Non-autonomous discrete Boussinesq equation: Solutions and consistency*

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A non-autonomous 3-component discrete Boussinesq equation is discussed. Its spacing parameters p_n and q_m are related to independent variables n and m, respectively. We derive bilinear form and solutions in Casoratian form. The plain wave factor is defined through the cubic roots of unity. The plain wave factor also leads to extended non-autonomous discrete Boussinesq equation which contains a parameter δ. Tree-dimendional consistency and Lax pair of the obtained equation are discussed.

Keywords: non-autonomous discrete Boussinesq equation, bilinear, solutions, Lax pair

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1. Introduction

Discrete integrable systems have made much progress in the past two decades. The property of multidimensional consistency[1–3] provides a practical approach to investigating the integrabilities for the discrete systems defined on an elementary quadrilateral as follows:

$$Q(u_{n,m},u_{n+1,m},u_{n,m+1},u_{n+1,m+1};p,q)=0, \tag{1}$$

where p and q are the spacing parameters of n and m directions, respectively. Roughly speaking, the above equation is multidimensionally consistent, which means that we can extend the number of independent variables by adding its copies in different directions without creating any inconsistency or multivaluedness. Multidimensionally consistent systems can be found by a searching approach. As a result, the Adler–Bobenko–Suris (ABS) list[4] and discrete Boussinesq (DBSQ) type equations[5] are found. If the spacing parameters are related to their directions, respectively, say $p=p_n=p(n)$ and $q=q_m=q(m)$, those lattice equations in the ABS list and the DBSQ type equations are still multidimensionally consistent. These equations are called non-autonomous since they contain n,m-dependent spacing parameters.

The non-autonomous ABS list has been studied by means of the singularity confinement test.[6] With regards to the solutions, these equations have been investigated through a bilinear approach[7] and by a comparison with the results of an autonomous case (e.g., Ref. [8]), deformations for solution elements from an autonomous case to a non-autonomous case are summarized in Table 1,[7] the procedure of which we call deautonomisation of lattice equations. We note that the ABS list (maybe except Q4 whose solutions are given via elliptic functions) corresponds to the second order dispersion relation, say, its plain wave factors are characterized by the roots of the second-order polynomials.

Table 1. Deautonomisation.

	Autonomous	Non-autonomous
Spacing paramaters	(a,b)	(a_n,b_m)
Linear function	$an+bm$	$\sum_{i=n_0}^{n-1}a_i+\sum_{j=m_0}^{m-1}b_j$
Discre exponential function	$\left(\frac{a+k}{a-k}\right)^n\left(\frac{b+k}{b-k}\right)^m$	$\prod_{i=n_0}^{n-1}\left(\frac{a_i+k}{a_i-k}\right)\prod_{j=m_0}^{m-1}\left(\frac{b_j+k}{b_j-k}\right)$

In this paper, we show that the above deautonomisation is also valid for the DBSQ equation. The dispersion relation of DBSQ-type equation has been shown to be characterized by the third order polynomials.[9,10] Now we consider the following discrete equation:

$$B_1\equiv\widetilde{w}-u\widetilde{u}+v=0, \tag{2a}$$

$$B_2\equiv\widehat{w}-u\widehat{u}+v=0, \tag{2b}$$

$$B_3\equiv w-u\widehat{\widetilde{u}}+\widehat{\widetilde{u}}-\frac{p_n-q_m}{\widehat{u}-\widetilde{u}}=0, \tag{2c}$$

where we have used the notation $\widetilde{u}=u_{n+1,m}$, $\widehat{v}=v_{n,m+1}$, n and m are the discrete independent variables, p_n and q_m are arbitrary functions of n and m, respectively, which serve as spacing

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parameters. When p_n and q_m are constants, this equation is referred to as the DBSQ equation.[11] Now the equation under consideration is the non-autonomous DBSQ equation.

Following the treatments for the autonomous DBSQ equation[12] and for the non-autonomous ABS case,[7] we construct the Casoratian solutions for the above non-autonomous DBSQ equation and investigate the relevant non-autonomous deformations.

The rest of this paper is organized as follows. In Section 2, we discuss the bilinearization of the non-autonomous DBSQ equation, and derive Casoratian solutions. The parameter extensions for the equation and solutions are then constructed in Section 3. Finally the three-dimensional (3D) consistensy and Lax pair of the parameter extended equation are investigated in Section 4.

