Cumulative solutions of nonlinear longitudinal vibration in isotropic solid bars*

Qian Zu-Wen(钱祖文)†

Institute of Acoustics, Chinese Academy of Sciences, Beijing 100190, *China*

(Received 9 September 2013; revised manuscript received 27 October 2013; published online 10 April 2014)

Based on the strain invariant relationship and taking the high-order elastic energy into account, a nonlinear wave equation is derived, in which the excitation, linear damping, and the other nonlinear terms are regarded as the first-order correction to the linear wave equation. To solve the equation, the biggest challenge is that the secular terms exist not only in the fundamental wave equation but also in the harmonic wave equation (unlike the Duffing oscillator, where they exist only in the fundamental wave equation). In order to overcome this difficulty and to obtain a steady periodic solution by the perturbation technique, the following procedures are taken: (i) for the fundamental wave equation, the secular term is eliminated and therefore a frequency response equation is obtained; (ii) for the harmonics, the cumulative solutions are sought by the Lagrange variation parameter method. It is shown by the results obtained that the second- and higher-order harmonic waves exist in a vibrating bar, of which the amplitude increases linearly with the distance from the source when its length is much more than the wavelength; the shift of the resonant peak and the amplitudes of the harmonic waves depend closely on nonlinear coefficients; there are similarities to a certain extent among the amplitudes of the odd- (or even-) order harmonics, based on which the nonlinear coefficients can be determined by varying the strain and measuring the amplitudes of the harmonic waves in different locations.

Keywords: nonlinear acoustics, compliance nonlinearity, mesoscopic nonlinearity, Duffing's oscillator

PACS: 43.25.+y, 43.35.+d, 91.60.Lj **DOI:** 10.1088/1674-1056/23/6/064301

1. Introduction

In general, the solid media are nonlinear. For a compact medium, the nonlinearity originates from the anharmonic interactions between the atoms, where one can regard the nonlinearity as the one on the atomic scale. On the other hand, if the medium is incompact (cracked and/or porous and with large compliance modulus), there is another nonlinearity besides the nonlinearity on an atomic scale, which can be referred to as the nonlinearity on a mesoscopic scale or compliant nonlinearity. Some experimental data published in journals (for example Ref. [1]) indicate that many rocks are porous and/or cracked, of which the mesoscopic nonlinearity is much larger than the atomic nonlinearity. Furthermore, resonant peak shifts far from the linear oscillator toward the left and numerous higher order harmonic waves are observed in several experiments. The experimental results also show that the phenomena mentioned above depend on strains, porosity, fluid saturation, environmental factors, etc.[2–4] On the other hand, the so-called non-equilibrium dynamical phenomena are observed in some experiments, for example, as the strain goes above a threshold, certain quantities (such as the resonant frequency) cannot return to the initial values in a timely manner during repeated measurements.[5,6] Even though there are many fascinating prospects of applications of sound propagation in earth materials and the nondestructive detection in solids, etc., the physical mechanism needs to be researched thoroughly.

Applying nonlinear acoustics to the relevant problems, the wave equation can be obtained only when the expression for elastic energy is invoked, from which the acoustic propagation in solid can be investigated. Due to certain mathematical difficulties in earlier studies, only the third-order elastic energy was processed, where the elastic constant corresponding to it is the second nonlinear coefficient,[7,8] i.e.

$$\beta = \frac{3}{2} + \frac{l+2m}{\lambda+2\mu}, \tag{1}$$

where λ and μ are the Lamé constants. Defined by Murnaghan, l and m are the third-order elastic constants. To the author's knowledge, there have been some experimental data about l and m for several metals (low-carbon steel and aluminum),[9–11] from which the second-order nonlinear coefficients are estimated to be less than 10. On the other hand, there are several authors (for instance in Refs. [2] and [12]) who analyzed and investigated the measured data in rock samples and estimated the data for $|\beta|$ that may be in a range of much larger than 10, obviously, much larger than the above-mentioned data of metals.

Furthermore, earlier results[1,13–15] also showed that the sound speed increases and the attenuation decreases when pressure is exerted on the rock sample. On the other hand, in the frequency sweep experiments conducted in a thin bar of such a type of material (for example, as shown in Ref. [3]), it

*Project supported by the National Natural Science Foundation of China (Grant No. 11274337).

†Corresponding author. E-mail: qianzw@mail.ioa.ac.cn

© 2014 Chinese Physical Society and IOP Publishing Ltd

is found that the shape of the amplitude–frequency response curve appears to be similar to that of the nonlinear soft spring.

