive the explicit transformations of the optimal $1 \to 3, 4, 5$ PPC in three dimensions, and then generalize to the cases of $1 \to M = 3n$, $3n+1$, $3n+2$ PPC. In Ref. [13], the authors derived the theoretical copy fidelities of the optimal $1 \to M = 3n$, $3n+1$, $3n+2$ PCC for qutrits with $n \geq 1$ being integer, however, they did not determine the explicit transformations. In our derivations, we determine the cloning coefficients and calculate the cloning fidelities. The fidelities obtained are coincident with the previous contributions.[13] With explicit transformation, one can realize PPC in experiments.

The paper is organized as follows. In Section 2, we briefly review some previous results of UQC and PCC. Then in Section 3, we present the explicit transformations of the optimal $1 \to M = 3n$, $3n+1$, $3n+2$ PCC of qutrits, as well as the maximal fidelities. The paper ends with a summary.

## 2. Previous contributions

If one wants to clone an input state $|\psi\rangle_1^{(\mathrm{in})} = \alpha|0\rangle_1 + \beta|1\rangle_1$, where $\alpha$ and $\beta$ are unknown complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, the optimal $1 \to 2$ UQC in two dimensions (qubits) is defined by[7]

$$\begin{aligned} |0\rangle_1|00\rangle_{2,\mathrm{a}} &\to \sqrt{2/3}|000\rangle_{1,2,\mathrm{a}} + \sqrt{1/6}(|01\rangle + |10\rangle)_{1,2}|1\rangle_{\mathrm{a}}, \\ |1\rangle_1|00\rangle_{2,\mathrm{a}} &\to \sqrt{2/3}|111\rangle_{1,2,\mathrm{a}} + \sqrt{1/6}(|10\rangle + |01\rangle)_{1,2}|0\rangle_{\mathrm{a}}, \end{aligned} \tag{1}$$

where the state $|0\rangle_2$ is the blank state, and $|0\rangle_{\mathrm{a}}$ is the ancillary system. For all transformations acting on the initial input system $U|\psi\rangle_1^{(\mathrm{in})}|00\rangle_{2,\mathrm{a}} = |\psi\rangle_{1,2,\mathrm{a}}^{(\mathrm{out})}$, the UQC given by Eq. (1) is optimal and symmetric. The UQC is qualified by the fidelity defined as $F_{1(2)}^{(\mathrm{UQC})} = {}^{(\mathrm{in})}_{1}\langle\psi|\rho_{1(2)}^{(\mathrm{out})}|\psi\rangle_1^{(\mathrm{in})}$, where $\rho_{1(2)}^{(\mathrm{out})}$ denotes the reduced density of clone 1 (2) by taking the partial trace over particles 2 and a (1 and a). The fidelity of each clone, $F_{1\to 2}^{(\mathrm{UQC})}(2) = 5/6$, is maximal and independent of the coefficients of the input state, $\alpha$ and $\beta$, and hence the cloning machine is universal. The UQC has been generalized to $N \to M$

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in two dimensions,[5] and to $1\to 2$ in $d$ dimensions,[7] as well as to the asymmetric case.[8]

To clone an input state $|\phi_1\rangle_1^{(\mathrm{in})} = (|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}|1\rangle)_1/\sqrt{2}$, where the angle $\varphi \in [0, 2\pi)$ is unknown, the optimal $1\to 2$ PCC in two dimensions (qubits) is defined as[10]

$$\begin{aligned}|0\rangle_1|00\rangle_{2,\mathrm{a}} &\to \sqrt{1/2}|000\rangle_{1,2,\mathrm{a}} + 1/2(|01\rangle + |10\rangle)_{1,2}|1\rangle_{\mathrm{a}},\\ |1\rangle_1|00\rangle_{2,\mathrm{a}} &\to \sqrt{1/2}|111\rangle_{1,2,\mathrm{a}} + 1/2(|10\rangle + |01\rangle)_{1,2}|0\rangle_{\mathrm{a}}.\end{aligned} \tag{2}$$