2. Bilinearization and Casoratian solutions

2.1. Bilinearization

Through direct substitution, we can verify that equation (2) admits the following simple solution:

$$u_0 = \sum_{i=n_0}^{n-1} a_i + \sum_{j=m_0}^{m-1} b_i + c_1, \tag{3a}$$

$$v_0 = \frac{1}{2}u_0^2 + \frac{1}{2}\left(\sum_{i=n_0}^{n-1} a_i^2 + \sum_{j=m_0}^{m-1} b_i^2 + c_2\right) + \gamma, \tag{3b}$$

$$w_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\left(\sum_{i=n_0}^{n-1} a_i^2 + \sum_{j=m_0}^{m-1} b_i^2 + c_2\right) - \gamma, \tag{3c}$$

where the spacing parameters p_n and q_n are parameterized by

$$p_n = r - a_n^3, \quad q_m = r - b_m^3, \tag{4}$$

and c_1, c_2, and γ are arbitrary constants.

Using Eq. (3) as a background solution and through the transformation

$$u = u_0 - \frac{g_{n,m}}{f_{n,m}}, \quad v = v_0 - u_0\frac{g_{n,m}}{f_{n,m}} + \frac{h_{n,m}}{f_{n,m}},$$
$$w = w_0 - u_0\frac{g_{n,m}}{f_{n,m}} + \frac{s_{n,m}}{f_{n,m}}, \tag{5}$$

we can bilinearize Eq. (2) as follows:

$$\mathscr{H}_1 = \widetilde{f}(h + a_n g) - \widetilde{g}(g + a_n f) + f\widetilde{s} = 0, \tag{6a}$$

$$\mathscr{H}_2 = \widehat{f}(h + b_m g) - \widehat{g}(g + b_m f) + f\widehat{s} = 0, \tag{6b}$$

$$\mathscr{H}_3 = \widetilde{f}\widehat{g} - \widehat{f}\widetilde{g} - (a_n - b_m)(\widetilde{f}\widehat{f} - f\widehat{\widetilde{f}}) = 0, \tag{6c}$$

$$\mathscr{H}_4 = (a_n^2 + a_n b_m + b_m^2)(f\widehat{\widetilde{f}} - \widetilde{f}\widehat{f}) + (a_n + b_m)(f\widehat{\widetilde{g}} - \widehat{\widetilde{f}}g) + \widehat{\widetilde{f}}s + f\widehat{\widetilde{h}} - g\widehat{\widetilde{g}} = 0. \tag{6d}$$

These non-autonomous bilinear equations are similar to the autonomous case in Ref. [12] and the connection with the nonlinear form is

$$B_1 = \frac{\mathscr{H}_1}{f\widetilde{f}}, \quad B_2 = \frac{\mathscr{H}_2}{f\widehat{f}},$$

$$B_3 = \frac{\mathscr{H}_3\mathscr{H}_4 + (a_n - b_m)f\widehat{\widetilde{f}}\mathscr{H}_4 + (a_n^2 + a_n b_m + b_m^2)\widetilde{f}\widehat{f}\mathscr{H}_3}{f\widetilde{f}\widehat{f}\widehat{\widetilde{f}}(\widehat{u} - \widetilde{u})}. \tag{7}$$

2.2. Casoratian solutions

Consider the following multi-indexed column vector:

$$\psi(n,m,l) = (\psi_1(n,m,l), \psi_2(n,m,l), \ldots, \psi_N(n,m,l))^{\mathrm{T}}. \tag{8}$$

We also introduce the bar-shift of ψ to denote $\overline{\psi}(n,m,l) = \psi(n,m,l+1)$. The generic $N \times N$ Casorati matrix is given by combining columns with different shifts l_i. The generic Casoratian is then the determinant

$$C_{n,m}(\psi; \{l_i\}) = |\psi(n,m,l_1), \psi(n,m,l_2), \ldots, \psi(n,m,l_N)|. \tag{9}$$

Such Casoratians are usually described by using the shorthand notation[13] in which only the shifts are given. Furthermore, for consecutive sequences, we use $\widehat{M} \equiv 0, 1, \ldots, M$. As an example, we have

$$|\widehat{N-1}| = |0, 1, \ldots, N-2, N-1|,$$
$$|\widehat{N-2}, N| = |0, 1, \ldots, N-2, N|.$$