In order to research the acoustic vibration in such a medium, we use the strain invariant to obtain the one-dimensional nonlinear wave equation, and obtain the approximate solution for a thin bar by means of the multiple scale perturbation technique.

2. Elastic energy and nonlinear wave equation

As is well known, there are three strain invariants (I_1, I_2, I_3)[7,16,17] in isotropic solids, which can be written as

$$\begin{aligned} I_1 &= \eta_{11}+\eta_{22}+\eta_{33}, \\ I_2 &= \begin{vmatrix} \eta_{11}\eta_{12} \\ \eta_{21}\eta_{22} \end{vmatrix} + \begin{vmatrix} \eta_{22}\eta_{23} \\ \eta_{32}\eta_{33} \end{vmatrix} + \begin{vmatrix} \eta_{33}\eta_{31} \\ \eta_{13}\eta_{11} \end{vmatrix}, \\ I_3 &= \begin{vmatrix} \eta_{11}\eta_{12}\eta_{13} \\ \eta_{21}\eta_{22}\eta_{23} \\ \eta_{31}\eta_{32}\eta_{33} \end{vmatrix}, \end{aligned} \tag{2}$$

where η_{ij} are the strains, which can be expressed as

$$\eta_{lk} = \frac{1}{2}\left[\frac{\partial x_i}{\partial a_l}\frac{\partial x_i}{\partial a_k} - \delta_{lk}\right] \tag{3}$$

and the elastic energy can be written as

$$\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5 + \cdots, \tag{4}$$

where ϕ_2 and ϕ_3 ... are the second- and third-order ... elastic energies, respectively, whose expressions are (see Refs. [7], [16], and [17])

$$\begin{aligned} \phi_2 &= \frac{\lambda+2\mu}{2}I_1^2 - 2\mu I_2, \\ \phi_3 &= \frac{l+2m}{3}I_1^3 - 2mI_1I_2 + nI_3. \end{aligned} \tag{5}$$

However, to the author's knowledge, the expressions for ϕ_4 and ϕ_5 are still unknown. So in this paper we can express them as

$$\begin{aligned} \phi_4 &= \gamma_1 I_1^4 + \gamma_2 I_1^2 I_2 + \gamma_3 I_2^2 + \gamma_4 I_1 I_3, \\ \phi_5 &= \delta_1 I_1^5 + \delta_2 I_2^2 I_1 + \delta_3 I_3 I_2 + \delta_4 I_4 I_1 + \delta_5 I_1^3 I_2 + \delta_6 I_1^2 I_3, \end{aligned} \tag{6}$$

where γ_i, δ_i, ... are the fourth- and fifth-order, ..., elastic constants.

Generally, the expressions associating the elastic energy with the strains and the subsequent wave equation can be derived; however, the amount of mathematical calculation is so tremendous that most researchers eventually give up. On the other hand, in a solid medium of finite size, the boundary conditions cannot be satisfied if one assumes that there exists only one simple vibration mode (for instance, only the longitudinal mode). Thus, in this paper we assume that the longitudinal scale of the system is always much larger than the transverse scale (a thin bar); in this way, we can consider its one-dimensional vibration and indicate the position x and the displacement (u, v, w) as follows:

$$\begin{aligned} &x = a + u, \\ &u = u(a,t), v = w = 0. \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (3) yields

$$\eta_{11} = u_a + \frac{1}{2}u_a^2, \tag{8}$$

and others $\eta_{ij} = 0$, where

$$u_a = \frac{\partial u}{\partial a}. \tag{9}$$

Substituting Eqs. (8) and (9) into Eqs. (5) and (6), one has

$$I_1 = \eta_{11}, \ \ I_2 = I_3 = 0, \tag{10}$$

and

$$\begin{aligned} &\phi_2 = \frac{1}{2}(\lambda+2\mu)\eta_{11}, \ \ \phi_3 = \frac{1}{3}(l+2m)(\eta_{11})^3, \\ &\phi_4 = \gamma_1\eta_{11}^4, \ \ \phi_5 = \delta_1\eta_{11}^5, \\ &\phi_6 = \xi_1\eta_{11}^6, \ \ \phi_7 = \zeta_1\eta_{11}^6, \ldots. \end{aligned} \tag{11}$$