The copy fidelity, $F_{1\to 2}^{(\mathrm{PPC})}(2) = 1/2 + 1/(2\sqrt{2}) > F_{1\to 2}^{(\mathrm{UQC})}(2)$, is maximal and independent of the phase factor $\varphi$. The PCC can be regarded as an efficient eavesdropping to the BB84 protocol.[39] The PPC was generalized to $1\to M$ in two dimensions[11] and to $1\to 2$ in $d$ dimensions.[12]

In Ref. [13], the authors investigated the optimal $1\to M$ PCC for qubits and qutrits, and derived the optimal fidelities. In two dimensions, the optimal fidelity is coincident with that in Ref. [11], which proves the optimality of the transformation.[11] In three dimensions (qutrits), the authors[13] also derived the formulation of the fidelity of the optimal $1\to M = 3n,\ 3n+1,\ 3n+2$ PCC of qutrits

$$\begin{aligned}F_{\nu_1,\nu_2} = \frac{1}{9}[&(r_0^{\nu_1,\nu_2})^2 + (r_1^{\nu_1,\nu_2})^2 + (r_2^{\nu_1,\nu_2})^2\\ &+ 2r_0^{\nu_1,\nu_2} r_1^{\nu_1,\nu_2} \Lambda_M(M-\nu_1-\nu_2, \nu_1)\\ &+ 2r_0^{\nu_1,\nu_2} r_2^{\nu_1,\nu_2} \Lambda_M(M-\nu_1-\nu_2, \nu_2)\\ &+ 2r_1^{\nu_1,\nu_2} r_2^{\nu_1,\nu_2} \Lambda_M(\nu_1+\nu_2, \nu_2)],\end{aligned} \tag{3}$$

where the notations involved in Eq. (3) can be referred to in Ref. [13]. By inserting the explicit parameters given in Ref. [13] into Eq. (3), the optimal fidelities can be explicitly expressed as

$$F_{1\to M=3n}^{(\mathrm{PPC})}(3) = \frac{1+7n+\sqrt{(1+n)(1+9n)}}{18n}, \tag{4}$$

$$F_{1\to M=3n+1}^{(\mathrm{PPC})}(3) = \frac{1}{3}\left(1+\frac{2n+2}{3n+1}\right), \tag{5}$$

$$F_{1\to M=3n+2}^{(\mathrm{PPC})}(3) = \frac{5+7n+\sqrt{(1+n)(17+9n)}}{18n+12}. \tag{6}$$

Quantum cloning has an intimate relation to the state estimation.[17–20] Werner derived the fidelity of the optimal $N\to M$ UQC for qudits as[6]

$$F_{N\to M}^{(\mathrm{UQC})}(d) = \frac{M-N+N(M+d)}{M(N+d)},$$

the estimation of the state $|\Psi\rangle^{(\mathrm{in})} = \sum_{i=0}^{d-1}\alpha_i \mathrm{e}^{\mathrm{i}\varphi_i}|i\rangle$ was given as[17]

$$F_{N,\mathrm{est}}^{(\mathrm{UQC})}(d) = \frac{N+1}{N+d} \geq F_{N\to M}^{(\mathrm{UQC})}(d),$$

where the equality holds as $M\to\infty$. For the PPC, the estimation of the state $|\Phi\rangle^{(\mathrm{in})} = d^{-1/2}\sum_{i=0}^{d-1}\mathrm{e}^{\mathrm{i}\varphi_i}|i\rangle$ was given as[18]

$$F_{1,\mathrm{est}}^{(\mathrm{PPC})}(d) = \frac{2d-1}{d^2}.$$

For qutrits, from Eqs. (4)–(6), one can obtain

$$\begin{aligned}F_{1,\mathrm{est}}^{(\mathrm{PPC})}(3) &= 5/9 = F_{1\to M=3n,3n+1,3n+2}^{(\mathrm{PPC})}(3)\Big|_{n\to\infty}\\ &> F_{1,\mathrm{est}}^{(\mathrm{UQC})}(3) = 1/2.\end{aligned} \tag{7}$$

To date, the explicit transformation of the PCC of qutrits has not been presented. In the next section, we will derive the explicit transformation of the optimal $1\to M = 3n,\ 3n+1,\ 3n+2$ PCC of qutrits, where the fidelities are expressed as Eqs. (4)–(6).