For the non-autonomous DBSQ equation (2), we are interested in the following Casoratian elements:

$$\psi_\nu(n,m,l) = \sum_{s=1}^{3} \rho_{\nu,s}^{(0)}(-\omega_s(k_\nu))^l \prod_{i=n_0}^{n-1}(a_i - \omega_s(k_\nu)) \times \prod_{j=m_0}^{m-1}(b_j - \omega_s(k_\nu)), \tag{10}$$

where n_0 and m_0 are some constant integers, and $\{\omega_s(k)\}$ are the roots of

$$\omega^3 - k^3 = 0, \tag{11}$$

which describes a third-order discrete dispersion relation.

Then, for the solution to the bilinear equation (6), we have the following proposition.

Proposition 1 Solution to the bilinear DBSQ equation (6) is given in terms of Casoratians by

$$f = |\widehat{N-1}|, \quad g = |\widehat{N-2}, N|,$$
$$h = |\widehat{N-2}, N+1|, \quad s = |\widehat{N-3}, N-1, N|, \tag{12}$$

composed of ψ in Eq. (10).

To prove the above proposition, we need the following two lemmas.

Lemma 1[13] Suppose that $\boldsymbol{B}$ is an $N \times (N-2)$ matrix and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are N-order column vectors, then

$$|\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{b}||\boldsymbol{B}, \boldsymbol{c}, \boldsymbol{d}| - |\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{c}||\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{d}| + |\boldsymbol{B}, \boldsymbol{a}, \boldsymbol{d}||\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{c}| = 0. \tag{13}$$

Lemma 2[13]

$$\sum_{j=1}^{N}|\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{j-1},\boldsymbol{b}\boldsymbol{a}_j,\boldsymbol{a}_{j+1},\ldots,\boldsymbol{a}_N| = \left(\sum_{j=1}^{N}b_j\right)|\boldsymbol{a}_1,\ldots,\boldsymbol{a}_N|, \tag{14}$$

where $\boldsymbol{a}_j=(a_{1j},\ldots,a_{Nj})^{\mathrm{T}}$ and $\boldsymbol{b}=(b_1,\ldots,b_N)^{\mathrm{T}}$ are N-order column vectors, and $\boldsymbol{b}\boldsymbol{a}_j$ denotes $(b_1a_{1j},\ldots,b_Na_{Nj})^{\mathrm{T}}$.

Proof for Proposition 1 The column vector $\psi(n,m,l)$ in Eq. (10) satisfies the shift relations

$$a_{n-1}\underset{\sim}{\psi}=\psi-\overline{\underset{\sim}{\psi}}, \tag{15a}$$

$$b_{m-1}\underset{\wedge}{\psi}=\psi-\overline{\underset{\wedge}{\psi}}, \tag{15b}$$

$$\Gamma_1\psi=\overline{\overline{\overline{\psi}}}, \tag{15c}$$

where $\Gamma_1=\mathrm{Diag}(-k_1^3,-k_2^3,\ldots,-k_N^3)$. For convenience, we introduce an up-shift operator E^ν and down-shift operator E_ν by

$$E^1\psi\equiv\widetilde{\psi},\ \ E_1\psi\equiv\underset{\sim}{\psi},\ \ E^2\psi\equiv\widehat{\psi},\ \ E_2\psi\equiv\underset{\wedge}{\psi}.$$

With this notation, the basic shift formulae for the Casoratians are

$$-\alpha_\mu^{N-2}E_\mu f=|\widehat{N-2},E_\mu\psi(N-2)|, \tag{16a}$$

$$-\alpha_\mu^{N-2}E_\mu[g+\alpha_\mu f]=|\widehat{N-3},N-1,E_\mu\psi(N-2)|, \tag{16b}$$

$$-\alpha_\mu^{N-2}E_\mu[h+\alpha_\mu g]=|\widehat{N-3},N,E_\mu\psi(N-2)|, \tag{16c}$$

$$(a_{n-1}-b_{m-1})a_{n-1}^{N-2}b_{m-1}^{N-2}\underset{\wedge}{f}=|\widehat{N-3},\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|, \tag{16d}$$