Based on Ref. [7] or Ref. [17], one has the relation between stress and elastic energies as follows:

$$T_{ik} = J_{il}\frac{\partial\phi}{\partial\eta_{kl}}, \ \ J_{il} = \frac{\partial x_i}{\partial a_l},$$

then one can obtain

$$\begin{aligned} T_{11} &= (\lambda+2\mu) \\ &\quad \times \{u_a + \beta u_a^2 + \gamma u_a^3 + \cdot\delta u_a^4 + \xi u_a^5 + \eta u_a^6 + \cdots\}, \\ T_{ij} &= 0 \ \ i,j \neq 1. \end{aligned} \tag{12}$$

The wave equation can be written as

$$\rho_0\frac{\partial^2 x}{\partial t^2} = \frac{\partial T_{11}}{\partial a}. \tag{13}$$

From Eq. (12) one has finally

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial a^2}, \tag{14}$$

where

$$\begin{aligned} c &= c_0\{1 + 2\beta u_a^+ 3\gamma u_a^2 + 4\delta u_a^3 \\ &\quad + 5\xi u_a^4 + 6\eta u_a^5 + \cdots\}^{1/2}, \end{aligned} \tag{15}$$

$$c_0 = \sqrt{\frac{\lambda+2\mu}{\rho_0}}, \tag{16}$$

$$\left.\begin{aligned} \gamma &= \frac{1}{2} + 2\frac{l+2m}{\lambda+2\mu} + \frac{4\gamma_1}{\lambda+2\mu}, \\ \delta &= \frac{5}{4}\frac{l+2m}{\lambda+2\mu} + \frac{10\gamma_1}{\lambda+2\mu} + \frac{5\delta_1}{\lambda+2\mu}, \\ \xi &= \frac{1}{4}\frac{l+2m}{\lambda+2\mu} + \frac{10\gamma_1}{\lambda+2\mu} + \frac{15\delta_1}{\lambda+2\mu} + \frac{6\xi_1}{\lambda+2\mu}, \\ \eta &= \cdots, \end{aligned}\right\} \tag{17}$$

where β, γ, ξ, η ... are the second-, third-, ..., order nonlinear coefficients, respectively. Equation (15) shows that the sound speed of the longitudinal waves depends on strain u_a, which is due to the nonlinearity of the medium. Substituting Eq. (15) into Eq. (14) and differentiating the resulting equation just obtained with respect to a, subsequently, yield

$$\begin{aligned}&\frac{\partial^2 u_a}{\partial t^2} - c_0^2 \frac{\partial^2 u_a}{\partial a^2} \\ &= c_0^2 \frac{\partial^2}{\partial a^2}\{\beta u_a^2 + \gamma u_a^3 + \delta u_a^4 + \xi u_a^5 + \eta u_a^6 + \cdots\} \\ &\quad - 2R\frac{\mathrm{d}u_a}{\mathrm{d}t} + E(a,t), \end{aligned} \tag{18}$$

where the linear damping and the excitation force $-2R\mathrm{d}u_a/\mathrm{d}t + E(a,t)$ are added to Eq. (18) phenomenologically. On the other hand, on the left-hand side of Eq. (18), let $c_0^2\partial^2 u_a/\partial a^2 \to \omega_0^2 u_a$ and in the brackets on the right-hand side of Eq. (18), let $\beta, \xi, \eta, \ldots = 0$ except $\gamma \neq 0$ and

$$c_0^2 \frac{\partial^2}{\partial a^2}\{\gamma u_a^3\} \to \gamma u_a^3,$$

then equation (18) becomes the Duffing's vibration equation.

3. Multiple-scale perturbation and steady period solutions[18,19]

In order to obtain the steady period solutions conveniently, multiple-scale perturbation is a flexible technique, where one begins with introducing new independent variables according to

$$T_n = \varepsilon^n t, \quad n = 0, 1, 2, \ldots, \tag{19}$$

where ε is a small quantity and $T_1 = \varepsilon t$ is the slow-scale (time) variable. It follows that the derivatives with respect to t become expansions in terms of the partial derivatives with respect to the T_n according to

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}T_0}{\mathrm{d}t}\frac{\partial}{\partial T_0} + \frac{\mathrm{d}T_1}{\mathrm{d}t}\frac{\partial}{\partial T_1} + \cdots = D_0 + \varepsilon D_1 + \cdots, \tag{20}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + \varepsilon 2D_0D_1 + \varepsilon^2[D_1^2 + 2D_0D_2] + \cdots. \tag{21}$$