## 3. Optimal phase-covariant cloning of qutrits

In order to clone an input state in the form of $|\phi_2\rangle = (\mathrm{e}^{\mathrm{i}\varphi_0}|0\rangle + \mathrm{e}^{\mathrm{i}\varphi_1}|1\rangle + \mathrm{e}^{\mathrm{i}\varphi_2}|2\rangle)/\sqrt{3}$ without knowing $\varphi_0$, $\varphi_1$, $\varphi_2 \in [0, 2\pi)$, we first consider the formulation of the maximal fidelity. For the input state $|\Phi\rangle^{(\mathrm{in})}$, if the reduced density matrix $\rho = (A)_{ij}$ of a clone is $A_{ij} = a_{ij}\mathrm{e}^{\mathrm{i}(\varphi_i-\varphi_j)}$, the fidelity is easily calculated as

$$F_{N\to M}^{(\mathrm{PPC})}(d) = {}^{(\mathrm{in})}\langle\Phi|\rho|\Phi\rangle^{(\mathrm{in})} = \frac{1}{d}\sum_{i,j=0}^{d-1} a_{ij}.$$

We use unitary transformations so the mapping is a completely positive trace-preserving one, i.e., $\sum_{i=0}^{d-1} a_{ii} = 1$. Therefore, in the case of $a_{ij} = a_{kl}$ $(i,j,k,l = 0,1,\ldots,d-1)$, the fidelity can be explicitly expressed as

$$F_{N\to M}^{(\mathrm{PPC})}(d) = \frac{1}{d}[1 + d(d-1)a_{ij}].$$

This is to say that the higher the valves of the off-diagonal elements of $\rho = (A)_{ij}$ are, the higher the fidelity is. In three dimensions, as $N = 1$, the formulation of the fidelity is reduced to

$$F_{1\to M}^{(\mathrm{PPC})}(3) = \frac{1}{3}(1 + 6a_{ij}). \tag{8}$$

In the following, we will construct some explicit unitary transformations, which should produce more items $a_{ij}$ in order to enhance the values of the off-diagonal elements.

### 3.1. Optimal $1\to M = 3n$ PPC

For the $1\to 3$ PCC, according to the symmetry, we can write the most general transformation as

$$|i\rangle \to a_1|(iii)\rangle|R_{iii}\rangle + a_2|(iij)\rangle|R_{iij}\rangle + a_3|(ijj)\rangle|R_{ijj}\rangle, \tag{9}$$

where $i,j,k = 0,1,2$ $(i\neq j\neq k)$. In Eq. (9), the first three particles $|(\cdots)\rangle$ are the three clones, and the fourth $|R_{\ldots}\rangle$ is the ancilla. The state $|(iij)\rangle$ denotes a completely symmetric state with two particles in the computational state $|i\rangle$ and one in $|j\rangle$, for example, $|(001)\rangle = |001\rangle + |010\rangle + |100\rangle$. The cloning coefficients must satisfy the normalization condition $a_1^2 + 6a_2^2 + 6a_3^2 + 6a_4^2 = 1$. In order to determine the explicit

transformation, the first step is to determine the form of the ancilla, and the second is to calculate the fidelity $F = f(a_l)$ as a function of the coefficients $a_l$ $(l = 1,2,3,4)$. The third is to maximize the fidelity under the normalization condition. Among the three steps, the first is crucial. If the choice of the form of the ancilla is not suitable, the items to produce the off-diagonal elements are not many, and thus the fidelity can not reach maximal. For simplicity and to avoid tedious calculations, we present some inclusive calculations.

