Research on the discrete variational method for a Birkhoffian system*

Liu Shi-Xing(刘世兴)[a], Hua Wei(花 巍)[b], and GuoYong-Xin(郭永新)[a]†

[a] College of Physics, Liaoning University, Shenyang 110036, China
[b] College of Physics and Technology, Shenyang Normal University, Shenyang 110034, China

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In this paper, we present a new integration algorithm based on the discrete Pfaff–Birkhoff principle for Birkhoffian systems. It is proved that the new algorithm can preserve the general symplectic geometric structures of Birkhoffian systems. A numerical experiment for a damping oscillator system is conducted. The result shows that the new algorithm can better simulate the energy dissipation than the R-K method, which illustrates that we can numerically solve the dynamical equations by the discrete variational method in a Birkhoffian framework for the systems with a general symplectic structure. Furthermore, it is demonstrated that the results of the numerical experiments are determined not by the constructing methods of Birkhoffian functions but by whether the numerical method can preserve the inherent nature of the dynamical system.

Keywords: Birkhoff's equations, discrete variational methods, general symplectic structure, discrete Birkhoff's equations

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1. Introduction

In 1927, Birkhoff[1] presented a new form of dynamic equation which is more general than Hamilton's equation, and gave a new class of the integral variational principle, which is more general than Hamilton's principle in his famous book "Dynamic Systems". In 1978, Santilli[2] suggested that this equation be named the Birkhoff equation, the system that is described by the equation be named the Birkhoffian system and the new integral variational principle be called the Pfaff–Birkhoff principle. Theoretically, all conservative and non-conservative, self-adjoint and non-self-adjoint, holonomic and non-holonomic constrained systems can always be represented in the form of Birkhoff's equations.[3] Birkhoff's equations have been applied to hadron physics, statistical mechanics, space mechanics,[2] engineering,[4] electro-magnetism,[5] and the analytical calculation of ordinary differential equations.[3] The research of Birkhoffian dynamics is an important field in modern analytical mechanics. A number of issues about the Birkhoffian systems have been reported, such as the Birkhoffian representation of a non-holonomic constrained system, the symmetry theory, global analysis, stability of motion,[3] geometrical description, variational principle,[6] generalized Birkhoff's equation theory,[7,8] etc. However, the complex structures of Birkhoff's equations bring great difficulties in solving its analytical solutions, thereby limiting the corresponding application in mechanics, physics, and engineering areas. Therefore, the study of numerical methods of solving Birkhoff's equations can help to solve the application problems in various dynamical systems. However, there are only a few researches about the algorithms for Birkhoff's equations,[9–14] and most of the numerical methods are constructed by generating a function theory,[9,10] which can only be applied to simple dynamical systems but fails in the complex dynamics. In Refs. [12] and [13], we constructed a discrete integrator for autonomous Birkhoffian systems by using the discrete mechanics method. In this work, we extend the idea of structure-preserving, which is not suitable for Birkhoffian systems.

The purpose of this paper is to study the numerical algorithm of Birkhoff's equations by adopting discrete variational mechanics. The theory of discrete variational mechanics has its roots in the optimal control literature of the 1960s.[15–17] In the framework of Lagrange mechanics, the basic idea of discrete variational mechanics is to consider a mechanical system with configuration manifold Q, velocity phase space TQ, and Lagrangian map $L : TQ \to R$. In discrete mechanics, the starting point is to replace TQ with $Q \times Q$, and take the discrete Lagrangian map $L_{\rm d} : Q \times Q \to R$. The kinetic equations of the discretization system can be obtained by the Hamilton principle.

The remainder of this paper is organized as follows. In Section 2, we introduce briefly the Pfaff–Birkhoff principle and Birkhoff's equations. In Section 3, we discrete the Pfaff–Birkhoff principle and Birkhoff's equations and present their geometric properties based on the discrete Pfaff–Birkhoff principle, and then construct the Birkhoff variational integrators. In Section 4, we give an example by computing a one-dimensional damped oscillator in the framework of Brikhoffian mechanics. Finally, we draw some conclusions from the

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†Corresponding author. E-mail: yxguo@lnu.edu.cn

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present study in Section 5. Throughout this paper, Einstein's summation convention is used and $\mu,\nu=1,\ldots,2n$.