One assumes that the solution of Eq. (18) can be represented by

$$\begin{aligned} u_a(a,t;\varepsilon) &= u_a^{(0)}(a,T_0,T_1) + \varepsilon u_a^{(1)}(a,T_0,T_1) \\ &\quad + \varepsilon^2 u_a^{(2)}(a,T_0,T_1) + \cdots. \end{aligned} \tag{22}$$

From Eq. (18), we can see that it becomes a linear wave equation when its left side equals zero and one can regard the terms on its right side as a correction to the linear wave equation. If the nonlinearity of the system is not so large that the excitation, linear damping as well as the other nonlinear terms are the first-order small quantities at least, thus, under the perturbation approximation, equation (18) can be written as

$$\begin{aligned}&\frac{\partial^2 u_a}{\partial t^2} - c_0^2 \frac{\partial^2 u_a}{\partial a^2} \\ &= \varepsilon c_0^2 \frac{\partial^2}{\partial a^2}\{\beta u_a^2 + \gamma u_a^3 + \delta u_a^4 + \xi u_a^5 + \eta u_a^6 + \cdots\} \\ &\quad - \varepsilon 2R\frac{\mathrm{d}u_a}{\mathrm{d}t} + \varepsilon E(a,t). \end{aligned} \tag{23}$$

If the excitation is realized by a sweep frequency technique, the instantaneous frequency can be expressed as follows:

$$\omega = \omega_0 + \varepsilon\Delta\omega, \tag{24}$$

then one has

$$\varepsilon E(a,t) = \varepsilon F\cos(\omega_0 T_0 + \Delta\omega T_1 - k_0 a), \quad k_0 = \frac{\omega_0}{c_0}, \tag{25}$$

where ω_0 is the natural angular frequency of the fundamental wave. Substituting Eqs. (20)–(22), (24), (25) into Eq. (23) and equating the coefficients of ε, ε^2, one has

$$\left[D_0^2 - c_0^2\frac{\partial^2}{\partial a^2}\right] u_a^{(0)} = 0, \tag{26}$$

$$\begin{aligned}&\left[D_0^2 - c_0^2\frac{\partial^2}{\partial a^2}\right] u_a^{(1)} \\ &= -2D_0D_1u_a^{(0)} - 2RD_0u_a^{(0)} + F\cos(\omega_0 T_0 + \Delta\omega T_1 - k_0 a) \\ &\quad + c_0^2\frac{\partial^2}{\partial a^2}\left\{\beta\left[u_a^{(0)}\right]^2 + \gamma\left[u_a^{(0)}\right]^3 + \cdots\right\}. \end{aligned} \tag{27}$$

The solution of Eq. (26) can be written as

$$u_a^{(0)} = A(T_1)\,\mathrm{e}^{\mathrm{i}(\omega_0 T_0 - k_0 a)} + \text{c.c.}, \tag{28}$$

where $A(T_1)$ is an underdetermined function. Substituting Eq. (28) into Eq. (27) yields

$$\begin{aligned}&\left[D_0^2 - c_0^2\frac{\partial^2}{\partial a^2}\right] u_a^{(1)} \\ &= \left\{-2\mathrm{i}\omega_0\left[A'(T_1) + RA(T_1)\right] - \omega_0^2[3\gamma A^2(T_1)\bar{A}(T_1)\right. \\ &\quad \left. + 10\xi A^3(T_1)\bar{A}^2(T_1) + \cdots] + \frac{1}{2}F\,\mathrm{e}^{\mathrm{i}\Delta\omega T_1}\right\}\mathrm{e}^{\mathrm{i}(\omega_0 T_0 - k_0 a)} \\ &\quad - \omega_0^2\{[4\beta A^2(T_1) + 16\delta A^3(T_1)\bar{A}(T_1) \\ &\quad + 15\eta A^4(T_1)\bar{A}^2(T_1) + \cdots]\mathrm{e}^{\mathrm{i}2(\omega_0 T_0 - k_0 a)} \\ &\quad + [9\gamma A^3(T_1) + 5\xi A^4(T_1)\bar{A}(T_1) + \cdots]\mathrm{e}^{\mathrm{i}3(\omega_0 T_0 - k_0 a)} \\ &\quad + [16\delta A^4(T_1) + 6\eta A^5(T_1)\bar{A}(T_1) + \cdots]\mathrm{e}^{\mathrm{i}4(\omega_0 T_0 - k_0 a)} \\ &\quad + [25\xi A^5(T_1) + \cdots]\mathrm{e}^{\mathrm{i}5(\omega_0 T_0 - k_0 a)} + \cdots\} + \text{c.c.} \end{aligned} \tag{29}$$