For the $1 \to 3$ PCC, we consider the following unitary transformation:

$$\begin{aligned}
|0\rangle \to\ & x_1[(|001\rangle + |010\rangle + |100\rangle)|1\rangle_{\rm a} \\
& + (|002\rangle + |020\rangle + |200\rangle)|2\rangle_{\rm a}] \\
& + y_1(|012\rangle + |021\rangle + |102\rangle + |120\rangle \\
& + |201\rangle + |210\rangle)|0\rangle_{\rm a}, \\
|1\rangle \to\ & x_1[(|110\rangle + |101\rangle + |011\rangle)|1\rangle_{\rm a} \\
& + (|112\rangle + |121\rangle + |211\rangle)|0\rangle_{\rm a}] \\
& + y_1(|102\rangle + |120\rangle + |012\rangle \\
& + |021\rangle + |201\rangle + |210\rangle)|2\rangle_{\rm a}, \\
|2\rangle \to\ & x_1[(|220\rangle + |202\rangle + |022\rangle)|2\rangle_{\rm a} \\
& + (|221\rangle + |212\rangle + |122\rangle)|0\rangle_{\rm a}] \\
& + y_1(|201\rangle + |210\rangle + |012\rangle \\
& + |021\rangle + |102\rangle + |120\rangle)|1\rangle_{\rm a}, \qquad (10)
\end{aligned}$$

where the first three particles in the right-hand side of Eq. (10) are the clones and the fourth is an ancilla. Equation (10) can also be rewritten in a compact form as

$$|i\rangle \to x_1 \sum_{\substack{j=0 \\ i\neq j}}^{2} |(iij)\rangle|i\oplus j\rangle_{\rm a} + y_1|(ijk)\rangle|j\oplus k\rangle_{\rm a}, \qquad (11)$$

where $j\oplus k$ in the ancilla state $|j\oplus k\rangle$ modulo 3. In the following, we set $i,j,k = 0,1,2$ and $i \neq j \neq k$.

Let us briefly explain the choice of the unitary transformation given by Eq. (10). Mathematically, a complete transformation should conclude the item $a_1|(iii)\rangle|R_{iii}\rangle_{\rm a} = a_1|iii\rangle|R_{iii}\rangle_{\rm a} = a_1|i\rangle|ii\rangle|R_{iii}\rangle_{\rm a}$. However, no matter what the form $|R_{iii}\rangle_{\rm a}$ of the ancilla is, after tracing out the common reducible state $|ii\rangle|R_{iii}\rangle_{\rm a}$, the reduced matrix of the state $a_1|(iii)\rangle|R_{iii}\rangle_{\rm a}$ only contains the diagonal elements, which has no contributions to the clone fidelity. On the other hand, the existence of the coefficient $a_1$ will inevitably decrease the values of $x_1$ and $y_1$ due to the normalization condition. So we have to choose the form of the clones as $|(iij)\rangle$ and $|(ijk)\rangle$. Therefore, not selecting the states $|(iii)\rangle$ is not an important step. For the first step to produce more off-diagonal elements, the valid form of the output state is $(a\,{\rm e}^{{\rm i}\phi_0}|0\rangle + b\,{\rm e}^{{\rm i}\phi_1}|1\rangle + c\,{\rm e}^{{\rm i}\phi_2}|2\rangle)|\cdot\cdot\rangle|R\rangle_{\rm a}$, where the common reducible state $|\cdot\cdot\rangle|R\rangle_{\rm a}$ of three particles, two clones and one ancilla can be reduced. In our choice of the form of the ancilla, the number of reducible states is the largest.

We next calculate the clone fidelity. The state $|(iij)\rangle$ contains two states $|i\rangle$, one state $|j\rangle$, and zero state $|k\rangle$. The numbers of the basis in states $|(iij)\rangle$ and $|(ijk)\rangle$ are easily calculated as $N_{x_1}[iij] = C_3^2C_1^1C_0^0 = C_{3n}^{n+1}C_{2n-1}^nC_{n-1}^{n-1} = 3$, $N_{y_1}[ijk] = C_3^1C_2^1C_1^1 = C_{3n}^nC_{2n}^nC_n^n = 6$, for $n = 1$, where

$$C_m^n = \frac{m!}{(m-n)!n!}.$$

Thus, the normalization condition is

$$6x_1^2 + 6y_1^2 = 1 \ \text{ or } \ 2N_{x_1}x_1^2 + N_{y_1}y_1^2 = 1. \qquad (12)$$