2. Pfaff–Birkhoff's principle and Birkhoff's equations

Consider a configuration manifold M^{2n}, with coordinate $\{a^\mu\}$ and time space R. Define the extended configuration manifold as $\hat{M}=R\times M^{2n}$. Take the Birkhoffian $B:\hat{M}\to R$ and Birkhoffian functions $R_\mu:\hat{M}\to R$, the Pfaffian is[3]

$$P(a,\dot{a},t)=R_\mu(a,t)\,\dot{a}^\mu-B(a,t). \tag{1}$$

The Pfaff action A is defined as

$$A=\int_{t_1}^{t_2}P(a,\dot{a},t)\,\mathrm{d}t=\int_{t_1}^{t_2}\left(R_\mu(a,t)\,\dot{a}^\mu-B(a,t)\right)\mathrm{d}t, \tag{2}$$

where $a^\mu=a^\mu(t)$ is the C^2 curve in configuration space. By using the contemporaneous variations $\delta A=0$ with fixed endpoint conditions $\delta a^\mu(t_1)=0,\delta a^\mu(t_2)=0$,[2] the Birkhoff's equations can be obtained as follows:

$$\left(\frac{\partial R_\nu(a,t)}{\partial a^\mu}-\frac{\partial R_\mu(a,t)}{\partial a^\nu}\right)\dot{a}^\nu-\left(\frac{\partial B(a,t)}{\partial a^\mu}+\frac{\partial R_\mu(a,t)}{\partial t}\right)=0. \tag{3}$$

The system described by Eq. (3) is named a Birkhoffian system. The process of variation is known as the Pfaff–Birkhoff principle, which is a universal form of the first-order integral variational principle. When $R_\mu=p_i$, $\dot{a}^\mu=\dot{q}^i$, and $B(a,t)=H(q,p,t)$, the principle degenerates into Hamilton's principle, so the Hamilton principle is a special case of the Pfaff–Birkhoff principle.

Consider a dynamical system described by first-order differential equations[2,3]

$$\dot{a}^\mu=\sigma^\mu(t,a). \tag{4}$$

Generally, there are three methods to construct dynamical functions $B(a,t)$ and $R_\mu(a,t)$ as follows.

Method 1 Let B be the total energy of Eq. (4) and then solve the Cauchy–Kovalevski (CK) equations in the functions R_μ. The CK equations are

$$\frac{\partial R_\mu}{\partial t}=\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\sigma^\nu-\frac{\partial B}{\partial a^\mu}.$$

Method 2 Via the method of the genotopic transformations staring from Eq. (4), construct a self-adjoint covariant form as

$$\left[\Omega_{\mu\nu}(t,a)\,\dot{a}^\nu+\Gamma_\mu(t,a)\right]_{SA}=0.$$

Then the Birkhoffian B and functions R_μ are given by

$$B(t,a)=-\left[\int_0^1\mathrm{d}\tau\left(\Gamma_\mu+\frac{\partial R_\mu}{\partial t}\right)(t,\tau a)\right]a^\mu,$$

$$R_\mu(t,a)=\left[\int_0^1\mathrm{d}\tau\tau\Omega_{\mu\nu}(t,\tau a)\right]a^\nu.$$

Method 3 Suppose that the obtained $2n$ first integrals I^μ of the first-order system (4) are independent in the sense of $\det(\partial I^\mu/\partial a^\nu)\neq 0$, then

$$R_\mu(t,a)=G_\nu\frac{\partial I^\nu}{\partial a^\mu},\quad B(t,a)=-G_\nu\frac{\partial I^\nu}{\partial t}.$$

Here, G_μ are functions of the integrals I^μ, which are constrained by the regularity condition

$$\det\left(\partial G_\mu/\partial I^\nu-\partial G_\nu/\partial I^\mu\right)\neq 0.$$