In order to eliminate the secular term from $u_a^{(1)}$, the coefficient associated with $\mathrm{e}^{\mathrm{i}(\omega_0 T_0 - k_0 a)}$ must be zero, i.e.,

$$\begin{aligned}&-2\mathrm{i}\omega_0\left[A'(T_1) + RA(T_1)\right] - \omega_0^2[3\gamma A^2(T_1)\bar{A}(T_1) \\ &\quad + 10\xi A^3(T_1)\bar{A}^2(T_1) + \cdots] + \frac{1}{2}F\,\mathrm{e}^{\mathrm{i}\Delta\omega T_1} = 0, \end{aligned}$$

$$A'(T_1) = \frac{\mathrm{d}A}{\mathrm{d}T_1}. \tag{30}$$

Equation (30) is just the same as Eq. (4.10) in Ref. [18]. Following Ref. [18] let

$$\begin{aligned} A(T_1) &= \frac{1}{2}B(T_1)\,\mathrm{e}^{\mathrm{i}\varphi(T_1)}, \\ \varphi(T_1) &= \Delta\omega T_1 - \psi(T_1), \end{aligned} \tag{31}$$

and substituting Eq. (31) into Eq. (30), one has

$$\begin{aligned} -2\mathrm{i}\omega_0\left[A'(T_1) + RA(T_1)\right] - \omega_0^2[3\gamma A^2(T_1)\bar{A}(T_1) \\ + 10\xi A^3(T_1)\bar{A}^2(T_1) + \cdots] + \frac{1}{2}F\,\mathrm{e}^{\mathrm{i}\Delta\omega T_1} = 0. \end{aligned} \tag{32}$$

In order to obtain the steady-state motion, let

$$B'(T_1) = \psi'(T_1) = 0,$$

and equation (32) becomes

$$\begin{aligned} RB(T_1) &= \frac{F}{2\omega_0}\sin\psi(T_1), \\ \Delta\omega B(T_1) - \omega_0\left[\frac{3\gamma}{8}B^3(T_1) + \frac{5}{16}\xi B^5(T_1) + \cdots\right] \\ &= -\frac{F}{2\omega_0}\cos\psi(T_1), \end{aligned} \tag{33}$$

Eliminating $\psi(T_1)$ from Eq. (33) yields

$$\begin{aligned} \left\{R^2 + \omega_0^2\left[\frac{\Delta\omega}{\omega_0} - \frac{3}{8}\gamma B^2 - \frac{5}{16}\xi B^4 - \frac{35}{128}\varsigma B^6 - \cdots\right]^2\right\}B^2 \\ = \frac{F^2}{4\omega_0^2}. \end{aligned} \tag{34}$$

This is the relationship between the fundamental amplitude and the driving force. If ξ, η, $\ldots = 0$ and only $\gamma \neq 0$, obviously, equation (34) becomes the frequency response equation for the Duffing oscillator. Substituting Eqs. (31) and (33) into Eq. (28), we find that the first approximation solution is given by

$$u_a^{(0)} = B\cos(\omega t - k_0 a - \gamma) + O(\varepsilon), \tag{35}$$

of which the amplitude–frequency response will be described by Eq. (34).

4. Cumulative solutions

Substituting Eq. (31) into Eq. (29) yields

$$\left[D_0^2 - c_0^2\frac{\partial^2}{\partial a^2}\right]u_a^{(1)} = \sum_{n=2}^{\infty}\alpha_n\,\mathrm{e}^{\mathrm{i}n(\omega_0 t - k_0 a + \varphi)} + \text{c.c.}, \tag{36}$$

where

$$\begin{aligned} \alpha_2 &= \beta B^2 + \delta B^4 + \frac{15}{16}\eta B^6 + \cdots, \\ \alpha_3 &= \frac{9}{8}\gamma B^3 + \frac{45}{32}\xi B^5 + \frac{189}{128}\varsigma B^7 + \cdots, \\ \alpha_4 &= \delta B^4 + \frac{3}{2}\eta B^6 + \cdots, \\ \alpha_5 &= \frac{25}{32}\xi B^5 + \frac{175}{128}\varsigma B^7 + \cdots, \\ \alpha_6 &= \frac{9}{16}\eta B^6 + \cdots, \\ \alpha_7 &= \frac{49}{128}\zeta B^7 + \cdots, \\ &\cdots \end{aligned} \tag{37}$$