$$(a_{n-1}-b_{m-1})a_{n-1}^{N-2}b_{m-1}^{N-2}\left[\underset{\wedge}{g}+(a_{n-1}+b_{m-1})\underset{\wedge}{f}\right]=-a_{n-1}^{N-2}\underset{\sim}{f}+b_{m-1}^{N-2}\underset{\wedge}{f}+|\widehat{N-4},N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|, \tag{16e}$$

$$\begin{aligned}&(a_{n-1}-b_{m-1})a_{n-1}^{N-2}b_{m-1}^{N-2}\left[\underset{\wedge}{s}+(a_{n-1}+b_{m-1})\underset{\wedge}{g}+[a_{n-1}^2+a_{n-1}b_{m-1}+b_{m-1}^2]\underset{\wedge}{f}\right]\\&=-b_{m-1}a_{n-1}^{N-2}\underset{\sim}{f}+a_{n-1}b_{m-1}^{N-2}\underset{\wedge}{f}+|\widehat{N-5},N-3,N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|,\end{aligned} \tag{16f}$$

where $\mu=1,2$, $\alpha_1=a_{n-1}$, and $\alpha_2=b_{m-1}$. We skip the proof of these shift relations. For details readers can refer to Refs. [7] and [8].

In the following, we prove Eqs. (6a) and (6d) as examples. For Eq. (6a), we consider the down-tilde-shifted $\mathscr{H}_1$, i.e.,

$$f[\underset{\sim}{h}+a_{n-1}\underset{\sim}{g}]-g[\underset{\sim}{g}+a_{n-1}\underset{\sim}{f}]+\underset{\sim}{f}s=0. \tag{17}$$

Using Eqs. (16a)–(16c) we have

$$\begin{aligned}&-a_{n-1}^{N-2}[f(\underset{\sim}{h}+a_{n-1}\underset{\sim}{g})-g(\underset{\sim}{g}+a_{n-1}\underset{\sim}{f})+\underset{\sim}{f}s]\\&=|\widehat{N-1}||\widehat{N-3},N,\underset{\sim}{\psi}(N-2)|-|\widehat{N-2},N||\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)|+|\widehat{N-2},\underset{\sim}{\psi}(N-2)||\widehat{N-3},N-1,N|,\end{aligned}$$

which is zero in the light of Lemma 1 by taking

$$\boldsymbol{B}=(\widehat{N-3}),\ \ \boldsymbol{a}=\psi(N-2),\ \ \boldsymbol{b}=\psi(N-1),$$
$$\boldsymbol{c}=\psi(N),\ \ \boldsymbol{d}=\underset{\sim}{\psi}(N-2).$$

For Eq. (6d), using relation (15c) and Lemma 2, we can genertate an explicit form for the identity $(\mathrm{Tr}(\Gamma_1)\underset{\sim}{f})\underset{\wedge}{f}=(\mathrm{Tr}(\Gamma_1)\underset{\wedge}{f})\underset{\sim}{f}$, i.e.,

$$\begin{aligned}&a_{n-1}^{N-2}\underset{\sim}{f}\Big[|\widehat{N-5},N-3,N-2,N-1,\underset{\wedge}{\psi}(N-2)|\\&-|\widehat{N-4},N-2,N,\underset{\wedge}{\psi}(N-2)|\\&+|\widehat{N-3},N+1,\underset{\wedge}{\psi}(N-2)|+b_{m-1}^{N+1}\underset{\wedge}{f}+g-b_{m-1}f\Big]\\&-b_{m-1}^{N-2}\underset{\wedge}{f}\Big[|\widehat{N-5},N-3,N-2,N-1,\underset{\sim}{\psi}(N-2)|\\&-|\widehat{N-4},N-2,N,\underset{\sim}{\psi}(N-2)|\\&+|\widehat{N-3},N+1,\underset{\sim}{\psi}(N-2)|+a_{n-1}^{N+1}\underset{\sim}{f}+g-a_{n-1}f\Big]=0.\end{aligned} \tag{18}$$

Then we rewrite Eq. (6d) into the following version:

$$\begin{aligned}&f\left[\underset{\wedge}{s}+(a_{n-1}+b_{m-1})\underset{\wedge}{g}+(a_{n-1}^2+a_{n-1}b_{m-1}+b_{m-1}^2)\underset{\wedge}{f}\right]\\&-g\left[\underset{\wedge}{g}+(a_{n-1}+b_{m-1})\underset{\wedge}{f}\right]\\&-(a_{n-1}^2+a_{n-1}b_{m-1}+b_{m-1}^2)\underset{\sim}{f}\underset{\wedge}{f}+\underset{\wedge}{f}h=0.\end{aligned}$$