3. Discrete variational method of Birkhoff's equations and variational integrators

3.1. Discrete Pfaff–Birkhoff principle

Define the discrete state space as $\hat{M}\times\hat{M}$ with discrete coordinates $\{(t_k,a_k)\}_{k=0}^N$. The discrete Pfaffian is a map $P_\mathrm{d}:\hat{M}\times\hat{M}\to R$ with coordinates as follows:

$$P_\mathrm{d}=P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1}). \tag{5}$$

Define the discrete path space as

$$\hat{C}_\mathrm{d}(\hat{M})=\hat{C}_\mathrm{d}\left(\{t_k\}_{k=0}^N,M^{2n}\right)=\left\{a_\mathrm{d}^\nu:\{t_k\}_{k=0}^N\to M^{2n}\right\},$$

and take $c_\mathrm{d}\in\hat{C}_\mathrm{d}$ as the discrete curves in discrete space. The discrete curves are sets $\{(t_k,a_k)\}_{k=0}^N$ of discrete points $(t_k,a_k)\in\hat{M}$.

Definition 1 Discrete action functional is a map $\hat{A}_\mathrm{d}:\hat{C}_\mathrm{d}\to R$, given in coordinates by

$$\hat{A}_\mathrm{d}(c_\mathrm{d})=\sum_{k=0}^{N-1}P_\mathrm{d}(t_k,t_{k+1},a_k,a_{k+1}). \tag{6}$$

Then the discrete Pfaff–Birkhoff principle is given below.

Theorem 1 Given a $C^k(k>1)$ extended discrete Pfaffian $P_\mathrm{d}:\hat{M}\times\hat{M}\to R$, then there will exist a unique C^{k-1} mapping $D_\mathrm{B}:\hat{M}\times\hat{M}\to T^*\hat{M}$ and unique one-forms $\hat{\Theta}_{p_\mathrm{d}}^+$ and $\hat{\Theta}_{p_\mathrm{d}}^-$ in the discrete state space $\hat{M}\times\hat{M}$, such that for all variations $\delta c_\mathrm{d}\in T\hat{C}_\mathrm{d}$ of $c_\mathrm{d}\in\hat{C}_\mathrm{d}$, we have

$$\begin{aligned}&\mathrm{d}\hat{A}_\mathrm{d}(c_\mathrm{d})\cdot\delta c_\mathrm{d}\\&=\sum_{k=1}^{N-1}D_\mathrm{B}P_\mathrm{d}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1})\cdot(\delta t_k,\delta a_k)\\&\quad+\hat{\Theta}_{P_\mathrm{d}}^+(t_{N-1},t_N,a_{N-1},a_N)\cdot(\delta t_{N-1},\delta t_N,\delta a_{N-1},\delta a_N)\\&\quad-\hat{\Theta}_{P_\mathrm{d}}^-(t_0,t_1,a_0,a_1)\cdot(\delta t_0,\delta t_1,\delta a_0,\delta a_1),\end{aligned} \tag{7}$$

where map D_B is called the extended discrete Birkhoffian map and the one-forms $\hat{\Theta}_{p_\mathrm{d}}^+$ and $\hat{\Theta}_{p_\mathrm{d}}^-$ are the extended discrete Birkhoffian one-forms. In coordinates, they have the expressions

$$D_\mathrm{B}P_\mathrm{d}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1})$$

$$
\begin{aligned}
&= [D_1P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}) + D_2P_{\mathrm{d}}(t_{k-1},t_k,a_{k-1},a_k)]\,dt_k \\
&+ [D_3P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}) + D_4P_{\mathrm{d}}(t_{k-1},t_k,a_{k-1},a_k)]\,da_k, \\
&\hat{\Theta}^{+}_{P_{\mathrm{d}}}(t_k,t_{k+1},a_k,a_{k+1}) \\
&= D_2P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1})\,dt_{k+1} \\
&+ D_4P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1})\,da_{k+1}, \qquad (8) \\
&\hat{\Theta}^{-}_{P_{\mathrm{d}}}(t_k,t_{k+1},a_k,a_{k+1}) \\
&= -D_1P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1})\,dt_k \\
&- D_3P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1})\,da_k. \qquad (9)
\end{aligned}
$$

Theorem 1 can be proved by using the contemporaneous variations methods.