Let

$$u_a^{(1)} = u_a^{(1,2)} + u_a^{(1,3)} + u_a^{(1,4)} + u_a^{(1,5)} + \cdots. \tag{38}$$

Substituting Eq. (38) into Eq. (36), one has

$$\begin{aligned} \left[D_0^2 - c_0^2\frac{\partial^2}{\partial a^2}\right]u_a^{(1,n)} = \alpha_n\,\mathrm{e}^{\mathrm{i}n[\omega_0 t - k_0 a + \varphi]} + \text{c.c.}, \\ n = 2, 3, 4, \ldots. \end{aligned} \tag{39}$$

This result shows that each harmonic wave has its secular term, which makes it more difficult to solve the problem. In order to find the periodic solutions of Eq. (26), a method is proposed as follows. For the fundamental wave solution, the conditions must be satisfied that the secular term is eliminated; however, for the harmonic waves, the cumulative solutions will be sought instead of eliminating secular terms. It is known that the cumulative solutions are also periodic. In the following, the Lagrange parameter variation method will be used to obtain the cumulative solutions. Let

$$u_a^{(1,n)} = f_n(a)\,\mathrm{e}^{\mathrm{i}n[\omega_0 t + \varphi]} + \text{c.c.}$$

Substituting it into Eq. (39) yields

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}a^2} + (nk_0)^2\right]f_n(a) = -\frac{\alpha_n}{c_0^2}\,\mathrm{e}^{-\mathrm{i}nk_0 a} + \text{c.c.}$$

The cumulative solutions can be (see Appendix A)

$$f_n(a) = -\frac{\alpha_n}{4n^2\omega_0^2}(1 + 2\mathrm{i}nk_0 a)\,\mathrm{e}^{-\mathrm{i}nk_0 a} + \text{c.c.},$$

and

$$\begin{aligned} u_a^{(1)} = -\frac{1}{2\omega_0^2}\sum_{n=2}^{\infty}\frac{\alpha_n}{n^2}\sqrt{1 + 4n^2(k_0 a)^2} \\ \times \cos[n(\omega_0 t - k_0 a + \varphi + \psi)], \\ \tan\psi = 2nk_0 a. \end{aligned} \tag{40}$$

If $k_0 a >> 1$, $\psi = \pi/2$, one has

$$u_a^{(1)} \approx \frac{1}{\omega_0^2}\sum_{n=2}^{\infty}\frac{\alpha_n}{n}k_0 a\cos\left[n\left(\omega_0 t - k_0 a + \varphi + \frac{\pi}{2}\right)\right]. \tag{41}$$

Since $u_a^{(1)}$ is the first-order quantity, all of the α_n are of the same order of magnitude. Furthermore, all amplitudes of the harmonic waves increase linearly with the distance from the source (cumulative).

5. Discussion

In order to research the mechanism of nonlinear vibration, the authors of Refs. [5] and [6] applied the Duffing oscillator to the experimental data and carried out numerical computations. As is well known, the Duffing oscillator is a special ideal model, a lumped system which only performs a lump motion with no stationary waves inside, so that it only has a natural frequency and therefore only the fundamental wave has the secular term. Moreover, it only has the third-order harmonic wave in $u_a^{(1)}$ that has no cumulative effects. However, various harmonic waves have indeed been observed in numerous experiments.[2,4] The investigation in this paper is performed on the longitudinal vibration for an isotropic solid bar, which is a logically reasonable model (with the stationary waves occurring in it) and, particularly, not only the fundamental wave but also various harmonic waves have the resonant frequencies $n\omega_0$ $(n = 1, 2, \ldots)$, indicating that all of them must have the secular terms and consequently make the problem much more complicated. In order to overcome these difficulties, a suggestion is made as follows. For the fundamental wave, the secular term is eliminated, however, for the harmonic waves the cumulative solutions are solved so that the useful results such as Eqs. (37), (40), and (44) are given, based on which the second- third-, ..., order coefficients can be measured through measuring the resonant frequency shift and amplitude of each of the harmonic waves at different locations.