With the help of Eq. (10), we can obtain the reduced density matrix $(A)_{ij}$ of each clone; so it is sufficient to calculate the fidelity of the first particle. Furthermore, due to the symmetry of Eq. (11), that is, the distributing basis, $|(iij)\rangle$ and $|(ijk)\rangle$, and the corresponding coefficients, $x_1$ and $y_1$, are symmetric, the modules of the off-diagonal elements are equal, i.e., $a_{ij} = a_{kl}$. Therefore, it is sufficient to calculate one off-diagonal element. The off-diagonal element $a_{01}$ is calculated as follows. For the parameter $x_1$, two computational bases $x_1|0\rangle(|01\rangle + |10\rangle)|1\rangle_{\rm a}$ with respect to $x_1\langle 1|(\langle 01| + \langle 10|)\langle 1|_{\rm a}$ can contribute to $a_{01}$, including the item $2x_1^2$; two states $x_1\,{\rm e}^{{\rm i}\varphi_0}|0\rangle(|02\rangle + |20\rangle)|2\rangle_{\rm a}$ with respect to $y_1\,{\rm e}^{-{\rm i}\varphi_1}\langle 1|(\langle 02| + \langle 20|)\langle 2|_{\rm a}$ and two states $y_1\,{\rm e}^{{\rm i}\varphi_0}|0\rangle(|12\rangle + |21\rangle)|0\rangle_{\rm a}$ with respect to $x_1\,{\rm e}^{-{\rm i}\varphi_1}\langle 1|(\langle 12| + \langle 21|)\langle 0|_{\rm a}$ can contribute to $a_{01}$, including the item $4x_1y_1\,{\rm e}^{{\rm i}(\varphi_0-\varphi_1)}$. So we can obtain $A_{01} = a_{01}\,{\rm e}^{{\rm i}(\varphi_0-\varphi_1)} = (2x_1^2 + 4x_1y_1)\,{\rm e}^{{\rm i}(\varphi_0-\varphi_1)}/3$, where $1/3$ and ${\rm e}^{{\rm i}(\varphi_0-\varphi_1)}$ are introduced by the amplitude $1/\sqrt{3}$ and the phase factors $\varphi_0$, $\varphi_1$ of the input state, respectively. The number of the state $|(iij)\rangle$ contributing to the off-diagonal elements is

$$N_{\rm off}[x_1^2] = N_{\rm off}[ij] = C_2^1C_1^1C_0^0 = C_{3n-1}^nC_{2n-1}^nC_{n-1}^{n-1} = 2,$$

and the number of the state $|(ijk)\rangle$ is

$$\begin{aligned}
N_{\rm off}[x_1y_1] &= N_{\rm off}[ij] + N_{\rm off}[jk] \\
&= 2C_2^1C_1^1C_0^0 = 2C_{3n-1}^nC_{2n-1}^nC_{n-1}^{n-1} = 4 \ \text{ for } \ n = 1.
\end{aligned}$$

Thus, the reduced density matrix, $(A)_{ij}$, of each clone can be expressed as

$$A_{ij} = \frac{1}{3}(2x_1^2 + 4x_1y_1)\,{\rm e}^{{\rm i}(\varphi_i-\varphi_j)}$$

or

$$a_{ij} = \frac{x_1^2N_{\rm off}[x_1^2] + x_1y_1N_{\rm off}[x_1y_1]}{3}. \qquad (13)$$

According to Eq. (8), the fidelity can be obtained as

$$f_{1\to3}^{\rm (PPC)}(3) = \frac{1}{3}[1 + 2(x_1^2N_{\rm off}[x_1^2] + x_1y_1N_{\rm off}[x_1y_1])]. \qquad (14)$$

We want to obtain the maximal fidelity given by Eq. (14) under the normalization condition given by Eq. (12). After some algebras, we determine the cloning coefficients as

$$x_1 = \frac{1}{2}\sqrt{\frac{1}{15}(5+\sqrt{5})}, \ \ y_1 = \frac{1}{2}\sqrt{\frac{1}{15}(5-\sqrt{5})}. \qquad (15)$$

The fidelity is then given by

$$F_{1\to3}^{(\mathrm{PPC})}(3)=\frac{1}{9}(4+\sqrt{5}), \tag{16}$$

which is covered by Eq. (4).