Furthermore, using Eqs. (16e) and (16f) we have

$$(a_{n-1}-b_{m-1})a_{n-1}^{N-2}b_{m-1}^{N-2}\times l.h.s. \tag{19a}$$

$$\begin{aligned}=\ &f|\widehat{N-5},N-3,N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\&-g|\widehat{N-4},N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\&+(a_{n-1}-b_{m-1})a_{n-1}^{N-2}b_{m-1}^{N-2}\underset{\wedge}{f}h\\&+a_{n-1}^{N-2}\underset{\sim}{f}[-b_{m-1}f+g+b_{m-1}^{N+1}\underset{\wedge}{f}]\\&-b_{m-1}^{N-2}\underset{\wedge}{f}[-a_{n-1}f+g+a_{n-1}^{N+1}\underset{\sim}{f}].\end{aligned} \tag{19b}$$

This is actually the left-hand side of Eq. (18), which is zero.

3. Parameter extension

We consider the following function:

$$\psi_\nu(n,m,l) = \sum_{s=1}^{3} \rho_{\nu,s}^{(0)} (\delta - \omega_s(k_\nu))^l \prod_{i=n_0}^{n-1} (a_i - \omega_s(k_\nu)) \times \prod_{j=m_0}^{m-1} (b_j - \omega_s(k_\nu)), \tag{20}$$

which is a δ-extension of Eq. (10). The column vector $\psi(n,m,l)$ with the above $\psi_\nu(n,m,l)$ satisfies δ-dependent shift relations as follows:

$$(a_{n-1} - \delta)\underset{\sim}{\psi} = \psi - \overline{\underset{\sim}{\psi}}, \tag{21a}$$
$$(b_{m-1} - \delta)\underset{\wedge}{\psi} = \psi - \overline{\underset{\wedge}{\psi}}, \tag{21b}$$
$$\Gamma_2 \psi = \overline{\overline{\overline{\psi}}} - 3\delta\overline{\overline{\psi}} + 3\delta^2\overline{\psi}, \tag{21c}$$

where $\Gamma_2 = \mathrm{Diag}(\delta^3 - k_1^3, \ldots, \delta^3 - k_N^3)$.

Like Proposition 1, we can find the following bilinear equation set:

$$\mathscr{H}_1^\delta = \widetilde{f}[h + (a_n - \delta)g] - \widetilde{g}[g + (a_n - \delta)f] + f\widetilde{s} = 0, \tag{22a}$$
$$\mathscr{H}_2^\delta = \widehat{f}[h + (b_m - \delta)g] - \widehat{g}[g + (b_m - \delta)f] + f\widehat{s} = 0, \tag{22b}$$
$$\mathscr{H}_3^\delta = \widetilde{f}\widehat{g} - \widehat{f}\widetilde{g} + (a_n - b_m)(\widetilde{f}\widehat{f} - f\widehat{\widetilde{f}}) = 0, \tag{22c}$$
$$\mathscr{H}_4^\delta = (a_n^2 + a_n b_m + b_m^2)(f\widehat{\widetilde{f}} - \widetilde{f}\widehat{f}) + (a_n + b_m + \delta)(\widehat{\widetilde{f}}g - f\widehat{\widetilde{g}}) + \widehat{\widetilde{f}}s + f\widehat{\widetilde{h}} - g\widehat{\widetilde{g}} = 0, \tag{22d}$$

admits the Casoratian solution

$$f = |\widehat{N-1}|, \quad g = |\widehat{N-2}, N|,$$
$$h = |\widehat{N-2}, N+1|, \quad s = |\widehat{N-3}, N-1, N|. \tag{23}$$

which are composed of the δ-independent ψ_ν given in Eq. (20).