From Eq. (34), we can obtain the resonant peak shift as follows:

$$\frac{\Delta\omega}{\omega_0} = \left\{\frac{3\gamma}{8} + \frac{5}{16}\xi B^2 + \frac{35}{128}\varsigma B^4 + \cdots\right\} B^2 \pm \frac{1}{\omega_0}\sqrt{\left(\frac{F}{2\omega_0 B}\right)^2 - R^2}. \tag{42}$$

Obviously, $\Delta\omega$ is real only when the quantity in the radical sign is not less than zero, thus, the maximal amplitude equals

$$B_m = \frac{F}{2\omega_0 R}, \tag{43}$$

and the maximal frequency shift is

$$\frac{\Delta\omega_m}{\omega_0} = \left[\frac{3\gamma}{8} + \frac{5}{16}\xi B_m^2 + \frac{35}{128}\varsigma B_m^4 + \cdots\right] B_m^2. \tag{44}$$

This result indicates that when a frequency sweep experiment is conducted in this system where the direction of the frequency shift of the resonant peak (or the sign of $\Delta\omega_m$) depends on the signs and the magnitudes of γ, δ, ξ, η, ζ, ..., unfortunately, we cannot find any data of various nonlinear coefficients and can only make the following analysis: (i) for infinitesimal strain, the absolute value of the frequency shift is proportional to B_m^2; (ii) if signs of ξ, η, ς, ... are negative and as maximum strain increases, the absolute value of the frequency shift is proportional to $(B_m)^n$ and $n > 2$; (iii) since the Lamé coefficients are positive and $\beta, \gamma < 0$, we can guess that if $\delta, \xi > 0$ and $\eta, \varsigma < 0$, in this way, the sign of the subsequent term in the amplitude of each harmonic wave will vary alternately with the increasing of B_m. Moreover, from Eq. (37), a fact is worth noticing that certain similarities exist among the amplitudes of odd- (or even-)order harmonic waves, for example, by comparing the amplitudes of various harmonics we can see that there is only an extra term βB_m^2 between second- and fourth-order harmonic waves and γB^3 between third- and fifth-order harmonic waves. Based on Eqs. (37), (40), and (44), the nonlinear coefficients can be measured by varying the strains and measuring the amplitudes of the harmonic waves at different locations.

The theoretical results are only valid in the quasi-equilibrium dynamical process, i.e., $B < \varepsilon_M$ as mentioned in Refs. [5] and [6].

6. Conclusions

The Duffing oscillator is a special ideal model-lumped system, and it cannot be applied to harmonic waves. In this paper, the longitudinal vibration in an isotropic solid bar is investigated. In order to overcome the secular-terms obstacles both in fundamental wave and in various harmonic waves, we propose that the secular term is eliminated for the former and the cumulative solutions are sought for the latter and therefore, the useful Eqs. (37) and (38) are obtained, which show that the frequency shift and the amplitude of harmonics depend closely on the nonlinear coefficient. Based on Eqs. (37), (40), and (44), the nonlinear coefficients can be measured by varying the strains and measuring the amplitudes of the harmonic waves at different locations.

Appendix A: Cumulative solutions

From Eq. (39) one has

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}a^2} + \kappa^2\right] f(a) = A\,\mathrm{e}^{-\mathrm{i}\kappa a}, \tag{A1}$$

where A is a constant. Obviously, this equation has cumulative solutions, which can be solved by Lagrange parameter variation method.[17] Let

$$f(a) = A_1(a)\,\mathrm{e}^{\mathrm{i}\kappa a} + A_2(a)\,\mathrm{e}^{-\mathrm{i}\kappa a}, \tag{A2}$$

then we will have

$$\frac{\mathrm{d}f(a)}{\mathrm{d}a} = \frac{\mathrm{d}A_1(a)}{\mathrm{d}a}\mathrm{e}^{\mathrm{i}\kappa a} + \frac{\mathrm{d}A_2(a)}{\mathrm{d}a}\mathrm{e}^{-\mathrm{i}\kappa a} + \mathrm{i}\kappa\left[A_1(a)\,\mathrm{e}^{\mathrm{i}\kappa a} - A_2(a)\,\mathrm{e}^{-\mathrm{i}\kappa a}\right].$$