3.2. Discrete Birkhoff equation and its geometric properties

From Eq. (7), it is clear that c_{d} is a solution if and only if $d\hat{A}_{\mathrm{d}}(c_{\mathrm{d}})\cdot\delta c_{\mathrm{d}} = 0$ at all points except the endpoints 0 and N. This statement at k reads

$$D_{\mathrm{B}}P_{\mathrm{d}}(t_{k-1},t_k,t_{k+1},a_{k-1},a_k,a_{k+1}) = 0. \qquad (10)$$

Equation (10) is named a discrete Birkhoff's equation. It can be separated into the time and configuration components to give

$$D_1P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}) + D_2P_{\mathrm{d}}(t_{k-1},t_k,a_{k-1},a_k) = 0, \qquad (11)$$

$$D_3P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}) + D_4P_{\mathrm{d}}(t_{k-1},t_k,a_{k-1},a_k) = 0. \qquad (12)$$

However, the discrete paths which satisfy Eq. (11) do not automatically satisfy Eq. (12). That is, both components contribute to restrictions on the space of solutions, and both are necessary for numerically solving Birkhoff's equations. This is because the extended action is a function of the entire extended path, not just some discrete associated curve.

Equations (11) and (12) give an extended discrete Birkhoffian map $\hat{F}_{P_{\mathrm{d}}}: \hat{M}\times\hat{M}\to\hat{M}\times\hat{M}$, which can be written in coordinates as

$$\hat{F}_{P_{\mathrm{d}}}(t_{k-1},t_k,a_{k-1},a_k) = (t_k,t_{k+1},a_k,a_{k+1}). \qquad (13)$$

If the points $\{(t_k,a_k)\}_{k=0}^{N}$ satisfy Eq. (13), the extended discrete path $c_{\mathrm{d}}\in\hat{C}_{\mathrm{d}}$ is a solution of Eqs. (11) and (12). In the discrete case, there are two one-forms (8) that arise from the boundary terms, and by using $d^2=0$, we have

$$\mathrm{d}\hat{\Theta}^{+}_{P_{\mathrm{d}}} = \mathrm{d}\hat{\Theta}^{-}_{P_{\mathrm{d}}}.$$

With the extended discrete Pfaffian, the extended discrete solution space $\hat{C}_{P_{\mathrm{d}}}$ can be taken as the initial condition space of systems $\hat{M}\times\hat{M}$, and define the extended discrete action as $\hat{A}_{\mathrm{d}}:\hat{M}\times\hat{M}\to R$, in coordinates

$$\hat{A}_{\mathrm{d}}(t_0,t_1,a_0,a_1) = \hat{A}_{\mathrm{d}}(c_{\mathrm{d}}).$$

Here, $c_{\mathrm{d}}\in\hat{C}_{P_{\mathrm{d}}}$, and it is the solution of discrete Birkhoff's equations with the initial conditions $c(0)=(t_0,a_0)$ and $c(1)=(t_1,a_1)$. From Eq. (6), we can conclude that the one-forms satisfy

$$\mathrm{d}\hat{A}_{\mathrm{d}} = \left(\hat{F}^{N-1}_{P_{\mathrm{d}}}\right)^{*}\left(\hat{\Theta}^{+}_{P_{\mathrm{d}}}\right) - \hat{\Theta}^{-}_{P_{\mathrm{d}}}.$$

Due to the fact that $d^2\hat{A}_{\mathrm{d}}=0$, the conservation law

$$\left(\hat{F}^{N-1}_{P_{\mathrm{d}}}\right)^{*}\left(\hat{\Omega}_{P_{\mathrm{d}}}\right) = \hat{\Omega}_{P_{\mathrm{d}}}$$

can be obtained, where $\hat{\Omega}_{P_{\mathrm{d}}} = -d\hat{\Theta}^{+}_{P_{\mathrm{d}}} = -d\hat{\Theta}^{-}_{P_{\mathrm{d}}}$ are the extended discrete Birkhoffian two-forms as well as the discrete Birkhoffian symplectic forms. Take an arbitrary subinterval of the discrete sequence $0,1,\ldots,N$ and perform the same procedure, we can obtain the conservation law in the discrete interval. So the discrete Birkhoffian map $\hat{F}_{P_{\mathrm{d}}}$ is discretely symplectic.