We generalize Eq. (11) to the $1\to M=3n$ case, the transformation is defined as

$$\begin{aligned}|i\rangle \to\ & X_1\sum_{\substack{j=0\\ j\neq i}}^{2}|(iij(012)^{\otimes(n-1)})\rangle|i\oplus j\rangle\\ &+Y_1\sum_{\substack{j,k=0\\ i\neq j\neq k}}^{2}|(ijk(012)^{\otimes(n-1)})\rangle|j\oplus k\rangle,\end{aligned} \tag{17}$$

where the notation $(012)^{\otimes(n-1)}$ denotes $\left(\underbrace{012,\cdots,012}_{3(n-1)}\right)$. The numbers of the states in Eq. (17) are

$$\begin{aligned}&N_{X_1}[iij(012)^{\otimes(n-1)}]=C_{3n}^{n+1}C_{2n-1}^{n}C_{n-1}^{n-1},\\ &N_{Y_1}[ijk(012)^{\otimes(n-1)}]=C_{3n}^{n}C_{2n}^{n}C_{n}^{n},\\ &N_{\mathrm{off}}[X_1^2]=N_{\mathrm{off}}[(ij(012)^{\otimes(n-1)})]=C_{3n-1}^{n}C_{2n-1}^{n}C_{n-1}^{n-1},\\ &N_{\mathrm{off}}[X_1Y_1]=N_{\mathrm{off}}[(ij(012)^{\otimes(n-1)})]+N_{\mathrm{off}}[(jk(012)^{\otimes(n-1)})]\\ &\qquad=2C_{3n-1}^{n}C_{2n-1}^{n}C_{n-1}^{n-1}.\end{aligned} \tag{18}$$

Similar to Eqs. (12) and (14), the normalization condition and the fidelity are obtained as

$$2N_{X_1}X_1^2+N_{Y_1}Y_1^2=1, \tag{19}$$

$$f_{1\to M=3n}^{(\mathrm{PPC})}(3)=\frac{1}{3}[1+2(X_1^2N_{\mathrm{off}}[X_1^2]+X_1Y_1N_{\mathrm{off}}[X_1Y_1])]. \tag{20}$$

When the cloning coefficients are

$$\begin{aligned}X_1&=\frac{1}{2\sqrt{N_{X_1}}}\sqrt{1+\sqrt{\frac{1+n}{1+9n}}},\\ Y_1&=\frac{(\sqrt{1+9n}-\sqrt{1+n})\sqrt{1+9n+\sqrt{(1+n)(1+9n)}}}{4\sqrt{n(1+9n)}\sqrt{N_{Y_1}}},\end{aligned} \tag{21}$$

the fidelity given by Eq. (4) is optimal.

### 3.2. Optimal $1\to M=3n+1$ PPC

In the $1\to4$ case, the PPC is economical,[13,14] which works without an ancilla. We define the following transformation:

$$\begin{aligned}|0\rangle \to\ & \frac{1}{\sqrt{12}}(|0012\rangle+|0021\rangle+|0120\rangle+|0102\rangle+|0210\rangle\\ &+|0201\rangle+|1002\rangle+|1020\rangle+|1200\rangle+|2001\rangle\\ &+|2010\rangle+|2100\rangle),\\ |1\rangle \to\ & \frac{1}{\sqrt{12}}(|1012\rangle+|1021\rangle+|1120\rangle+|1102\rangle+|1210\rangle\\ &+|1201\rangle+|0112\rangle+|0121\rangle+|0211\rangle+|2110\rangle\\ &+|2101\rangle+|2011\rangle),\\ |2\rangle \to\ & \frac{1}{\sqrt{12}}(|2012\rangle+|2021\rangle+|2120\rangle+|2102\rangle+|2210\rangle\\ &+|2201\rangle+|0212\rangle+|0122\rangle+|0221\rangle+|1202\rangle\\ &+|1220\rangle+|1022\rangle),\end{aligned} \tag{22}$$