Now, going further, if we still make use of the bilinearization formula (5) where u_0, v_0, and w_0 are given in Eq. (3), we recover the following nonlinear equation:

$$\widetilde{w'} = u'\widetilde{u'} - v' + \delta(\widetilde{u'} - u' - a_n), \tag{24a}$$
$$\widehat{w'} = u'\widehat{u'} - v' + \delta(\widehat{u'} - u' - b_m), \tag{24b}$$
$$w' = u'\widehat{\widetilde{u'}} - \widehat{\widetilde{v'}} + \frac{a_n^3 - b_m^3}{\widetilde{u'} - \widehat{u'}} - \delta(\widehat{\widetilde{u'}} - u' - a_n - b_m). \tag{24c}$$

This equation is called δ-non-autonomous DBSQ equation. It is still 3D consistent. Obviously, if taking $\delta = 0$, it reduces to Eq. (2). Besides, both Eq. (2) and Eq. (24) are related through

$$u' = u, \tag{25a}$$
$$v' = v - \delta(u - u_0), \tag{25b}$$
$$w' = w + \delta(u - u_0). \tag{25c}$$

The above transformation looks trivial because it is invertible. However, it is indeed not trivial. In fact, it is shown in Ref. [14] that the δ-DBSQ equation admits rational solutions with the help of the non-zero parameter δ; then through the above invertible transformation, rational solutions to the DBSQ equation can be given.

4. 3D consistency and Lax pair of non-autonomous δ-DBSQ equation

Now we consider the cube in Fig. 1, where F denotes the three components (u', v', w').

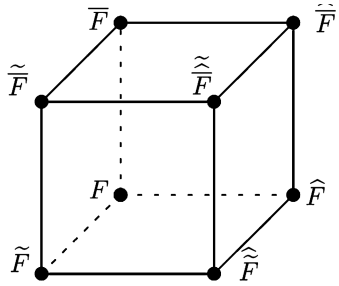


Fig. 1. The multi-dimensionally consistent cube.

Through Eq. (24), we have

$$\widehat{\widetilde{u'}} = \frac{\widetilde{v'} - \widehat{v'} + \delta(\widetilde{u'} - \widehat{u'} + b_m - a_n)}{\widetilde{u'} - \widehat{u'}}, \tag{26}$$

and

$$\overline{w'} = u'\overline{u'} - v' + \delta(\overline{u'} - u' - k), \tag{27a}$$
$$w' = u'\widetilde{\overline{u'}} - \widetilde{\overline{v'}} + \frac{a_n^3 - k^3}{\widetilde{u'} - \overline{u'}} - \delta(\widetilde{\overline{u'}} - u' - a_n - k), \tag{27b}$$
$$w' = u'\widehat{\overline{u'}} - \widehat{\overline{v'}} + \frac{k^3 - b_m^3}{\overline{u'} - \widehat{u'}} - \delta(\widehat{\overline{u'}} - u' - k - b_m). \tag{27c}$$

The initial values are given at $F, \widetilde{F}, \widehat{F}, \overline{F}$, but due to the linear equations (2a), (2b), and (27a), we only need to give (u', v', w'), $(\widetilde{u'}, \widetilde{v'})$, $(\widehat{u'}, \widehat{v'})$, $(\overline{u'}, \overline{v'})$. Given these values, we can compute $(\widehat{\widetilde{u'}}, \widehat{\widetilde{v'}}, \widehat{\widetilde{w'}})$, $(\widehat{\overline{u'}}, \widehat{\overline{v'}}, \widehat{\overline{w'}})$, $(\widetilde{\overline{u'}}, \widetilde{\overline{v'}}, \widetilde{\overline{w'}})$ using Eq. (26) and its tilde-bar and hat-bar versions, along with Eqs. (2c), (27b), and (27c). After this there are three different ways to compute the remaining $(\widehat{\widetilde{u'}}, \widehat{\widetilde{v'}}, \widehat{\widetilde{w'}})$ but the obtained values agree with each other as shown below and the system is therefore 3D consistent,

$$\widehat{\widetilde{u'}} = u' + \frac{(b_m^3 - a_n^3)\overline{u'} + (a_n^3 - k^3)\widehat{u'} + (k^3 - b_m^3)\widetilde{u'}}{\widetilde{u'}[\widehat{v'} - \overline{v'} + \delta(k - b_m)] + \widehat{u'}[\overline{v'} - \widetilde{v'} + \delta(a_n - k)] + \overline{u'}[\widetilde{v'} - \widehat{v'} + \delta(b_m - a_n)]}, \tag{28a}$$