Assuming that

$$\frac{\mathrm{d}A_1(a)}{\mathrm{d}a}\mathrm{e}^{\mathrm{i}\kappa a} + \frac{\mathrm{d}A_2(a)}{\mathrm{d}a}\mathrm{e}^{-\mathrm{i}\kappa a} = 0, \tag{A3}$$

one has

$$\frac{\mathrm{d}^2 f(a)}{\mathrm{d}a^2} = -\kappa^2 f(a) + \mathrm{i}\kappa \left[\frac{\mathrm{d}A_1(a)}{\mathrm{d}a} \mathrm{e}^{\mathrm{i}\kappa a} - \frac{\mathrm{d}A_2(a)}{\mathrm{d}a} \mathrm{e}^{-\mathrm{i}\kappa a} \right].$$

Substituting them into Eq. (A1) yields

$$\frac{\mathrm{d}A_1(a)}{\mathrm{d}a} \mathrm{e}^{\mathrm{i}\kappa a} - \frac{\mathrm{d}A_2(a)}{\mathrm{d}a} \mathrm{e}^{-\mathrm{i}\kappa a} = \frac{A}{\mathrm{i}\kappa} \mathrm{e}^{-\mathrm{i}\kappa a}. \tag{A4}$$

From Eq. (A3) one has

$$\frac{\mathrm{d}A_2(a)}{\mathrm{d}a} = -\frac{A}{2\mathrm{i}\kappa} \tag{A5}$$

or

$$A_2(a) = -\frac{A}{2\mathrm{i}\kappa} a. \tag{A6}$$

Substituting Eq. (A5) into Eq. (A3) yields

$$A_1(a) = \frac{A}{4\kappa^2} \mathrm{e}^{-2\mathrm{i}\kappa a}. \tag{A7}$$

Inserting Eqs. (A6) and (A7) into Eq. (A2), the cumulative solutions can be obtained as

$$f(a) = \frac{A}{4\kappa^2} (1 + 2\mathrm{i}\kappa a) \mathrm{e}^{-\mathrm{i}\kappa a}. \tag{A8}$$

References

[1] Gist G A 1994 *J. Acoust. Soc. Am.* **96** 1158

[2] Meegan G D, Johnson P A, Guyer R A and McCall K R 1993 *J. Acoust. Soc. Am.* **94** 3387

[3] Johnson P L, Bernard Z and Rasolofosaon P N J 1996 *J. Geophys. Res.* **101**(B5) 11533

[4] Van Den Abeele, Koen E-A and Johnson P A 1996 *J. Acoust. Soc. Am.* **99** 3334

[5] Pasqualini D, Heitmann K, TenCate J, Tencate J A, Habib S, Higdon D and Johnson P A 2007 *J. Geophy. Res.* **112**(B1) B01204

[6] TenCate J A, Pasqualini D, Habib S, Heitmann K, Higdon D and Johnson P A 2004 *Phys. Rev. Lett.* **93** 065501

[7] Murnaghan F D 1951 *Finite Deformation of an Elastic Solid* (New York: Wiley Dover)

[8] Breazeal M A and Jacob P 1984 *Physical Acoustics*, Vol. 17, ed. Mason W P and Thurston R N (New York: Academic) pp. 2–60

[9] Smith R T, Stern R and Stephens W B 1966 *J. Acoust. Soc. Am.* **40** 1002

[10] Wang R Q, Li Y C and Zhang J D 1988 *Proceedings of the Third Western/Pacific Regional Acoustics Conference* (WESTPAC III88) Shanghai, China, p. 203

[11] Qian Z W and Jiang W H 1998 *Proc. 16th International Congress on Acoustics and 135th Meeting Acoustical Society of America*, Vol. III, ed. Kuhl P K and Crum L A, Seattle, Washington, USA, p. 1727

[12] Liu C, Feng X, He M, Yang B J and Zhao J G 2000 *Prog. Geophys.* **15** 68 (in Chinese)

[13] Demonico S N 1977 *Geophysics* **42** 1339

[14] Winkler K W and Mulphy W F 1984 *J. Acoust. Soc. Am.* **76** 820

[15] Winkler K W 1983 *J. Geophys. Research* **88** 9493

[16] Green Jr R E 1973 *Ultrasonic Investigation of Mechanical Properties in Treatise on Materials Science and Technology*, Vol. 3 (New York: Academic Press)

[17] Qian Z W 2009 *Nonlinear Acoustics*, 2nd edn. (Beijing: Science Press) Chap. 17 (in Chinese)

[18] Nayfeh A H and Mook D T 1979 *Nonlinear Oscillations* (Toronto: John Wiley) Chap. 4

[19] Chen Y S 2002 *Nonlinear Vibration* (Beijing: Higher Education Press) Chap. 7 (in Chinese)