3.3. Birkhoff's variational integrator

We can regard the discrete Birkhoff's equations in Eq. (10) as integrators of a given Birkhoffian system, which can be named Birkhoff's integrators. From Subsection 3.2, we see that the discrete Birkhoff map $\hat{F}_{P_{\mathrm{d}}}$ can preserve the general symplectic structure, so it is a symplectic integrator, and the discrete Pfaffian $P_{\mathrm{d}}:\hat{M}\times\hat{M}\to R$ satisfies

$$
\begin{aligned}
&P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}) \\
&= \int_{t_k}^{t_{k+1}} P(t,a(t),\dot{a}(t))\,\mathrm{d}t + O\left(|t_{k+1}-t_k|^{r+1}\right),
\end{aligned}
$$

where $a:[t_k,t_{k+1}]\to\hat{M}$ is any solution of Birkhoff's Eq. (3) for P. Here, r is known as the order of the discrete Pfaffian, and we require $r\geq 1$ for P_{d} to be consistent. To construct the integrators, we define the maps by using Eqs. (10) and (11)

$$
\begin{aligned}
&\hat{F}^{+}P_{\mathrm{d}}:(t_k,t_{k+1},a_k,a_{k+1}) \\
&\to (t_{k+1},b_{k+1}=D_4P_{\mathrm{d}},a_{k+1},E_{k+1}=-D_2P_{\mathrm{d}}), \\
&\hat{F}^{-}P_{\mathrm{d}}:(t_k,t_{k+1},a_k,a_{k+1}) \\
&\to (t_{k+1},b_k=-D_3P_{\mathrm{d}},a_{k+1},E_k=D_1P_{\mathrm{d}}), \qquad (14)
\end{aligned}
$$

where b_k,b_{k+1},E_k,E_{k+1} are the intermediate variables, and map (14) satisfies

$$\hat{F}_{P_{\mathrm{d}}} = \hat{F}^{+}P_{\mathrm{d}}\circ(\hat{F}^{-}P_{\mathrm{d}})^{-1}.$$

Regarding $\hat{F}_{P_{\mathrm{d}}}$ as an integrator of systems, it maps

$$\hat{F}_{P_{\mathrm{d}}}:(t_k,E_k,a_k,b_k)\to(t_{k+1},E_{k+1},a_{k+1},b_{k+1}),$$

and is defined by the relations

$$
\begin{aligned}
b_k &= -D_3P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}), \\
E_k &= D_1P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}), \qquad (15) \\
b_{k+1} &= D_4P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}), \\
E_{k+1} &= -D_2P_{\mathrm{d}}(t_k,t_{k+1},a_k,a_{k+1}), \qquad (16)
\end{aligned}
$$

together with the requirement that $t_{k+1} > t_k$ for each point k. In computation, the implicit equation (15) is solved and advanced to t_{k+1} and a_{k+1} under the restriction $t_{k+1} > t_k$, then with t_{k+1} and a_{k+1} the explicit equation (16) is propagated to give b_{k+1} and E_{k+1}, so the map $\hat{F}_{P_\mathrm{d}} : (t_k, E_k, a_k, b_k) \to (t_{k+1}, E_{k+1}, a_{k+1}, b_{k+1})$ can be obtained. As has been stated, the method defined by Eq. (10) preserves a discrete symplectic form and satisfies the discrete energy evolution equations. There are many methods to discrete the Pfaffian, such as Euler's mid-point rule, the Verlet scheme, the R-K method,[15–17] etc. In this paper, the Euler mid-point rule will be used.

4. Discrete Birkhoffization of one-dimensional damped oscillator and numerical analysis

Consider a one-dimensional damped linear oscillator system

$$\ddot{x} + \gamma\dot{x} + x = 0, \quad \gamma = \mathrm{const} > 0.$$

It is a non-conservative system and cannot be expressed in a standard Hamilton equation, so it has no simple symplectic structure. We cannot adopt the symplectic-preserving algorithms to calculate it. But it can be expressed in Birkhoff's equations.