$$\begin{aligned}\widehat{\widetilde{v'}} =\ & v' + \Big\{[a_n^3(\widetilde{u'}-\delta)+\delta(a_n+b_m+k)(\widetilde{v'}+\delta a_n)](\widehat{u'}-\overline{u'})\\ & + [b_m^3(\widehat{u'}-\delta)+\delta(a_n+b_m+k)(\widehat{v'}+\delta b_m)](\overline{u'}-\widetilde{u'}) + [a_n^3(\widetilde{u'}-\delta)+\delta(a_n+b_m+k)(\widetilde{v'}+\delta k)](\widehat{u'}-\overline{u'})\Big\}\\ & \Big/\Big\{\widetilde{u'}[\widehat{v'}-\overline{v'}+\delta(k-b_m)]+\widehat{u'}[\overline{v'}-\widetilde{v'}+\delta(a_n-k)]+\overline{u'}[\widetilde{v'}-\widehat{v'}+\delta(b_m-a_n)]\Big\}, \end{aligned} \tag{28b}$$

$$\begin{aligned}\widehat{\widetilde{w'}} =\ & w' + \Big\{\overline{v'}[a_n^3-b_m^3+\delta(a_n+b_m+k)(\widehat{u'}-\widetilde{u'})]+\widehat{v'}[a_n^3-k^3+\delta(a_n+b_m+k)(\widetilde{u'}-\overline{u'})]\\ & + \widetilde{v'}[k^3-b_m^3+\delta(a_n+b_m+k)(\overline{u'}-\widehat{u'})]+\delta[k^3(b_m-a_n-2(\widehat{u'}-\widetilde{u'}))\\ & + b_m^3(a_n-k-2(\widetilde{u'}-\overline{u'}))+a_n^3(k-b_m-2(\overline{u'}-\widehat{u'}))]\\ & - \delta^2(a_n+b_m+k)[k(\widehat{u'}-\widetilde{u'})+b_m(\widetilde{u'}-\overline{u'})+a_n(\overline{u'}-\widehat{u'})]\Big\}\\ & \Big/\Big\{\widetilde{u'}[\widehat{v'}-\overline{v'}+\delta(k-b_m)]+\widehat{u'}[\overline{v'}-\widetilde{v'}+\delta(a_n-k)]+\overline{u'}[\widetilde{v'}-\widehat{v'}+\delta(b_m-a_n)]\Big\}. \end{aligned} \tag{28c}$$

It is known that 3D consistency can be used to derive Lax pairs.[2] The procedure can be described as follows. Write Eq. (24) in the generic form as follows:

$$Q(\boldsymbol{F},\widetilde{\boldsymbol{F}},\widehat{\boldsymbol{F}},\widehat{\widetilde{\boldsymbol{F}}},a_n,b_m)=0, \tag{29}$$

where $\boldsymbol{F}=(u',v',w')^{\mathrm{T}}$. Owing to the 3D consistency, the Bäcklund transformation (BT) of Eq. (29) can be given by

$$Q(\boldsymbol{F},\widetilde{\boldsymbol{F}},\overline{\boldsymbol{F}},\widetilde{\overline{\boldsymbol{F}}},a_n,k)=0, \tag{30a}$$

$$Q(\boldsymbol{F},\overline{\boldsymbol{F}},\widehat{\boldsymbol{F}},\widehat{\overline{\boldsymbol{F}}},k,b_m)=0, \tag{30b}$$

where $\overline{\boldsymbol{F}}$ serves as a new solution to Eq. (29). Then, taking $\overline{\boldsymbol{F}}=(g'/f',h'/f',s'/f')^{\mathrm{T}}$ and $\boldsymbol{\Phi}_1=(f',g',h',s')^{\mathrm{T}}$, from the BT (30) we will find the form

$$\widetilde{\boldsymbol{\Phi}}_1=\boldsymbol{L}_1\boldsymbol{\Phi}_1,\ \widehat{\boldsymbol{\Phi}}_1=\boldsymbol{M}_1\boldsymbol{\Phi}_1, \tag{31}$$

where

$$\boldsymbol{L}_1=\begin{pmatrix} \widetilde{u'} & -1 & 0 & 0\\ -\widetilde{v'}-\delta(\widetilde{u'}+k-a_n) & \delta & 1 & 0\\ l_{31} & -w'+\delta(2u'+k+a_n)+\delta^2 & u'-\delta & 0\\ \delta(a_n\widetilde{u'}-\widetilde{v'}-\delta(\widetilde{u'}+k-a_n)) & -\widetilde{v'}-\delta(k-\delta) & \widetilde{u'}+\delta & 0 \end{pmatrix}, \tag{32}$$

and

$$l_{31}=k^3-a_n^3-u'\widetilde{v'}+\widetilde{u'}w'-\delta(\widetilde{u'}(2u'+k+a_n)+u'(k-a_n)-\widetilde{v'})+\delta^2(\widetilde{u'}+k-a_n).$$