or in a compact form

$$|i\rangle\to x_2|(i012)\rangle. \tag{23}$$

The number of the basis in the state $|(i012)\rangle$ is $N_{x_2}[i012]=C_4^2C_2^1C_1^1=12$, so the normalization condition is

$$N_{x_2}x_2^2=1\Rightarrow x_2^2=1/12. \tag{24}$$

The number of the items to produce the off-diagonal elements is $N_{\mathrm{off}}[x_2^2]=N_{\mathrm{off}}[012]=C_3^1C_2^1C_1^1=6$, and thus the fidelity can be directly calculated as

$$F_{1\to4}^{(\mathrm{PPC})}(3)=\frac{1}{3}(1+2x_2^2N_{\mathrm{off}}[x_2^2])=\frac{2}{3}, \tag{25}$$

which is covered by Eq. (5). In the $1\to M=3n+1$ case, the transformation directly takes the form

$$|i\rangle\to X_2|(i(012)^{\otimes n})\rangle. \tag{26}$$

For the state $|(i(012)^{\otimes n})\rangle$, we have

$$\begin{aligned}&N_{X_2}[i(012)^{\otimes n}]=C_{3n+1}^{n+1}C_{2n}^{n}C_{n}^{n},\ \ N_{X_2}X_2^2=1,\\ &N_{\mathrm{off}}[X_2^2]=N_{\mathrm{off}}[(012)^{\otimes n}]=C_{3n}^{n}C_{2n}^{n}C_{n}^{n}.\end{aligned} \tag{27}$$

The fidelity reads

$$\begin{aligned}F_{1\to M=3n+1}^{(\mathrm{PPC})}(3)&=\frac{1}{3}(1+2X_2^2N_{\mathrm{off}}[X_2^2])\\ &=\frac{1}{3}\left(1+\frac{2n+2}{3n+1}\right),\end{aligned} \tag{28}$$

which is given by Eq. (5).

### 3.3. Optimal $1\to M=3n+2$ PPC

The optimal $1\to5$ PCC transformation is defined as

$$\begin{aligned}&|0\rangle\to x_2[|(01012)\rangle|1\rangle+|(02012)\rangle|2\rangle_{\mathrm{a}}]+y_2|(00012)\rangle|0\rangle_{\mathrm{a}},\\ &|1\rangle\to x_2[|(10012)\rangle|0\rangle+|(12012)\rangle|2\rangle_{\mathrm{a}}]+y_2|(11012)\rangle|1\rangle_{\mathrm{a}},\\ &|2\rangle\to x_2[|(21012)\rangle|1\rangle+|(20012)\rangle|0\rangle_{\mathrm{a}}]+y_2|(22012)\rangle|2\rangle_{\mathrm{a}}.\end{aligned} \tag{29}$$

It can be rewritten as

$$|i\rangle\to x_3\sum_{\substack{j=0\\ j\neq i}}^{2}|(ij012)\rangle|j\rangle_{\mathrm{a}}+y_3|(ii012)\rangle|i\rangle_{\mathrm{a}}. \tag{30}$$

Note that equation (30) has a formulation similar to that of Eq. (11), so the formulations of the normalization condition and the fidelity are the same as the $1\to3n$ PPC. The constants needed are calculated as follows:

$$N_{x_3}[ij(012)]=C_5^2C_3^2C_1^1,\quad N_{y_3}[ii(012)]=C_5^3C_2^1C_1^1,$$

$$N_{\rm off}[x_3^2] = N_{\rm off}[i(012)] = C_4^2 C_2^1 C_1^1,$$
$$N_{\rm off}[x_3 y_3] = N_{\rm off}[j(012)] + N_{\rm off}[i(012)] = 2C_4^2 C_2^1 C_1^1. \quad (31)$$