Let $a^1 = x, a^2 = \dot{x}$, Method 1 and Method 2 of constructing Birkhoffian functions can be used to obtain the Birkhoffian B and functions R_μ respectively, then two different Birkhoffian representations are obtained as follows:

(i)

$$B = \frac{1}{2}\left[\left(a^1\right)^2 + \left(a^2\right)^2\right],$$
$$R_1 = \frac{1}{\gamma}\left(a^2\gamma + a^1\right)\mathrm{e}^{\gamma t} - a^1 t, \quad R_2 = \frac{1}{\gamma}\mathrm{e}^{\gamma t}a^2 - a^2 t.$$

The corresponding Pfaffian is

$$\begin{aligned} P^1(a,\dot{a},t) &= R_1(t,a)\dot{a}^1 + R_2(t,a)\dot{a}^2 - B(t,a) \\ &= \left(\frac{1}{\gamma}(a^2\gamma + a^1)\mathrm{e}^{\gamma t} - a^1 t\right)\dot{a}^1 \\ &\quad + \left(\frac{1}{\gamma}\mathrm{e}^{\gamma t}a^2 - a^2 t\right)\dot{a}^2 - \frac{1}{2}\left[\left(a^1\right)^2 + \left(a^2\right)^2\right]. \end{aligned}$$

Using the Euler mid-point rule, the Pfaffian P^1 can be expressed in discrete Pfaffian as follows:

$$\begin{aligned} P^1_\mathrm{d} &= \left(\frac{1}{\gamma}\left(\frac{\gamma}{2}\left(a^2_k + a^2_{k+1}\right) + \frac{a^1_k + a^1_{k+1}}{2}\right)\exp\left[\gamma\left(\frac{t_k + t_{k+1}}{2}\right)\right]\right. \\ &\quad \left. - \frac{\left(a^1_k + a^1_{k+1}\right)}{2}\frac{\left(t_k + t_{k+1}\right)}{2}\right)\left(a^1_{k+1} - a^1_k\right) \\ &\quad + \left(\frac{1}{\gamma}\exp\left[\gamma\left(\frac{t_k + t_{k+1}}{2}\right)\right]\frac{\left(a^2_k + a^2_{k+1}\right)}{2}\right. \\ &\quad \left. - \frac{\left(a^2_k + a^2_{k+1}\right)}{2}\left(\frac{t_k + t_{k+1}}{2}\right)\right)\left(a^2_{k+1} - a^2_k\right) \\ &\quad - \frac{\left(t_{k+1} - t_k\right)}{2}\left[\left(\frac{a^1_k + a^1_{k+1}}{2}\right)^2 + \left(\frac{a^2_k + a^2_{k+1}}{2}\right)^2\right]. \end{aligned}$$

(ii)

$$B = \frac{1}{2}\mathrm{e}^{\gamma t}\left[\left(a^1\right)^2 + \left(a^2\right)^2 + \gamma a^1 a^2\right],$$
$$R_1 = \frac{1}{2}\mathrm{e}^{\gamma t}a^2, R_2 = -\frac{1}{2}\mathrm{e}^{\gamma t}a^1.$$

The corresponding Pfaffian is

$$\begin{aligned} P^2(a,\dot{a},t) &= R_1(t,a)\dot{a}^1 + R_2(t,a)\dot{a}^2 - B(t,a) \\ &= \left(\frac{1}{2}\mathrm{e}^{\gamma t}a^2\right)\dot{a}^1 - \left(\frac{1}{2}\mathrm{e}^{\gamma t}a^1\right)\dot{a}^2 \\ &\quad - \frac{1}{2}\mathrm{e}^{\gamma t}\left[\left(a^1\right)^2 + \left(a^2\right)^2 + \gamma a^1 a^2\right]. \end{aligned}$$