The $\boldsymbol{M}_1$ is similar to $\boldsymbol{L}_1$ with the shift $\widetilde{\ }$ replaced by $\widehat{\ }$, while replacing a_n by b_m.

Since the last column of $\boldsymbol{L}_1$ is $\boldsymbol{0}$, we can drop off this column. For the non-autonomous δ-DBSQ equation (24), we find the following Lax pair:

$$\widetilde{\boldsymbol{\Phi}}=\boldsymbol{L}\boldsymbol{\Phi},\ \widehat{\boldsymbol{\Phi}}=\boldsymbol{M}\boldsymbol{\Phi}, \tag{33}$$

where

$$\boldsymbol{L}=\begin{pmatrix} \widetilde{u'} & -1 & 0\\ -\widetilde{v'}-\delta(\widetilde{u'}+k-a_n) & \delta & 1\\ l_{31} & -w'+\delta(2u'+k+a_n)+\delta^2 & u'-\delta \end{pmatrix}, \tag{34}$$

and $\boldsymbol{M}$ is similar to $\boldsymbol{L}$ with the shift $\widetilde{\ }$ replaced by $\widehat{\ }$ while replacing a_n by b_m. The compatibility $\widehat{\boldsymbol{L}}\boldsymbol{M}=\widetilde{\boldsymbol{M}}\boldsymbol{L}$ gives rise to the following equations:

$$\widehat{\widetilde{u'}}=\frac{\widetilde{v'}-\widehat{v'}+\delta(\widetilde{u'}-\widehat{u'}+b_m-a_n)}{\widetilde{u'}-\widehat{u'}}, \tag{35a}$$

$$w'=u'\widehat{\widetilde{u'}}-\widehat{\widetilde{v'}}+\frac{a_n^3-b_m^3+\delta(\widetilde{u'}-\widehat{u'}+b_m-a_n)}{\widetilde{u'}-\widehat{u'}}-\delta(\widehat{\widetilde{u'}}-u'-a_n-b_m). \tag{35b}$$

This equation is not the same as Eq. (24). Such a Lax pair is a weak Lax pair (e.g., Ref. [15]).

5. Conclusions

In the paper we discuss a non-autonomous 3-component DBSQ equation. We derive bilinear form and solutions in Casoratian form. The plain wave factor is defined through the cubic roots of unity. Compared with the results of the autonomous DBSQ equation given in Ref. [12], we find deautonomisation that we described in Table 1 in Section 1 is also valid for the equations with a 3rd-order dispersion relation with the corresponding defined plain wave factor. This indicates that the deautonomisation described in Table 1 might be universal for a large group of discrete soliton equations. We also introduce a parameter δ and obtaine an extended non-autonomous DBSQ equation. This equation is 3D consistent and the Lax pair obtained using the consistency is a weak one. An interesting fact is that although this equation is nonautonomous, in the Lax pair, the spectral parameter r is independent of n and m. The same fact appears in the non-autonomous ABS lattice equations. This is different from the non-autonomous or non-isospectral continuous and semi-discrete integrable systems where the spectra are usually time-dependent (e.g., Refs. [28] and [29]).

Integrable non-autonomous systems have their own importance. Actually, most of discrete Painlevé equations are non-autonomous ordinary difference equations, and these equations are more or less, related to the integrable non-autonomous partial difference equations through reductions. The non-autonomous DBSQ equation is also hoped to play a role in studying discrete Painlevé equations. Besides, recently, Lou *et al.*[33] proposed a method to generate the discrete integrable systems and a classical Lie symmetry approach was applied to integrable systems with discrete spacial variables.[34,35] We hope that these developments can be applied to non-autonomous discrete integrable systems.

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