The normalization condition and the fidelity are

$$60x_2^2 + 20y_2^2 = 1 \ \text{ or } \ 2N_{x_3}y_3^2 + N_{y_3}x_3^2 = 1, \quad (32)$$
$$f_{1\to5}^{(\rm PPC)}(3) = \frac{1}{3}[1 + 2(x_3^2 N_{\rm off}[x_3^2] + x_3 y_3 N_{\rm off}[x_3 y_3])]$$
$$= \frac{1}{3}[1 + 2(12x_3^2 + 24x_3 y_3)]. \quad (33)$$

The cloning coefficients are determined as

$$x_2 = \frac{1}{\sqrt{120}}\sqrt{1 + \frac{1}{\sqrt{13}}}, \ \ y_2 = \frac{1}{\sqrt{40}}\sqrt{1 - \frac{1}{\sqrt{13}}}. \quad (34)$$

The fidelity then is

$$F_{1\to5}^{(\rm PPC)}(3) = \frac{1}{15}(6 + \sqrt{13}), \quad (35)$$

which is covered by Eq. (6). From Eq. (30), the transformation of the $1 \to M = 3n+2$ PPC can be generalized as

$$|i\rangle \to X_3 \sum_{\substack{j=0 \\ j\neq i}}^{2} |(ij(012)^{\otimes n})\rangle |j\rangle_{\rm a} + Y_3 |(ii(012)^{\otimes n})\rangle |i\rangle_{\rm a}. \quad (36)$$

The constants needed are calculated as follows:

$$N_{X_3}[(ij(012)^{\otimes n})] = C_{3n+2}^{n+1} C_{2n+1}^{n+1} C_n^n,$$
$$N_{Y_3}[(ii(012)^{\otimes n})] = C_{3n+2}^{n+2} C_{2n}^n C_n^n,$$
$$N_{\rm off}[X_3^3] = N_{\rm off}^{(3n+2)}[(i(012)^{\otimes n})] = C_{3n+1}^{n+1} C_{2n}^n C_n^n,$$
$$N_{\rm off}[X_3 Y_3] = N_{\rm off}^{(3n+2)}[(i(012)^{\otimes n})] + N_{\rm off}^{(3n+2)}[(j(012)^{\otimes n})]$$
$$= 2C_{3n+1}^{n+1} C_{2n}^n C_n^n. \quad (37)$$

Similar to Eqs. (32) and (33), the normalization condition and the fidelity are obtained as

$$2N_{X_3}^{(3n+2)} X_3^2 + N_{Y_3}^{(3n+2)} Y_3^2 = 1,$$
$$f_{1\to M=3n+2}^{(\rm PPC)}(3) = \frac{1}{3}[1 + 2(X_3^2 N_{\rm off}[X_3^2] + X_3 Y_3 N_{\rm off}[X_3 Y_3])]. \quad (38)$$

The fidelity is optimal as given by Eq. (6) when the cloning coefficients are

$$X_3 = \frac{1}{2\sqrt{N_{X_3}}}\sqrt{1 + \frac{1+n}{\sqrt{(1+n)(17+9n)}}},$$
$$Y_3 = [\sqrt{(1+n)(17+9n)} - 1 - n]$$
$$\times \sqrt{17 + 9n + \sqrt{(1+n)(17+9n)}}$$
$$\times [4\sqrt{N_{Y_3}}\sqrt{(17+9n)(1+n)(2+n)}]^{-1}. \quad (39)$$

We here present the explicit transformation of the optimal $1 \to M = 3n, 3n+1, 3n+2$ PPC for qutrits. We do not give the proof of the optimality, since it has been given in a previous contribution.[13]

## 4. Conclusion

In summary, we have started with the explicit transformations of the optimal $1 \to 3, 4, 5$ PPC for qutrits, and then generalized the PPC to the $1 \to M = 3n$, $3n+1$, $3n+2$ case. The fidelities are coincident with the previous results.

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