In the same way, the discrete Pfaffian P^2_d is

$$\begin{aligned} P^2_\mathrm{d} &= \frac{1}{2}\exp\left[\gamma\frac{t_{k+1} + t_k}{2}\right]\left(\frac{a^2_{k+1} + a^2_k}{2}\right)\left(a^1_{k+1} - a^1_k\right) \\ &\quad - \frac{1}{2}\exp\left[\gamma\frac{t_{k+1} + t_k}{2}\right]\left(\frac{a^1_{k+1} + a^1_k}{2}\right)\left(a^2_{k+1} - a^2_k\right) \\ &\quad - \frac{1}{2}\left(t_{k+1} - t_k\right)\exp\left[\gamma\frac{t_{k+1} + t_k}{2}\right] \\ &\quad \times\left[\left(\frac{a^1_k + a^1_{k+1}}{2}\right)^2 + \left(\frac{a^2_k + a^2_{k+1}}{2}\right)^2\right. \\ &\quad \left. + \gamma\left(\frac{a^1_k + a^1_{k+1}}{2}\right)\left(\frac{a^2_k + a^2_{k+1}}{2}\right)\right]. \end{aligned}$$

The discrete Pfaffians P^1_d and P^2_d are substituted into Eqs. (15) and (16) respectively. The initial conditions and the damping coefficient are

$$t_0 = 0, \quad E_0 = 8, \quad a^1_0 = 0, \quad a^2_0 = 4,$$
$$b^1_0 = 0, b^2_0 = 1, \gamma = 0.05.$$

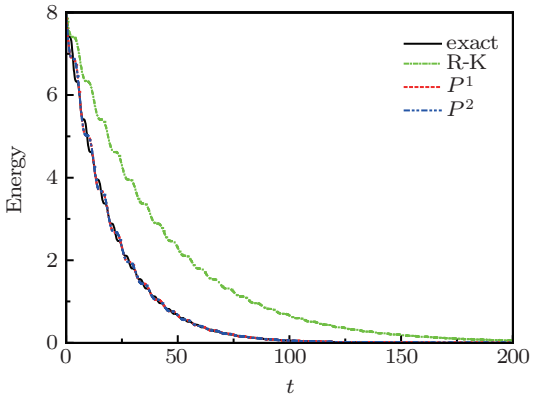


Fig. 1. (color online) Energies computed with variational, R-K methods and compared with the exact solution.

In Fig. 1, the time evolutions of the energy by the discrete variational method and 2-order R-K method are shown and compared with the exact solution. We can see clearly that the variational methods are better than the R-K method in simulating the energy dropping of the one-dimensional damped

oscillator system, and the results of the variational methods accord well with the exact solution. In Fig. 2, we present the relative errors of the exact solution and the numerical solution obtained by different methods of constructing Birkhoffian functions. From the figure, we can see that the energy errors obtained by different methods of constructing Birkhoffian functions are nearly the same.

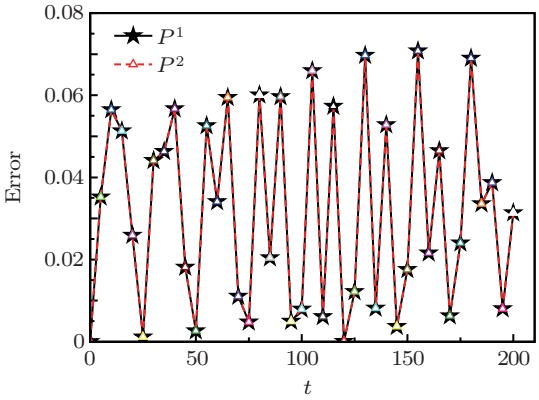


Fig. 2. (color online) Comparison of energy error between two constructing methods.

5. Conclusions

The above analysis and numerical results illustrate that one can solve numerically the dynamical equations by discrete variational methods in the Birkhoffian framework for the systems with a non-simple symplectic structure. Since the discrete variational methods can preserve well the general symplectic structures of dynamical systems, we can obtain more precise numerical results than those from the traditional methods such as R-K. At the same time, we present that the different methods of constructing Birkhoffian functions are completely equivalent, and the numerical results, good or bad, are independent of the constructing methods of Birkhoffian functions, but are mainly dependent on whether the numerical methods are able to preserve the inherent nature of the dynamical system. The example shows that it is significant to study the system with a non-standard symplectic structure by discrete variational methods in the Birkhoffian framework.

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